The Teichmüller theory of the solenoid

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2000 Mathematics Subject Classification: 30F60, 30F45, 32H02, 32G05; 30C62

Keywords: Solenoid, inverse limit, baseleaf preserving mapping class group, virtual automorphism, Modular group, Teichmüller space, commensurator, Teichmüller extremal, Whitehead move, triangulation complex.

Contents

[*]Most of the recent results in this chapter were obtained in various collaborations of the author with S. Bonnot, A. Epstein, V. Markovic and R. Penner. I am indebted to them for the insights and efforts in our joint explorations. I am grateful to F. Bonahon, F. Gardiner, W. Goldman, L. Keen, M. Lyubich, J. Millson, J. Milnor, D. Sullivan and S. Wolpert for the various discussion we had regarding the surveyed material. This chapter was written while I was visiting the Department of Mathematics of the University of Maryland. I am grateful to the department for its warm hospitality during my visit.

1 Introduction

The compact solenoid $\mathcal{S}$ (also called the universal hyperbolic solenoid) was introduced by Sullivan [43] as the universal object in the category of all pointed, unbranched, finite sheeted coverings of a (base) closed surface of genus at least two ($\mathcal{S}$ can be thought of as a "universal closed surface"). The compact solenoid $\mathcal{S}$ is independent (as a topological space) of the choice of the base surface in the definition (as long as the genus is at least two).

More explicitly, the compact solenoid $\mathcal{S}$ is the inverse limit of the system of all pointed, unbranched, finite sheeted coverings of a closed surface of genus at least two. In fact, the inverse limit of an infinite tower of pointed, unbranched, finite sheeted coverings gives a homeomorphic object as long as the intersection of all fundamental groups in the tower (when considered as subgroups of the fundamental group of the base surface) is trivial. A particularly interesting tower is obtained by defining the n-th covering in the tower to have fundamental group equal to the intersection of all subgroups of index at most n.

Another description of the compact solenoid $\mathcal{S}$ is that it is a principal fiber bundle over a closed surface of genus at least two whose fibers are homeomorphic to a Cantor set with a topological group structure such that the base surface fundamental group is realized as a dense subgroup of the fiber group. If the base surface is given a fixed hyperbolic metric then the compact solenoid is explicitly realized as follows (see Section 2.2 or [37]). Let G be a subgroup of the Möbius group acting on the unit disk $\mathbf{D}$ which uniformizes the base surface and let $\omega \subset \mathbf{D}$ be a fundamental polygon for G. Then there exists a Cantor set $\hat{G}$ with the structure of a topological group and an injective homomorphism $G \hookrightarrow \hat{G}$ whose image is dense in $\hat{G}$ (the group $\hat{G}$ is defined in Section 2 and later in the Introduction). The compact solenoid $\mathcal{S}$ is the quotient of $\omega \times \mathbf{D}$ by the action of finitely many elements of G which pairwise identify the sides of ω, where the action of these elements on $\omega \times \mathbf{D}$ is given by a side pairing

Möbius action on the ω-factor and by right multiplication in the group $\hat{G}$ using the identification of G with its image in $\hat{G}$ (on the $\hat{G}$-factor). Thus a neighborhood of a point in the compact solenoid $\mathcal{S}$ is given by the product of the open fundamental polygon $\overset{\circ}{\omega}$ and a Cantor set $\hat{G}$. The boundary sides of ω are identified with the corresponding boundary sides of ω but on different "levels", i.e. the second factors (in $\hat{G}$) are different. If we follow the identifications of a single $\omega \times \{t\}$, for a fixed $t \in \hat{G}$, in the quotient we approach any point of $\mathcal{S}$ arbitrary close (because the image of G in $\hat{G}$ is dense).

The compact solenoid $\mathcal{S}$ is locally homeomorphic to a 2-disk times a Cantor set; each leaf (i.e. a path component) of $\mathcal{S}$ is dense in $\mathcal{S}$ and it is homeomorphic to the unit disk; a distinguished leaf of $\mathcal{S}$ is called the *baseleaf*. Moreover, $\mathcal{S}$ has a unique transverse measure, i.e. a holonomy invariant measure on each transverse set, which is induced by the Haar measure on the fiber group. The holonomy map is given by the action of the base surface group on the fiber group via its natural identification as a subgroup of the fiber.

We give some motivation for the study of the compact solenoid $\mathcal{S}$. The Ehrenpreis conjecture [13] states that for any two closed non-conformal Riemann surfaces of the same genus greater than 1 and for any $\epsilon > 0$ there exist two finite sheeted, unbranched, conformal covers that are $(1+\epsilon)$-quasiconformal. Since the universal cover of both surfaces is the unit disk, this is a question whether finite covers approximate the universal cover. Instead of considering two Riemann surfaces at the same time and finding their appropriate finite covers, it is (at least) conceptually more appropriate to have all Riemann surfaces in a single space. The space is the union of properly normalized embeddings in the universal Teichmüller space $T(\mathbf{D})$ of the Teichmüller spaces of all closed Riemann surfaces covering the base surface. The group of all isomorphisms between finite index subgroups of the fundamental group (called the *commensurator* of the surface group) acts naturally on the above union and the Ehrenpreis conjecture is equivalent to the statement that the action has dense orbits in the union [31]. It is natural to take the closure of the union in the universal Teichmüller space to obtain a Banach manifold and the action of the commensurator extends naturally to the closure [31].

Sullivan noticed the connection with the compact solenoid $\mathcal{S}$: instead of considering Riemann surfaces of different genera as points in a single space $T(\mathbf{D})$ as well as their limit points in $T(\mathbf{D})$, it is natural to form a single topological space (the compact solenoid $\mathcal{S}$) using all finite coverings of a base surface and to express Riemann surfaces of different genus as well as their limit points in $T(\mathbf{D})$ as different complex structures on the compact solenoid $\mathcal{S}$. Then the Ehrenpreis conjecture is equivalent to the statement that the action of the commensurator group $\mathrm{Comm}(\pi_1(S))$ of the base surface group

$\pi_1(S)$ on the Teichmüller space $T(\mathcal{S})$ of the compact solenoid $\mathcal{S}$ has dense orbits [5], [31].

C. Odden [37] showed that the Modular group $\text{Mod}(\mathcal{S})$ of the compact solenoid $\mathcal{S}$ is isomorphic to the commensurator group $\text{Comm}(\pi_1(\mathcal{S}))$ of the fundamental group $\pi_1(S)$ of the base surface S. This is in an analogy with the classical statement that the group of outer isomorphisms of the closed surface group is the mapping class group of the surface [8], [32], [2]. Thus, the compact solenoid $\mathcal{S}$ is a natural space for which the commensurator group is its Modular group. From the group theoretic point of view, the Modular group $Mod(\mathcal{S}) \equiv \text{Comm}(\pi_1(S))$ is describing the "hidden symmetries" of the surface group [25].

We are also interested in studying complex structures on the compact solenoid $\mathcal{S}$ from the view point of the complex analytic theory of Teichmüller spaces. The Teichmüller space $T(\mathcal{S})$ is a first example of a Teichmüller space which is an infinite-dimensional but separable complex Banach manifold. Recall that Teichmüller spaces of Riemann surfaces are either finite dimensional complex manifolds or infinite-dimensional non-separable Banach manifolds. It appears that the complex analytic and the metric structure of $T(\mathcal{S})$ is quite different from the Teichmüller spaces of geometrically finite as well as infinite Riemann surfaces. Inverse limit spaces commonly appear in dynamics [43], [44], [23], [30] and the compact solenoid is a first non-trivial example of an inverse limit with interesting Teichmüller space.

The non-compact solenoid $\mathcal{S}_{nc}$ (also called the punctured solenoid) is the inverse limit of the system of all pointed, unbranched, finite sheeted coverings of a base punctured surface with negative Euler characteristic [36]. The covering surfaces are punctured with the covering maps sending punctures to punctures. If we fill in the punctures, the covering maps become finitely branched at the punctures. Therefore, the branching in the covering tower is restricted by allowing it only over the punctures of the base surface (unlike for towers of rational maps where branching appears to be "wild" [23]). The inverse limit $\mathcal{S}_{nc}$ is a non-compact space because we do not include the backward orbits of punctures in the space, each leaf is homeomorphic to the unit disk $\mathbf{D}$ and the ends of each leaf are universal covers of neighborhoods of punctures on surfaces, i.e. the ends are horoballs with the induced non-standard topology. The analog of the Ehrenpreis conjecture for punctured surfaces asks whether every two finite Riemann surfaces have finite covers which are $(1+\epsilon)$-quasiconformal. This is equivalent to the statement that the Modular group $\text{Mod}(\mathcal{S}_{nc})$ has dense orbits in the Teichmüller space $T(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$. In analogy to the compact case, the Modular group $\text{Mod}(\mathcal{S}_{nc})$ is isomorphic to a subgroup of the commensurator of the base punctured surface group which preserves the peripheral elements [36]. The existence of ends of leaves allows for a combinatorial decomposition of the (decorated) Teichmüller space of $\mathcal{S}_{nc}$ [36] which gives a better understanding of the Modular group $\mathrm{Mod}(\mathcal{S}_{nc})$ of the non-compact solenoid than of the Modular group of the compact solenoid $\mathcal{S}$.

In this chapter, we survey results on the Teichmüller space $T(\mathcal{S})$ of the compact solenoid $\mathcal{S}$ regarding its metric structure with respect to the Teichmüller metric and its complex structure. We also survey results on the Modular group $\mathrm{Mod}(\mathcal{S})$ of the compact solenoid $\mathcal{S}$ and the Modular group $\mathrm{Mod}(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$. We give more details below.

In Section 2 we give different equivalent definitions of the compact solenoid $\mathcal{S}$. In addition to defining $\mathcal{S}$ as an inverse limit space, we define it as a principal fiber bundle space as follows. For a fixed Fuchsian group G uniformizing a closed Riemann surface of genus at least two, we define a profinite group completion $\hat{G}$ of G with respect to the profinite metric. The profinite metric on G is defined by [37]

$$d_{pf}(A,B) = e^{-\frac{1}{n}},$$

where AB^{-1} is an element of all subgroups of G of index at most n, and there exists a subgroup of G of index $n+1$ which does not contain AB^{-1}. Then the G-tagged compact solenoid $\mathcal{S}_G$ is the quotient of $\mathbf{D}\times\hat{G}$ by an action of G, where G acts by Möbius maps on the unit disk component $\mathbf{D}$ and shifts the levels by acting on $\hat{G}$ by right translation in the group. The G-tagged solenoid $\mathcal{S}_G$ is homeomorphic to the compact solenoid $\mathcal{S}$. The natural map from $\mathbf{D}\times\hat{G}$ to the Riemann surface $\mathbf{D}/G$ obtained by "forgetting" the second coordinate and by mapping the first coordinate to its orbit under G projects to a map from the quotient $(\mathbf{D}\times\hat{G})/G=\mathcal{S}_G$ onto $\mathbf{D}/G$; the fibers of the map are homeomorphic to $\hat{G}$. Thus $\mathcal{S}_G$ is a $\hat{G}$-fiber bundle over $\mathbf{D}/G$.

A complex structure on the compact solenoid $\mathcal{S}$ is by definition an atlas whose transition maps when restricted to local leaves are holomorphic and are continuous for the transverse variations of local leaves [43]. Candel [7] proved a uniformization theorem for laminations which, in particular, implies that each transversely continuous conformal structure on $\mathcal{S}$ has a unique transversely continuous hyperbolic metric representative. The compact solenoid $\mathcal{S}$ is a fiber bundle over a closed surface such that the restriction to any leaf of the fiber map is the universal covering of the base surface. Thus, any complex structure on the base surface lifts to a complex structure on the compact solenoid $\mathcal{S}$. Such a complex structure has a sub-atlas where the transition maps are constant in the transverse directions. Nag and Sullivan [31] showed that every such complex structure is obtained by forming a G-tagged solenoid $\mathcal{S}_G$, for some uniformizing Fuchsian group G of a closed Riemann surface (see Section 3 for more details). The G-tagged punctured solenoid is formed similarly by using the punctured Riemann surface uniformizing Fuchsian group G (see Section 4).

The Teichmüller space $T(\mathcal{S}_G)$ of the compact solenoid $\mathcal{S}_G$ consists of all marked complex solenoids $f : \mathcal{S}_G \to \mathcal{X}$ up to post-composition by conformal maps and up to homotopy, where G is fixed and f is a differentiable, quasiconformal map (see Definitions 5.1 and 5.3). (Equivalently, the Teichmüller space $T(\mathcal{S}_G)$ is a quotient of the space of smooth Beltrami coefficients on $\mathcal{S}_G$ continuous in the transverse directions.) The Teichmüller distance of $[f] \in T(\mathcal{S}_G)$ to the basepoint $[id] \in T(\mathcal{S}_G)$ is the infimum of the logarithm of the quasiconformal constants of maps in the homotopy class $[f]$. The Teichmüller metric is not degenerate, namely $T(\mathcal{S}_G)$ is a Hausdorff space (see [43]; see Section 5 for an alternative proof).

The restrictions of complex structures on the marked solenoids $f : \mathcal{S}_G \to \mathcal{X}$ to the baseleaf l of $\mathcal{S}_G$ defines a map $\pi_l : T(\mathcal{S}_G) \to T(\mathbf{D})$. This map is a homeomorphism onto its image (a proof is sketched in [43]; see Section 5.2 for an alternative proof). In fact, a consequence of Theorem 7.1 and McMullen's solution [29] to the Kra's conjecture is that π_l is a bi-Lipschitz map onto its image with constant 1/3.

The study of the Teichmüller metric on $T(\mathcal{S}_G)$ starts with the Reich-Strebel inequality (see [41] and Section 6) which estimates the (complex) distortion (i.e. the Beltrami coefficient) of a quasiconformal self-map of the compact solenoid $\mathcal{S}_G$ which is homotopic to the identity in terms of the leafwise Euclidean structures given by the restrictions of holomorphic quadratic differentials on the leaves of $\mathcal{S}_G$. The Reich-Strebel inequality is a non-trivial generalization of Grötzsch's length-area method for determining extremal maps between rectangles. In this chapter we give a different proof of the Reich-Strebel inequality from the proof in [41] (see Theorem 6.1 and its proof).

The consequences of the Reich-Strebel inequality give a better understanding of the Teichmüller metric on $T(\mathcal{S}_G)$. In Section 6, we summarize consequences related to infinitesimal structure of $T(\mathcal{S}_G)$ from [41]. In particular, a Beltrami differential is tangent to a trivial path of Beltrami coefficients (i.e. it represents a trivial infinitesimal deformation) if and only if it is zero when paired with all holomorphic quadratic differentials on $\mathcal{S}_G$ (see Theorem 6.2). In section 7, we analyze extremal maps in a given homotopy (Teichmüller) class. A consequence of the Reich-Strebel inequality is that Teichmüller-type maps (i.e. vertical stretch maps in the natural parameter of a holomorphic quadratic differential on $\mathcal{S}_G$) are extremal in their corresponding homotopy (Teichmüller) classes (see Theorem 7.1). Moreover, the natural inclusion map from the Teichmüller space of a closed surface into $T(\mathcal{S}_G)$ obtained by lifting a complex structure on the surface to $\mathcal{S}_G$ is an isometry (see Corollary 7.3).

We also give an account of the question of the existence of the Teichmüller-type extremal maps in a given Teichmüller class considered in a joint work of the author with A. Epstein and V. Markovic [14]. The results for Riemann surfaces fall into two cases; either every point in the Teichmüller space has a Teichmüller-type representative for closed and finite punctured surfaces-Teichmüller's theorem, or an open, dense subset of the Teichmüller space has Teichmüller-type representatives for geometrically infinite surfaces [21]. Therefore, in both cases, at least a large subset of the Teichmüller space has Teichmüller-type representatives. For the Teichmüller space $T(\mathcal{S}_G)$ of the compact solenoid $\mathcal{S}_G$ the situation is quite different. In fact, a generic point in $T(\mathcal{S}_G)$ does not have Teichmüller-type representatives, i.e. only a set of the first kind in $T(\mathcal{S}_G)$ in the sense of Baire has Teichmüller-type representatives (see [14], or Theorem 7.4 together with a brief account of the proof.) We also give a necessary condition for a point in $T(\mathcal{S}_G)$ to have a Teichmüller-type representative (see [14], or Corollary 7.5).

In Section 8, we survey basic results on the Modular group (see [37], [27]). The Modular group $\mathrm{Mod}(\mathcal{S}_G)$ is isomorphic to the commensurator group of the base surface group (see [37] or Theorem 8.3). In a joint work with V. Markovic, we established that there exist orbits of $\mathrm{Mod}(\mathcal{S}_G)$ in $T(\mathcal{S}_G)$ with accumulation points [27]; and that finite subgroups of $\mathrm{Mod}(\mathcal{S}_G)$ are cyclic and mapping class like (i.e. they are lifts of self-maps of closed surfaces) [27].

In Section 9, we give a quasiconformal definition of the Teichmüller space $T(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$ and an equivalent representation-theoretic definition from our joint work with R. Penner (see [36]). In Section 10, we define the decorated Teichmüller space $\tilde{T}(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$ and give its parametrization in terms of lambda lengths (see our work with R. Penner [36] or Theorem 10.3). We also describe a convex hull construction for decorations on the punctured solenoid and show that a dense, open subset of $\tilde{T}(\mathcal{S}_{nc})$ is combinatorially interesting (see [36] or Theorem 10.6 for the punctured solenoid; for punctured surfaces see [15], [34]; see [33] for the universal Teichmüller space; see [20] for a related construction for punctured surfaces).

In Section 11, we give a generating set for $\mathrm{Mod}(\mathcal{S}_{nc})$ in terms of Whitehead homeomorphisms and $PSL_2(\mathbb{Z})$ (see our work with R. Penner [36] or theorems 11.3, 11.4 and 11.5). Moreover, we define a natural triangulation 2-complex, show that it is connected and simply connected, and show that the Modular group $\mathrm{Mod}(\mathcal{S}_{nc})$ acts cellularly on it (see our joint work with S. Bonnot and R. Penner [6] or Theorem 11.6). Using the triangulation complex, we give a presentation for $\mathrm{Mod}(\mathcal{S}_{nc})$ (see [6] or Theorem 11.7).

2 The compact solenoid

In this section we give two equivalent definitions of the compact solenoid $\mathcal{S}$ which is usually called the universal hyperbolic solenoid [43], [31], [37].

Let (S_0, x_0) be a fixed closed surface of genus at least two with basepoint x_0. Consider all finite degree, unbranched, pointed covers $\pi_i : (S_i, x_i) \to (S_0, x_0)$ up to isomorphisms of covers. The family of such covers has a natural partial ordering "$\leq$" defined by

$$(\pi_i, S_i, x_i) \leq (\pi_j, S_j, x_j)$$

if there exists a pointed, unbranched, finite degree cover $\pi_{i,j} : (S_j, x_j) \to (S_i, x_i)$ such that

$$\pi_j = \pi_i \circ \pi_{i,j}.$$

Given two arbitrary covers $\pi_i : (S_i, x_i) \to (S_0, x_0)$ and $\pi_j : (S_j, x_j) \to (S_0, x_0)$ from the above family, there exists a third cover $\pi_k : (S_k, x_k) \to (S_0, x_0)$ such that $(\pi_i, S_i, x_i), (\pi_j, S_j, x_j) \leq (\pi_k, S_k, x_k)$. Namely, the family of all covers $\pi_i : (S_i, x_i) \to (S_0, x_0)$ is inverse directed; thus the inverse limit of the family is well defined. Sullivan [43] introduced the *compact solenoid* $\mathcal{S}$ by

$$\mathcal{S} = \varprojlim (S_i, x_i).$$

By definition, $\mathcal{S} \subset \Pi_{i \in I} S_i$, where I is the index set of coverings, consists of all $y = (y_i)_{i \in I} \in \Pi_{i \in I} S_i$ such that whenever $(S_i, x_i) \leq (S_j, x_j)$ then $\pi_{i,j}(y_j) = y_i$. The product space $\Pi_{i \in I} S_i$ is compact in the Tychonov topology because each S_i is compact. The subset $\mathcal{S}$ is closed in $\Pi_{i \in I} S_i$ and therefore it is also a compact space.

The compact solenoid $\mathcal{S}$ is universal in the sense that it does not depend on the base surface S_0. Namely, if we take the inverse limit of all finite degree unbranched covers of another closed surface S_0' of genus at least two then it is homeomorphic to $\mathcal{S}$. (This follows from the fact that the inverse limit of any given cofinal subsystem of covers is homeomorphic to the inverse limit of the original system of covers. Recall that a subsystem of covers is *cofinal* if any surface in the original system is covered by a surface of the subsystem.) To show that $\mathcal{S}$ is independent of the base surface, it is enough to note that any two such inverse systems of covers have homeomorphic cofinal subsystems because there exist two surfaces in these two systems that are homeomorphic.

The universal property of the compact solenoid enables us to consider a tower of covers of a closed surface of genus at least two instead of all finite covers (as long as the intersection of all fundamental groups in the tower when identified via pointed covers with subgroups of the base surface is the trivial group; this is required for a tower to be cofinal by the residual finiteness of the base surface group). For example, we can consider the system of covers given by the tower of covers whose n-th level surface has fundamental group equal to the intersection of all index at most n subgroups of the base surface (see [37]). The choice of the subgroup at the n-th level uniquely (up to isomorphism) determines the pointed cover of the base surface. For each $m > n$ the group at level m is a subgroup of the group at level n. Thus the system of covers is a tower and its inverse limit is homeomorphic to $\mathcal{S}$.

For convenience, we work with the above tower of covers from now on. Thus we can replace the index set I for the covers by the natural numbers $\mathbb{N}$, where $\pi_j : S_j \to S_0$ factors through a cover $\pi_i : S_i \to S_0$ if and only if $j > i$. Then a point y in $\mathcal{S}$ is given by a backward sequence $y = (y_0, y_1, y_2, \ldots)$ with respect to the tower of covers, namely $y_i \in S_i$ and $\pi_{i,i+1}(y_{i+1}) = y_i$ for $i \in \mathbb{N} \cup \{0\}$. A neighborhood of a point in $\mathcal{S}$ is homeomorphic to a (2-disk)$\times$(Cantor set). To see this, note that by the definition of the Tychonov topology a neighborhood of a point $y = (y_0, y_1, y_2, \ldots)$ in $\Pi_{i\in\mathbb{N}} S_i$ is the set $V(y)$ consisting of all $z = (z_0, z_1, z_2, \ldots)$ such that each z_i is in a small ball $U_i(y_i)$ with center $y_i \in S_i$ for all $i < i_0$, with $i_0 \in \mathbb{N}$ fixed, and the rest of the coordinates of z are arbitrary. If $y \in \mathcal{S}$ then a neighborhood $V(y) \subset \mathcal{S}$ is given by successively taking a single lift $U_i(y_i) \subset S_i$ of a ball $U_0(y_0) \subset S_0$ for all $i < i_0$, where $i_0 \in \mathbb{N}$ is fixed, and the rest of the coordinates of the points in $V(y)$ belong to all lifts $\pi_i^{-1}(U_0)$ such that π_{i,i_0} maps them into U_{i_0}, for $i \geq i_0$. The lifts to the tower $\{S_i\}_{i\in\mathbb{N}}$ of a ball in S_0 are enumerated by the locally finite tree of all possibilities of lifts from S_i to S_{i+1} for $i \in \mathbb{N}$. The local structure of $\mathcal{S}$ is given by taking a 2-disk for each infinite path (without backtracking) in the tree with the induced product topology, where the 2-disk has the standard topology, and points in two 2-disks for two different infinite paths are close if they are close as points in the 2-disk and if the infinite paths follow the same finite paths for a long time. The set of all infinite paths without backtracking is a Cantor set and we have completely described the local structure of $\mathcal{S}$.

A path component of $\mathcal{S}$ is called a *leaf*. A *local leaf* is a path component in any local chart of the above form (2-disk)$\times$(Cantor set), namely a local leaf is a 2-disk. Therefore, a (global) leaf of the compact solenoid $\mathcal{S}$ is a surface. There is a natural projection $\Pi_i : \mathcal{S} \to S_i$, for $i \in \mathbb{N} \cup \{0\}$, to any surface in the tower of covers given by

$$\Pi_i(y_0, y_1, y_2, \ldots) = y_i,$$

for $y = (y_0, y_1, y_2, \ldots) \in \mathcal{S}$. Since the intermediate covers $\pi_{i,i+1}$ in the tower are unbranched, the restriction of the projection Π_i to each leaf is an unbranched covering. We claim that each leaf is simply connected. If a leaf of

$\mathcal{S}$ is not simply connected, then a closed curve which is not homotopic to a point maps under each Π_i to a curve on S_i which is not homotopic to a point. However, the covers in the tower are chosen so that each closed homotopically non-trivial curve on any surface cannot be lifted to a closed curve in a high enough cover. (If one considers all finite covers in the definition of $\mathcal{S}$, this follows because the fundamental group of S_0 is residually finite.) Thus each leaf is a simply connected unbranched cover of closed surfaces. Namely, each leaf of $\mathcal{S}$ is homeomorphic to the unit disk and the restriction of the natural projection to each leaf is the universal covering map.

2.1 The profinite completion

Denote by G the fundamental group of S_0. We define the profinite metric on G as follows. Let G_n be the intersection of all subgroups of G of index at most n. There are only finitely many such subgroups and their intersection G_n is also of finite index. (It is possible that $G_n = G_{n+1}$ for some n and we ignore the repeating groups.) From now on, $\{G_n\}_{n\in\mathbb{N}}$ is a sequence of decreasing (as sets) subgroups of G of finite index. Each G_n is a characteristic subgroup of G and in particular a normal subgroup. Since G is residually finite, it follows that $\cap_{n\in\mathbb{N}} G_n = \{id\}$. We define the *profinite distance* of $A, B \in G$ by

$$d_{pf}(A,B) = e^{-\frac{1}{n}},$$

where $AB^{-1} \in G_n - G_{n+1}$. In particular, an element of G is close to the identity in the profinite metric d_{pf} if it belongs to G_n for n large.

We denote by $\hat{G}$ the metric completion of G in the profinite metric d_{pf} (see [37]). Each point of $\hat{G}$ is an equivalence class of Cauchy sequences in (G, d_{pf}). The multiplication of two sequences is given by multiplying corresponding elements and the product of two Cauchy sequences is Cauchy. The operation of multiplying equivalence classes of Cauchy sequences is well-defined and $\hat{G}$ is a group with respect to multiplication. The group $\hat{G}$ is homeomorphic to the Cantor set and there is a natural injective homomorphism of G into $\hat{G}$ obtained by mapping $A \in G$ into the equivalence class of the constant sequence $(A, A, A, \ldots)$. The image of G is dense in $\hat{G}$.

Since $\hat{G}$ is a compact topological group, there exists a unique left and right translation measure m on $\hat{G}$ such that $m(\hat{G}) = 1$. The measure m is called Haar measure and it is a positive Radon measure.

2.2 The G-tagged compact solenoid

At this point we fix a Fuchsian group G such that the Riemann surface $\mathbf{D}/G$ has genus at least two, where $\mathbf{D}$ is the unit disk. We describe the compact solenoid $\mathcal{S}$ using the profinite group $\hat{G}$. Consider the product $\mathbf{D}\times\hat{G}$. The action of $A\in G$ on $\mathbf{D}\times\hat{G}$ is defined by

$$A(z,t)=(Az,tA^{-1}),$$

where $(z,t)\in\mathbf{D}\times\hat{G}$ and A acts by hyperbolic isometries on the disk component and by right multiplication by A^{-1} on the group $\hat{G}$ component. It is a fact that the quotient $(\mathbf{D}\times\hat{G})/G$ is homeomorphic to the compact solenoid $\mathcal{S}$ (see [37]). The natural projection $\Pi:(\mathbf{D}\times\hat{G})/G\to\mathbf{D}/G$ is given by forgetting the second coordinate. Thus the fiber over a point in $\mathbf{D}/G$ is homeomorphic to $\hat{G}$. The orbit in $\mathbf{D}\times\hat{G}$ under G of a single disk $\mathbf{D}\times\{t\}$ is a leaf of the solenoid. We define the orbit of $\mathbf{D}\times\{id\}$ to be the baseleaf and the orbit of $(0,id)$ to be the basepoint. After fixing the baseleaf and the basepoint, each fiber has an identification with $\hat{G}$ and the projection $\Pi:(\mathbf{D}\times\hat{G})/G\to\mathbf{D}/G$ is a $\hat{G}$-bundle.

We define the *G-tagged compact solenoid* $\mathcal{S}_G$ by

$$\mathcal{S}_G=(\mathbf{D}\times\hat{G})/G.$$

Let $\omega\subset\mathbf{D}$ be a fundamental polygon for the action of G on $\mathbf{D}$. Then $\omega\times\hat{G}$ is a fundamental set for the action of G on $\mathbf{D}\times\hat{G}$. The action of G is identifying a boundary side of $\omega\times\{t\}$ with a boundary side of $\omega\times\{tA^{-1}\}$, where $A\in G$ identifies the boundary side of ω onto another boundary side of ω. The group G is countable while $\hat{G}$ is an uncountable set. Since G glues together the ω-pieces to make a single leaf, we conclude that $\mathcal{S}_G\approx\mathcal{S}$ has uncountably many leaves.

The holonomy of the leaves of the G-tagged compact solenoid $\mathcal{S}_G$ is given by the right translation of the group G in the group $\hat{G}$. Since m is an invariant measure on $\hat{G}$, we conclude that m induces a holonomy invariant transverse measure on the compact solenoid $\mathcal{S}_G$.

3 Complex structures and hyperbolic metrics on the compact solenoid

A local chart of the compact solenoid $\mathcal{S}$ is homeomorphic to a (2-disk)$\times$(Cantor set). A transition function between two local charts is a homeomorphism from (an open subset of a 2-disk)$\times$(Cantor set) onto another such set. In particular, the restriction of the transition map to each 2-disk is a homeomorphism and the family of homeomorphisms varies continuously in the Cantor set direction for the C^0-topology on continuous maps.

A *complex structure* on the compact solenoid $\mathcal{S}$ is a choice of charts such that transition maps are holomorphic when restricted to each local leaf and vary continuously in the Cantor set direction for the C^0-topology. Since maps are holomorphic, the continuous variation in the C^0-topology implies continuous variation in the C^∞-topology.

A *hyperbolic metric* on the compact solenoid $\mathcal{S}$ is an assignment of a metric of curvature -1 to each local leaf such that it varies continuously in the Cantor set direction and that there is a choice of an atlas whose transition functions are leafwise isometries (and vary continuously in the Cantor set direction).

It follows from the work of Candel [7] that any conformal structure on the compact solenoid $\mathcal{S}$ contains a unique hyperbolic metric. Any complex structure on the compact solenoid $\mathcal{S}$ corresponds to a conformal structure and any conformal structure gives a unique complex structure by the continuous dependence on the parameters of the solution of the Beltrami equation (see Ahlfors-Bers [1]).

The construction of the G-tagged compact solenoid above provides an example of a complex structure on $\mathcal{S}$ as well as a hyperbolic metric (by simply inducing the complex structure and the hyperbolic metric on the leaves of $\mathcal{S}$ from the unit disk $\mathbf{D}$). The local charts of $\mathcal{S}$ are chosen to be of the form $D \times \hat{G}$, where $D \subset \mathbf{D}$ is a small hyperbolic disk such that no two points of $D \times \hat{G}$ are in the same orbit of G. The complex structure on the unit disk $\mathbf{D}$ gives complex charts for $\mathcal{S}$ where the transition maps between any two charts $D \times \hat{G}$ and $D_1 \times \hat{G}$ are constant in $\hat{G}$ (namely, they are given by Möbius maps $A \in G$) and therefore continuous. The hyperbolic metric on $\mathbf{D}$ gives a hyperbolic metric on $\mathcal{S}$ which is also constant in the $\hat{G}$ direction.

Complex structures on $\mathcal{S}$ whose transition maps are locally constant in the Cantor set direction are called *transversely locally constant* (TLC) complex structures. (It is enough to find a subfamily of charts which cover $\mathcal{S}$ for which transition maps are constant in the Cantor set direction.) Similarly, a hyperbolic metric on $\mathcal{S}$ is TLC if there exists a cover of $\mathcal{S}$ by charts in which the hyperbolic metric is locally constant in the Cantor set direction. It is a fact that any TLC complex structure (hyperbolic metric) is obtained by taking a G-tagged solenoid, where G is a Fuchsian group uniformization of a closed surface of a (possibly large) genus greater than two. This follows by the compactness of $\mathcal{S}$ and the fact that each transverse direction corresponds to the profinite completion of a finite index subgroup of the fundamental group of a genus two surface (see also [31]). Therefore, the set of all TLC complex structures on $\mathcal{S}$ is given by lifting complex structures on Riemann surfaces. Sullivan [43] showed that any complex structure on $\mathcal{S}$ can be approximated by TLC complex structure in the C^0-topology, which is equivalent to the C^∞-topology.

4 The G-tagged non-compact solenoid

We introduce the non-compact solenoid $\mathcal{S}_{nc}$ (see [36]). Since we require that the topological ends of leaves are well-behaved, our construction immediately assigns a hyperbolic metric on $\mathcal{S}_{nc}$. It will follow that $\mathcal{S}_{nc}$ has finite area in an appropriate sense.

Let $G < PSL_2(\mathbb{Z})$ be such that $\mathbf{D}/G$ is the once punctured torus modular group. Denote by $\hat{G}$ the profinite completion of G. Recall that the action of G on $\mathbf{D}\times\hat{G}$ is given by $A(z,t)=(Az,tA^{-1})$ for $z\in\mathbf{D}$, $t\in\hat{G}$ and $A\in G$. We define the *non-compact solenoid* $\mathcal{S}_{nc}$ by

$$\mathcal{S}_{nc}=(\mathbf{D}\times\hat{G})/G.$$

A leaf of $\mathcal{S}_{nc}$ is the orbit under G of a single disk $\mathbf{D}\times\{t\}$. Let ω be a fundamental polygon for the action of G on $\mathbf{D}$ such that the boundary edges are infinite geodesics which project to the geodesics on the torus $\mathbf{D}/G$ connecting the puncture to itself. Then $\omega\times\hat{G}$ is a fundamental set for the action of G on $\mathbf{D}\times\hat{G}$. The identifications by G on $\omega\times\hat{G}$ are identifying only the boundary edges in pairs on different levels according to the G-action on $\hat{G}$.

Any compact subset of $\mathcal{S}_{nc}$ is a subset of a compact set of the form $((\mathbf{D}_r\cap\omega)\times\hat{G})/G$, where $\mathbf{D}_r$, $0<r<1$, is the Euclidean disk of radius r with center 0 and $\mathbf{D}_r\cap\omega$ is a compact subset of ω. The non-compact parts on each leaf $(\mathbf{D}\times\{t\})/G\equiv\mathbf{D}$ as sets are given by the G-orbit of a single horoball ζ in $\mathbf{D}$ centered at the fixed point of a parabolic element of G. In the induced topology on the set $G\{\zeta\}$ each horoball accumulates onto itself since it is preserved by the action of an infinite cyclic group generated by the parabolic element of G with the fixed point at the center of the horoball. A fundamental set η in the horoball for the action of the cyclic group is the intersection of the horoball and the region between a geodesic with endpoint at the center of the horoball and its image under the generating parabolic map. Then the corresponding points in η and $C^n(\eta)$ are close for $n\in\mathbb{Z}$ with $|n|$ large, where $C\in G$ is the generating parabolic element with fixed point at the center of the horoball. Moreover, the corresponding points in the horoballs ζ and

$A(\zeta)$ are close provided that $A \in G_n$ for n large. Therefore, each leaf of $\mathcal{S}_{nc}$ has countably many topological ends corresponding to the fixed points of the parabolics in G. The non-compact solenoid $\mathcal{S}_{nc}$ has only one end given by the equivalence class of the set $(G(\zeta) \times \hat{G})/G$.

Given a local chart of the form (2-disk)×(Cantor set), there is a transversal identification of local charts. We consider only local charts of the form $D \times \hat{G} \subset \mathbf{D} \times \hat{G}$ for D a hyperbolic disk sufficiently small such that the projection map from $D \times \hat{G}$ to $\mathcal{S}_{nc}$ is a homeomorphism. The transverse identification of local leaves is an isometry because the hyperbolic metric is constant in $\hat{G}$ and it extends to an isometric identification of global leaves. This identification is specified by fixing two local leaves of two global leaves. In the above identification, the ends of leaves correspond to each other. The ends are called "*punctures*" by abuse of notation. We say that two punctures of $\mathcal{S}_{nc}$ are close if they correspond to each other under an identification of the leaves on which they reside where the identification is specified by two local leaves which are close in a given chart.

The above construction gives a hyperbolic metric on the leaves of $\mathcal{S}_{nc}$ which is transversely locally constant. We will consider an arbitrary non-compact solenoid $\mathcal{X}$ with a hyperbolic metric on leaves which varies continuously for the transverse variations together with a marking map $f : \mathcal{S}_{nc} \to \mathcal{X}$. The marking f is a homeomorphism which is quasiconformal and differentiable on leaves, varies continuously in the transverse direction for the C^1-topology on differentiable maps and for the quasiconformal topology when global leaves are identified using local charts as above. In particular, the supremum of quasiconformal constants over the leaves is bounded. The end of $\mathcal{S}_{nc}$ is homeomorphically mapped onto the end of $\mathcal{X}$. Moreover, the intersection of a leaf of $\mathcal{S}_{nc}$ with the end is quasi-isometrically mapped onto the corresponding leaf of $\mathcal{X}$. Therefore, our notion of ends being close on the TLC non-compact solenoid $\mathcal{S}_{nc}$ is transferable to an arbitrary non-compact marked solenoid $f : \mathcal{S}_{nc} \to \mathcal{X}$.

5 The Teichmüller space of the compact solenoid

We define the Teichmüller space $T(\mathcal{S})$ of the compact solenoid $\mathcal{S}$. Let G be a fixed Fuchsian group such that $\mathbf{D}/G$ is a closed Riemann surface of genus at least two. Let $\mathcal{S}_G$ be the G-tagged compact solenoid with the induced complex structure from $\mathbf{D}/G$. The complex structure on the solenoid $\mathcal{S}_G$ is a TLC complex structure.

Definition 5.1. A homeomorphism $f : \mathcal{S} \to \mathcal{X}$ of a complex compact solenoid $\mathcal{S}$ onto a complex compact solenoid $\mathcal{X}$ is said to be *quasiconformal* if it is differentiable and quasiconformal on each leaf and if it varies continuously in the transverse direction in the C^1-topology on the C^1-maps.

By the above definition, the composition $g \circ f$ of two quasiconformal maps $f : \mathcal{S} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ is quasiconformal.

Remark 5.2. Since $\mathcal{S}$ is compact, it follows that the continuity in the C^1-topology for the variations on the local leaves implies the continuity in the quasiconformal topology on the global leaves. It is necessary to require smoothness of quasiconformal maps in order to preserve quasiconformality under the composition. One is tempted to require that Beltrami coefficients of leafwise quasiconformal maps vary continuously in the transverse direction in the essential supremum norm. However, the chain rule for Beltrami coefficients shows that the composition of such two maps does not satisfy the same continuity property unless the quasiconformal maps have additional C^1 smoothness and continuity in the transverse direction in the C^1-topology.

Definition 5.3. The *Teichmüller space* $T(\mathcal{S}_G)$ of the compact G-tagged solenoid $\mathcal{S}_G$ consists of all quasiconformal maps $f : \mathcal{S}_G \to \mathcal{X}$ up to an equivalence. Two quasiconformal maps $f, g : \mathcal{S}_G \to \mathcal{X}, \mathcal{Y}$ are *Teichmüller equivalent* if there exists a conformal map $c : \mathcal{X} \to \mathcal{Y}$ such that $g^{-1} \circ c \circ f : \mathcal{S}_G \to \mathcal{S}_G$ is homotopic to the identity. Denote by $[f] \in T(\mathcal{S}_G)$ the Teichmüller class of the quasiconformal map $f : \mathcal{S}_G \to \mathcal{X}$, i.e. all quasiconformal maps homotopic to f up to post-composition by conformal maps.

Since the transverse set for $\mathcal{S}$ is totally disconnected, any homotopy does not mix the leaves. Any two homotopic quasiconformal maps of a complex compact solenoid are isotopic through uniformly bounded quasiconformal maps [27, Theorem 3.1].

Definition 5.4. The *Teichmüller distance* d_T on $T(\mathcal{S}_G)$ is given by

$$d_T([f],[g]) = \inf_{f_1 \in [f], g_1 \in [g]} 1/2 \log K(f_1 \circ g_1^{-1}),$$

where $K(f)$ is the supremum of the quasiconformal constants of the restrictions of $f : \mathcal{S} \to \mathcal{X}$ to the leaves of $\mathcal{S}$. Since f is transversely continuous in the C^1-topology and since each leaf is dense in $\mathcal{S}$, we conclude that $K(f)$ is equal to the quasiconformal constant on each leaf of $\mathcal{S}$. In particular, the restriction of f to each leaf has the same quasiconformal constant.

Sullivan [43] showed that the Teichmüller (pseudo-)metric is a proper metric, i.e. that it is not degenerate. We give an alternative proof in this section.

5.1 The universal coverings of complex compact solenoids

Recall that the G-tagged complex solenoid $\mathcal{S}_G$ is given by the quotient of $\mathbf{D}\times\hat{G}$ under the action of G. The complex structure and the hyperbolic metric on $\mathcal{S}_G$ are inherited from $\mathbf{D}$ and they are transversely locally constant. The natural quotient map

$$\pi : \mathbf{D}\times\hat{G} \to (\mathbf{D}\times\hat{G})/G \equiv \mathcal{S}_G$$

is a local homeomorphism which is leafwise conformal and which varies continuously in $\hat{G}$ for the C^0-topology on continuous maps (which is equivalent to the C^∞-topology on conformal maps). The space $\mathbf{D}\times\hat{G}$ is globally much simpler (a product) than $\mathcal{S}_G$. Thus we consider $\mathbf{D}\times\hat{G}$ as a complex "universal covering" of a TLC solenoid $\mathcal{S}_G$ and we consider G as the covering group with its action on $\mathbf{D}\times\hat{G}$.

Let $f : \mathcal{S}_G \to \mathcal{X}$ be a quasiconformal map, where $\mathcal{X}$ is a hyperbolic compact solenoid not necessarily TLC. We form a complex universal covering for $\mathcal{X}$ using the marking map f. We recall (see [41]) that there exists a chart $(U\times T,\psi)$ of $\mathcal{X}$, where U a disk with center 0, such that $\psi\circ f(\{0\}\times\hat{G}) = \{0\}\times T$. Then f induces a homeomorphism of $\hat{G}$ and T. Consider a family of maps $\pi_t^X : \mathbf{D}\to\mathcal{X}$, for $t\in T$, such that π_t is an isometry onto a leaf of $\mathcal{X}$ (with its hyperbolic metric), $\psi\circ\pi_t^X(0) = 0$ and $(\psi\circ\pi_t^X)'(0) > 0$. Then the maps π_t^X fit together to a single map

$$\pi^X : \mathbf{D}\times T \to \mathcal{X},$$

defined by $\pi^X(\cdot,t) := \pi_t^X(\cdot)$. The map π^X is a local homeomorphism and a leafwise isometry. We consider $\mathbf{D}\times T$ as a hyperbolic (or a complex) "universal covering" with π^X as a cover map [41].

There is a well-defined lift

$$\tilde{f} : \mathbf{D}\times\hat{G} \to \mathbf{D}\times T$$

of the map $f : \mathcal{S}_G \to \mathcal{X}$ given by the formula $\tilde{f}(z,t) := (\pi_t^X)^{-1}\circ f\circ\pi(z,t)$ (see [41]).

The group G acts on $\mathbf{D}\times\hat{G}$ as a covering group of $\mathcal{S}_G$ and we use $\tilde{f}$ to introduce a conformal covering group for $\mathcal{X}$. Since

$$\pi^X\circ\tilde{f} = f\circ\pi,$$

it follows that

$$\pi^X\circ\tilde{f}\circ A = \pi^X\circ\tilde{f}$$

for all $A \in G$. Let $(z,t) \in \Delta \times \widehat{G}$, $\tilde{f}(z,t) = (w_1,t_1) \in \Delta \times T$ and $(\tilde{f} \circ A)(z,t) = (w_2,t_2) \in \Delta \times T$. By the above,

$$\pi^X(w_2,t_2) = \pi^X(w_1,t_1).$$

Consequently,

$$(\pi^X)^{-1}(\pi^X(w_1,t_1))$$

contains (w_2,t_2). Note that $(\pi^X_{t_2})^{-1} \circ \pi^X_{t_1}$ is an isometry of $\Delta \times \{t_1\}$ onto $\Delta \times \{t_2\}$ and $((\pi^X_{t_2})^{-1} \circ \pi^X_{t_1})(w_1) = w_2$. We induce an action of $A \in G$ on T by its natural action (by right multiplication) on $\widehat{G}$ via the identification $\psi \circ f \circ \pi : \widehat{G} \equiv T$. We introduce a covering map A_X on the universal covering $\Delta \times T$ of $\mathcal{X}$ corresponding to A by

$$A_X(z,t) = ((\pi^X_{tA^{-1}})^{-1} \circ \pi^X_t(z), tA^{-1}),$$

where $t, tA^{-1} \in T \equiv \widehat{G}$. The covering map A_X is an isometry on each leaf. Moreover, A_X is transversely continuous and $\tilde{f} \circ A = A_X \circ \tilde{f}$ from the definition (see [41]). Then we define $G_X := \tilde{f} G \tilde{f}^{-1}$ to be the covering group of $\mathcal{X}$, namely $(\mathbf{D} \times T)/G_X$ is conformally equivalent to $\mathcal{X}$.

5.2 Beltrami coefficients and holomorphic quadratic differentials on the compact solenoid

Given a quasiconformal map $f : \mathcal{S}_G \to \mathcal{X}$, there is a corresponding leafwise smooth (i.e. C^1) Beltrami coefficient $\mu = \frac{\bar{\partial} f}{\partial f}$ which is continuous for the transverse variations in the local charts for the C^1-topology on C^1-maps. The lift $\tilde{f}$ has Beltrami coefficient $\tilde{\mu}$ (which is the lift of μ) and it satisfies

$$\tilde{\mu}(z,t) = \tilde{\mu}(Az, tA^{-1}) \frac{\overline{A'(z)}}{A'(z)}, \tag{5.1}$$

for $A \in G$.

More generally, if $g : \mathcal{X} \to \mathcal{Y}$ is a quasiconformal map of complex compact solenoids then there exists a lift $\tilde{g} : \mathbf{D} \times T_1 \to \mathbf{D} \times T_2$ to their universal covers. The Beltrami coefficient ν of g lifts to the Beltrami coefficient $\tilde{\nu}$ on $\mathbf{D} \times T_1$ such that

$$\tilde{\nu}(z,t) = \tilde{\nu}(A_X(z,t)) \frac{\overline{A'_X(z,t)}}{A'_X(z,t)},$$

for $A_X \in G_X$, where $A_X(z,t) = (A^t_X(z), tA^{-1})$ and $A'_X(z,t)$ is the leafwise derivative. Note that $A'_X(z,t)$ depends on t.

By the compactness of $\mathcal{S}_G$, the continuity in the local charts for the transverse variations of a Beltrami coefficient μ on $\mathcal{S}_G$ implies that

$$\|\tilde{\mu}(\cdot,t)-\tilde{\mu}(\cdot,t_1)\|_\infty \to 0 \tag{5.2}$$

as $t\to t_1$, for all $t_1\in\hat{G}$. In the opposite direction, a Beltrami coefficient $\tilde{\mu}$ on $\mathbf{D}\times\hat{G}$ which is leafwise C^1, which is continuous in the C^1-topology for the variations in $\hat{G}$ on the compact subsets of $\mathbf{D}\times\hat{G}$ and which satisfies (5.2) is the lift of the Beltrami coefficient of a quasiconformal map $f:\mathcal{S}_G\to\mathcal{X}$, where $\mathcal{X}$ is determined by $\tilde{\mu}$.

Let $\tilde{f}^{\tilde{\mu}}$ denote the leafwise solutions to the Beltrami equation with the coefficient $\tilde{\mu}$ on $\mathbf{D}\times\hat{G}$ normalized such that 1, i and -1 are fixed on each leaf. Then $\tilde{f}^{\tilde{\mu}}$ conjugates the action of G on $\mathbf{D}\times\hat{G}$ to the action of $G^{\tilde{\mu}}$ on $\mathbf{D}\times\hat{G}$. Let $\mathcal{X}^\mu=(\mathbf{D}\times\hat{G})/G^{\tilde{\mu}}$ be the induced complex solenoid. Then $\tilde{f}^{\tilde{\mu}}$ projects to a quasiconformal map $f^\mu:\mathcal{S}^G\to\mathcal{X}^\mu$.

Definition 5.5. A *transversely locally constant* (TLC) Beltrami coefficient on a TLC compact solenoid $\mathcal{S}$ is a leafwise Beltrami coefficient which is constant in the transverse direction in some atlas of local charts.

Definition 5.6. A *holomorphic quadratic differential* φ on a complex compact solenoid $\mathcal{X}$ is a leafwise holomorphic quadratic differential which varies continuously in the local chart in the transverse direction in the C^0-topology. Equivalently, a leafwise holomorphic function $\tilde{\varphi}$ on the universal cover $\mathbf{D}\times T$ of $\mathcal{X}$ is a lift of a holomorphic quadratic differential if

$$\tilde{\varphi}(z,t)=\tilde{\varphi}(A_X(z,t))A_X'(z,t)^2 \tag{5.3}$$

for $A_X\in G_X$ and if

$$\|\tilde{\varphi}(\cdot,t)-\tilde{\varphi}(\cdot,t_1)\|_{Bers}\to 0 \tag{5.4}$$

as $t\to t_1$, where $\|f\|_{Bers}:=\sup_{z\in\mathbf{D}}|\rho^{-2}(z)f(z)|$ with ρ the Poincaré density on $\mathbf{D}$ (see [41]).

Definition 5.7. A *transversely locally constant* (TLC) holomorphic quadratic differential on a complex compact solenoid $\mathcal{S}$ is a leafwise holomorphic quadratic differential which is constant in the transverse direction in some atlas of local charts.

Using the above notion of Beltrami coefficients on the universal cover of $\mathcal{S}_G$ we give an equivalent definition of the Teichmüller space $T(\mathcal{S}_G)$.

Definition 5.8. The *Teichmüller space* $T(\mathcal{S}_G)$ of the compact G-tagged solenoid $\mathcal{S}_G$ consists of all smooth Beltrami coefficients $\tilde{\mu}$ on $\mathbf{D}\times\hat{G}$ which vary continuously in the C^1-topology on compact subsets of $\mathbf{D}\times\hat{G}$ and which satisfy (5.1) and (5.2) modulo an equivalence relation. Two Beltrami coefficients $\tilde{\mu}$ and $\tilde{\nu}$ are (Teichmüller) equivalent if there exists a conformal map $c : \mathcal{X}^\mu \to \mathcal{X}^\nu$ such that $(f^\nu)^{-1} \circ c \circ f^\mu : \mathcal{S}_G \to \mathcal{S}_G$ is isotopic to the identity map.

5.3 The restriction map π_l

We recall the definition of the restriction map $\pi_l : T(\mathcal{S}_G) \to T(\mathbf{D})$ from [43]. Given a quasiconformal map $f : \mathcal{S}_G \to \mathcal{X}$, the restriction to the baseleaf $f|_l : l \to f(l)$ maps l to the leaf $f(l) \subset \mathcal{X}$. We fix a conformal identification $l \equiv \mathbf{D}$ and take an arbitrary conformal identification $f(l) \equiv \mathbf{D}$. Then $f|_l : \mathbf{D} \to \mathbf{D}$ is well-defined up to post-composition with a conformal map of $\mathbf{D}$ (because of the choice $f(l) \equiv \mathbf{D}$). This gives a well-defined element of the universal Teichmüller space $T(\mathbf{D})$. Sullivan [43] showed that $\pi|_l$ is injective. We give a different proof.

Theorem 5.9. *Let l be the baseleaf of the G-tagged compact solenoid $\mathcal{S}_G$. Let $\pi_l : T(\mathcal{F}_G) \to T(\mathbf{D})$ be defined by the restriction of the hyperbolic metric on $\mathcal{S}_G$ to the baseleaf l, where l is identified with the unit disk $\mathbf{D}$. Then π_l is injective.*

Proof. It is enough to show that if $\pi_l([f])$ is trivial in $T(\mathbf{D})$ then $[f] \in T(\mathcal{S}_G)$ is trivial. Let $\tilde{f} : \mathbf{D} \times \hat{G} \to \mathbf{D} \times T$ be a lift of $f : \mathcal{S}_G \to \mathcal{X}$ to the universal coverings. Denote by G_X the covering group of $\mathcal{X}$. Our assumption implies that $\tilde{f}|_{S^1 \times \{id\}}$ is a Möbius map. By the invariance of $\tilde{f}$ we conclude that $\tilde{f}|_{S^1 \times \{A\}} = (A_X)^{-1} \circ \tilde{f}|_{S^1 \times \{id\}} \circ A$ is also a Möbius map, for each $A \in G$. Since G is dense in $\hat{G}$, we conclude that $\tilde{f}$ is a Möbius map on the boundary $S^1 \times \{t\}$, $t \in \hat{G}$, of each leaf.

Thus, when restricted to a leaf, $\tilde{f}$ is homotopic to a Möbius map (where different leaves can give different Möbius maps). We need to show that there is a homotopy F_t, $0 \leq t \leq 1$, of $\tilde{f}$ to Möbius maps on leaves such that $F_1 = \tilde{f}$, $F_t|_{S^1 \times \hat{G}} = \tilde{f}|_{S^1 \times \hat{G}}$ for each t, and the Beltrami coefficients $\tilde{\mu}_t$ of F_t satisfy (5.1) and (5.2), for each t.

Let $\tilde{\mu}$ be the Beltrami coefficient of $\tilde{f}$. Then we consider a path of Beltrami coefficients $t \mapsto \tilde{\nu}_t = t\tilde{\mu}$, for $0 \leq t \leq 1$, which satisfy (5.1) and (5.2). Then $\tilde{\nu}_t$ converges to the trivial Beltrami coefficient as $t \to 0$ and the path of properly normalized solutions $t \mapsto \tilde{f}^{\tilde{\nu}_t}$ give a homotopy from $\tilde{f}^{\tilde{\nu}_1} = \tilde{f}$ to the Möbius maps. However, we are not guaranteed that $\tilde{f}^{\tilde{\nu}_t}$ extends to the Möbius maps (determined by $\tilde{f}$) on the boundaries $S^1 \times \hat{G}$ for $0 < t < 1$. Let h_t be the boundary map for $\tilde{f} \circ (\tilde{f}^{\tilde{\mu}_t})^{-1}$. Let g_t be the leafwise barycentric extensions of h_t, for $0 \leq t \leq 1$. Then $g_1 = g_0 = id$ because $h_1 = h_0 = id$ on the boundary (by the properties of the barycentric extension [9]). The Beltrami coefficients of $g_t \circ \tilde{f}^{\tilde{\nu}_t}$ satisfy (5.1) and (5.2) for each t (again by the properties of the barycentric extension [9]) and the path $t \mapsto g_t \circ \tilde{f}^{\tilde{\nu}_t}$ gives a homotopy from $\tilde{f} = g_1 \circ \tilde{f}^{\tilde{\nu}_1}$ to the leafwise Möbius maps. (The idea of using barycentric extensions to find homotopies first appears in [12] for the plane domains, and it is utilized in [27] to show that homotopic maps of the compact solenoid are isotopic as well.) □

Sullivan [43] showed that the Teichmüller metric on $T(\mathcal{S}_G)$ is a proper metric. We use the above theorem to give an alternative argument.

Theorem 5.10. *The Teichmüller metric on the Teichmüller space $T(\mathcal{S}_G)$ of the compact solenoid is a proper metric, i.e. $T(\mathcal{S}_G)$ is a Hausdorff space for the Teichmüller metric.*

Proof. Note that $\pi_l : T(\mathcal{S}_G) \to T(\mathbf{D})$ is a contracting map with respect to the Teichmüller metrics on $T(\mathcal{S}_G)$ and $T(\mathbf{D})$. Since the Teichmüller metric on $T(\mathbf{D})$ is a proper metric the theorem follows. □

5.4 The complex Banach manifold structure on $T(\mathcal{S}_G)$

The Teichmüller space $T(\mathcal{S}_G)$ embeds as an open subset in a complex Banach vector space as follows. Denote by $\tilde{f} : \mathbf{D} \times \hat{G} \to \mathbf{D} \times T$ the lift to the universal covering of a quasiconformal map $f : \mathcal{S}_G \to \mathcal{X}$. The Bers embedding for the universal Teichmüller space assigns to each $\tilde{f}|_{\mathbf{D}\times\{t\}}$ a holomorphic quadratic differential $\tilde{\varphi}|_{\mathbf{D}\times\{t\}}$. The holomorphic quadratic differential $\tilde{\varphi}$ satisfies (5.4) because of the continuous dependence on the parameters of the solutions to the Beltrami equation [1] and it satisfies (5.3) because the Beltrami coefficient of $\tilde{f}$ satisfies (5.1).

We denote by $B(\mathcal{S}_G)$ the space of all holomorphic quadratic differentials on $\mathcal{S}_G$ which vary continuously in the transverse direction in the local charts for the C^0-topology. Note that $B(\mathcal{S}_G)$ is conformally isometric to the space of all leafwise holomorphic functions on $\tilde{\varphi} : \mathbf{D} \times \hat{G} \to \mathbb{C}$ that are uniformly leafwise Bers bounded, i.e. $\sup_{t\in\hat{G}} \|\tilde{\varphi}|_{\mathbf{D}\times\{t\}}\|_{Bers} < \infty$, and that vary continuously in the transverse direction for Bers norm, i.e. $\|\tilde{\varphi}|_{\mathbf{D}\times\{t\}} - \tilde{\varphi}|_{\mathbf{D}\times\{t_1\}}\|_{Bers} \to 0$ as $t \to t_1$, for each $t_1 \in \hat{G}$, and that are invariant under the action of G, i.e. they satisfy (5.3) (see [41]).

Therefore, we obtained a map $\Pi : T(\mathcal{S}_G) \to B(\mathcal{S}_G)$ which is injective because the Bers map for the universal Teichmüller space is injective and the restriction map π_l is injective. Moreover, Π is a homeomorphism onto an open subset of $B(\mathcal{S}_G)$ (see [41] for details). Note that $\Pi : T(\mathcal{S}_G) \to B(\mathcal{S}_G)$ is the quotient of the holomorphic map $\tilde{\Pi} : U_s^\infty(\mathcal{S}_G) \to B(\mathcal{S}_G)$, where $U_s^\infty(\mathcal{S}_G)$ is the unit ball in the space $L_s^\infty(\mathcal{S}_G)$ of all leafwise smooth, transversely continuous Beltrami differentials with the essential supremum norm and where $\tilde{\Pi}$ is obtained by taking the leafwise Bers embedding construction as above. Thus we define $\Pi : T(\mathcal{S}_G) \to B(\mathcal{S}_G)$ to be a complex global chart for $T(\mathcal{S}_G)$. For details see Sullivan [43].

Since $T(\mathcal{S}_G)$ has a complex structure, there is a well-defined Kobayashi pseudometric on $T(\mathcal{S}_G)$. The *Kobayashi pseudometric* is the largest metric on $T(\mathcal{S}_G)$ which makes all holomorphic maps from the unit disk with the Poincaré metric into $T(\mathcal{S}_G)$ weakly contracting. It is a well-known fact that the Kobayashi metric coincides with the Teichmüller metric for the Teichmüller spaces of Riemann surfaces (see [40], [17]). We showed that the same is true for $T(\mathcal{S}_G)$ [41].

Theorem 5.11. *On the Teichmüller space $T(\mathcal{S}_G)$ of the universal hyperbolic solenoid $\mathcal{S}_G$, the Kobayashi pseudometric equals the Teichmüller metric. In particular, the Kobayashi pseudometric is a metric.*

6 The Reich-Strebel Inequality

The study of the Teichmüller metric on Teichmüller spaces of Riemann surfaces depends on the Reich-Strebel inequality which is a (highly non-trivial) generalization of the length-area method for finding extremal maps between quadrilaterals. We give a proper generalization of the Reich-Strebel inequality for the marked compact solenoid $\mathcal{X}$ from [41]. If φ is a transversely continuous holomorphic quadratic differential then $|\varphi|$ is a leafwise area form on $\mathcal{X}$. The product $|\varphi|dm$ is a measure on $\mathcal{X}$. Recall that m is the Haar measure on the profinite completion group $\hat{G}$ of G and that the transverse sets in the local charts for $\mathcal{X}$ are identified with $\hat{G}$.

Definition 6.1. Let φ be a holomorphic quadratic differential on a complex compact solenoid $\mathcal{X}$. Then

$$\|\varphi\|_{L^1(\mathcal{X})} := \iint_{\mathcal{X}} |\varphi| dm$$

Definition 6.2. The space of all holomorphic quadratic differentials on a complex compact solenoid $\mathcal{X}$ is called $A(\mathcal{X})$.

The proof of the Reich-Strebel inequality for the closed solenoid used a careful approximation argument (of holomorphic quadratic differentials and complex solenoids by TLC holomorphic quadratic differentials on TLC complex solenoids) in [41] and we give a different proof below utilizing an idea of Gardiner [17, Section 2] for the proof in the closed surface case.

Definition 6.3. A Beltrami coefficient μ on a complex solenoid $\mathcal{X}$ is called *Teichmüller trivial* if it is equivalent to the trivial coefficient 0, i.e. the solution of the Beltrami equation is homotopic to a conformal map.

Theorem 6.4. (Reich-Strebel inequality) *Let φ be a holomorphic quadratic differential on the solenoid $\mathcal{X}$ and let μ be a Teichmüller trivial Beltrami coefficient. Then*

$$\|\varphi\|_{L^1(\mathcal{X})} \leq \int_{\mathcal{X}} \frac{\left|1+\mu\frac{\varphi}{|\varphi|}\right|^2}{1-|\mu|^2}|\varphi|dm. \tag{6.1}$$

Proof. Let $\varphi \in A(\mathcal{X})$ and let $f : \mathcal{X} \to \mathcal{X}$ be the quasiconformal map whose Beltrami coefficient is μ. Then f is homotopic to the identity on $\mathcal{X}$ and its restriction to each leaf is homotopic to the identity. Since φ is a holomorphic function on each leaf, the set of zeroes of φ on each leaf is at most countable and they accumulate at the boundary of the leaf. Thus, the set of critical vertical (as well as horizontal) trajectories is countable on each leaf and does not influence the integration of $|\varphi|$ on compact subsets of a leaf.

For a given closed arc $\alpha \subset \mathcal{X}$, we denote by $h_\varphi(\alpha)$ the height of α, namely the length in the metric $|Im(\sqrt{\varphi(z,t)}dz)|$ given in the local chart. We claim that there exists $M > 0$ such that for any compact segment β on a non-critical vertical trajectory we have

$$h_\varphi(\beta) \leq h_\varphi(f(\beta)) + M. \tag{6.2}$$

Let $t \mapsto f_t$ be a homotopy from $f_0 = id$ to $f_1 = f$. Recall that if γ is a path in $\mathcal{S}$ connecting the endpoints of β then $h_\varphi(\beta) \leq h_\varphi(\gamma)$ (see, for example, [42] or [17, Lemma 2, page 41]).

Let p be the initial point and let q be the terminal point of β. We define a path γ connecting the endpoints of β by taking $\gamma_0 : t \mapsto f_t(p)$ followed by $f(\beta)$ followed by $\gamma_1 : t \mapsto f_{1-t}(p)$. Then

$$h_\varphi(\beta) \leq h_\varphi(\gamma_0) + h_\varphi(f(\beta)) + h_\varphi(\gamma_1).$$

We consider the displacement function $d : \mathcal{X} \to \mathbb{R}$ for the map f. Since f is homotopic to the identity and since the transverse set is totally disconnected, it follows that f fixes each leaf. Then $d(p)$, for $p \in \mathcal{X}$, is defined by taking the leafwise distance in the metric $\sqrt{|\varphi|}|dz|$ from p to $f(p)$. The displacement function d is continuous because φ varies continuously for the transverse variations in charts and f is continuous as well for the transverse variations. Since $\mathcal{S}$ is a compact space, there exists a maximum M_1 for the displacement function d. Then, from the above inequality, we obtain the desired inequality (6.2) by using the above triangle inequality for heights, by observing that the height h_φ of a curve is shorter that the distance in the above metric $\sqrt{|\varphi|}|dz|$ and by taking $M = 2M_1$.

We claim that each ray of a non-critical vertical trajectory of φ is of infinite length. To see this, assume that a ray r of a non-critical vertical trajectory is of finite length in the $\sqrt{|\varphi|}|dz|$ metric, namely $h_\varphi(r) = h < \infty$. Then let $0 < u_n < h$ be an increasing sequence of parameters for r with $u_n \to h$ such that $r(u_n)$ converges to a point $q \in \mathcal{X}$ (there is a convergent sequence by the compactness of $\mathcal{X}$). Then either q belongs to the same leaf as r or to a different leaf. We consider both cases below.

If q belongs to the same leaf as r then a standard argument shows that q must be a zero of φ [42], [17]. This implies that r is critical which is a contradiction.

If q belongs to another leaf, then q must be a zero of φ as well. Otherwise, there would exist a neighborhood of q in $\mathcal{X}$ in which φ does not have any zeroes. This neighborhood can be chosen to consists of small disks with fixed radius in the metric $\sqrt{|\varphi|}|dz|$ centered at a transverse neighborhood of q. But r has to enter this neighborhood intersecting the disks of half the radius infinitely many times. This implies that r has an infinite length which is a contradiction.

Therefore q is a zero of φ. Then there exists a neighborhood of q in $\mathcal{X}$ consisting of disks with small fixed radius around a transverse neighborhood of q such that all zeroes of φ in this neighborhood are in the disks of $1/3$ the radius. Note that r has to enter infinitely many times in the smaller disks and exit the larger disk. In particular, r crosses infinitely many times the annulus whose outer boundary is the boundary of the larger disk and whose inner boundary is the boundary of the smaller disk. The holomorphic quadratic differential has no zeros in the annulus. It follows that the length of r is infinite, which is again a contradiction. Thus r has infinite length.

At this point we modify the standard arguments in [17] to the compact solenoid $\mathcal{X}$. On the set of points $p \in \mathcal{X}$ which do not lie on the critical vertical trajectories of φ, we define the function

$$g(p) = h_\varphi(f(\beta_p))$$

where β_p is a compact vertical segment with center p and length b. By a change of variable, we obtain

$$g(p) = \int_{\beta_p} |Im(\sqrt{\psi}dz)|,$$

where $\psi = (\varphi \circ f)f_z^2(1 - \frac{\mu\varphi}{|\varphi|})^2$ is a quadratic differential on $\mathcal{X}$. At this point, we write all the integration in terms of the natural parameter $\zeta = \xi + i\eta$ for φ. Then

$$\int_{\mathcal{X}} g(p) d\xi d\eta dm = b \int_{\mathcal{X}} |Im\sqrt{\psi}| d\xi d\eta dm$$

by Fubini's theorem. By (6.2), we obtain $b - M \leq \int_{\beta_p} |Im(\sqrt{\psi}d\zeta)|$ for $p \in \mathcal{X}$. By integrating both sides of the above inequality over $\mathcal{X}$ with respect to measure $d\xi d\eta dm$, we obtain

$$\frac{b-M}{b} \int_{\mathcal{X}} d\xi d\eta dm \leq \int_{\mathcal{X}} |Im\sqrt{\psi(\zeta)}| d\xi d\eta dm.$$

By letting $b \to \infty$ and inserting $|\sqrt{\varphi(\zeta)}| = 1$ under the integral on the right, we obtain

$$\int_{\mathcal{X}} |\varphi| dm \leq \int_{\mathcal{X}} |\sqrt{\varphi}\sqrt{\psi}| dm,$$

and after substituting the expression for ψ and using Cauchy-Schwarz's inequality, we obtain the desired inequality called *Reich-Strebel inequality.* □

We consider equivalence classes of Beltrami coefficients on the compact solenoid $\mathcal{S}_G$ as elements of the Teichmüller space $T(\mathcal{S}_G)$. If $f : \mathcal{S}_G \to \mathcal{X}$ is a marked solenoid and μ is a Beltrami coefficient on $\mathcal{X}$, then there is a marked solenoid $f^\mu \circ f : \mathcal{S}_G \to \mathcal{X}^\mu$ such that the Beltrami coefficient of $f^\mu : \mathcal{X} \to \mathcal{X}^\mu$ is μ. Then the class of the Beltrami coefficient of $f^\mu \circ f$ determines a point in $T(\mathcal{S}_G)$. In this sense, we consider the class of a Beltrami coefficient on a marked solenoid $\mathcal{X}$ as an element of $T(\mathcal{S}_G)$.

A derivative of a path of Beltrami coefficients on a marked compact solenoid $\mathcal{X}$ is called a *Beltrami differential* (when the derivative exists) and it is considered as a representative of a tangent vector to $T(\mathcal{S}_G)$ at the point $\mathcal{X}$. A Beltrami differential has finite essential supremum norm while a Beltrami coefficient have essential supremum norm less than 1.

One important question is when do two Beltrami differentials on $\mathcal{X}$ represent the same tangent vector. The Reich-Strebel inequality gives the answer (see [41]) similar to the Riemann surface case. We say that a Beltrami coefficient μ on a complex solenoid X is *infinitesimally trivial* if

$$\int_X \mu\varphi dm = 0$$

for each holomorphic quadratic differential φ on $\mathcal{X}$.

Theorem 6.5. *A smooth Beltrami differential ν on a complex solenoid $\mathcal{X}$ is infinitesimally trivial if and only if there exists a holomorphic curve μ_s of Teichmüller trivial smooth Beltrami coefficients on $\mathcal{X}$ such that $\mu_s = s\nu + O(s^2)$ in the essential supremum norm on $\mathcal{X}$.*

Denote by $L_s^\infty(\mathcal{X})$ the space of all smooth Beltrami differentials on $\mathcal{X}$ that vary continuously in the transverse direction for the C^∞-topology. Denote by $N(\mathcal{X})$ the space of infinitesimally trivial smooth Beltrami differentials. The above theorem identifies the space of tangent vectors at $[f : \mathcal{S}_G \to \mathcal{X}] \in T(\mathcal{S}_G)$ with $L_s^\infty(\mathcal{X})/N(\mathcal{X})$. Since $L_s^\infty(\mathcal{X})$ and $N(\mathcal{X})$ are not complete, it is not obvious that the tangent space is a complete vector space.

Recall that $A(\mathcal{X})$ is the space of all (transversely continuous) holomorphic quadratic differentials on $\mathcal{X}$. We introduce a surjective continuous linear map $P : L_s^\infty(\mathcal{X}) \to A(\mathcal{X})$, where $A(\mathcal{X})$ is equipped with the Bers norm. Note that $L_s^\infty(\mathcal{X})$ is identified with the space of all essentially bounded leafwise smooth function $\tilde{\mu}$ on the universal cover $\mathbf{D} \times T$ of $\mathcal{X}$ that are continuous for the transverse variations in the C^∞-topology and for the essential supremum norm, i.e.

$$\|\tilde{\mu}(z,t) - \tilde{\mu}(z,t_1)\|_\infty \to 0$$

as $t \to t_1$ for all $t_1 \in T$, and that satisfy

$$\tilde{\mu}(z,t) = \tilde{\mu}(A_X(z,t))\frac{\overline{A_X'(z,t)}}{A_X'(z,t)}$$

for all $A_X \in G_X$.

Then $P : L_s^\infty(\mathcal{X}) \to A(\mathcal{X})$ is defined by taking leafwise Bers' reproducing formula and noting that the invariance of $\tilde{\mu}$ with respect to G_X gives the invariance of the leafwise holomorphic functions $P(\tilde{\mu})$ with respect to G_X. The transverse continuity of $P(\tilde{\mu})$ follows by the continuity of the Bers' reproducing formula.

We showed [41] that P induces a linear isomorphism $\bar{P}$ from the tangent space at the point $[f : \mathcal{S}_G \to \mathcal{X}] \in T(\mathcal{S}_G)$ onto $A(\mathcal{X})$.

Corollary 6.6. *The map $P : L_s^\infty(\mathcal{X}) \to A(\mathcal{X})$ induces a continuous linear isomorphism from the normed space $L_s^\infty(\mathcal{X})/N(\mathcal{X})$ onto the Banach space $A(\mathcal{X})$*

equipped with the Bers norm. Consequently, the tangent space $L_s^\infty(\mathcal{X})/N(\mathcal{X})$ *at any point* $[f:\mathcal{S}_G\to\mathcal{X}]\in T(\mathcal{S}_G)$ *is a complex Banach space.*

Thus the tangent space to $T(\mathcal{S}_G)$ has a nice interpretation in terms of the harmonic Beltrami differentials as in the case of Teichmüller spaces of Riemann surfaces. We considered [41] to which extent the duality between the integrable holomorphic quadratic differentials and tangent vectors carries from Teichmüller spaces of Riemann surfaces to $T(\mathcal{S}_G)$. It is worth noting that $A(\mathcal{X})$ is a complete space in the Bers norm and it is not complete in the L^1-norm. This accounts for the difference from the Riemann surface case.

Theorem 6.7. *The dual* $A^*(\mathcal{X})$ *for* L^1*-norm on* $A(\mathcal{X})$ *is strictly larger than the tangent space at* $[f:\mathcal{S}\to\mathcal{X}]\in T(\mathcal{S})$.

Denote by $A^1(\mathcal{X})$ the space of integrable, a.e. leafwise holomorphic quadratic differentials on $\mathcal{X}$. Then $A(\mathcal{X})\subsetneq A^1(\mathcal{X})$ and we showed [41] the density statement in the L^1-norm.

Theorem 6.8. *The closure of* $A(\mathcal{X})$ *for the* L^1*-norm is equal to* $A^1(\mathcal{X})$.

7 The Teichmüller-type extremal maps

The Teichmüller distance between a point $[f:\mathcal{S}_G\to\mathcal{X}]\in T(\mathcal{S})$ and the basepoint $[id]\in T(\mathcal{S}_G)$ is the infimum of the logarithms of the quasiconformal constants of all maps homotopic to f. A map $f_1\in[f]$ is called *extremal* if it has the least quasiconformal constant in the homotopy class $[f]$. If $f_1\in[f]$ is extremal then

$$d([f],[id])=1/2\log K(f_1).$$

A Beltrami coefficient μ on $\mathcal{S}_G$ is called *extremal* if its corresponding quasiconformal map is extremal.

Given a holomorphic quadratic differential φ on $\mathcal{X}$, the Beltrami coefficient $k\frac{|\varphi|}{\varphi}$ is called a *Teichmüller-type* Beltrami coefficient. The corresponding quasiconformal map $f^{k\frac{|\varphi|}{\varphi}}$ is called a *Teichmüller-type* map; the quasiconformal constant of $f^{k\frac{|\varphi|}{\varphi}}$ is $K=\frac{1+k}{1-k}$; in the natural parameter $\zeta=\sqrt{\varphi}$, $f^{k\frac{|\varphi|}{\varphi}}$ is given by stretching the horizontal direction by a factor $\sqrt{K}$ and by shrinking the vertical direction by a factor $1/\sqrt{K}$.

An important consequence of the Reich-Strebel inequality is that the Teichmüller-type Beltrami coefficients are extremal in their classes [41]. In fact, a path of Teichmüller-type Beltrami coefficients gives a geodesic in $T(\mathcal{S}_G)$.

Theorem 7.1. *Let $f : \mathcal{S}_G \to \mathcal{X}$ be a quasiconformal map and let $\varphi \neq 0$ be a holomorphic quadratic differential on $\mathcal{X}$. Then the path $t \mapsto t\frac{|\varphi|}{\varphi}$, $-1 < t < 1$, of Teichmüller type Beltrami coefficients on $\mathcal{X}$ gives a geodesic (in the Teichmüller metric) through the point $[f] \in T(\mathcal{S}_G)$. In addition, any two points on this geodesic have no other geodesics connecting them.*

Remark 7.2. Note that $\varphi \in A(\mathcal{X})$ can have zeros on $\mathcal{X}$ which makes Teichmüller-type Beltrami coefficient discontinuous at these points. Strictly speaking a Teichmüller-type Beltrami coefficient does not belong to a Teichmüller class of smooth Beltrami coefficients on $\mathcal{X}$. However, this is a technical difficulty which was addressed in [41]. In fact, any zero of φ on a leaf of $\mathcal{X}$ has a neighborhood in $\mathcal{X}$ such that each local leaf has at least one zero. It can happen that a multiple zero of φ on one leaf is a limit of several simple zeros of φ on nearby leaves. The idea is to replace the Teichmüller-type Beltrami coefficient in such small neighborhoods of zeros of φ by a smooth Beltrami coefficient such that the new global Beltrami coefficient on $\mathcal{X}$ is smooth. This can be done in such a way that the restriction to each leaf of the original Teichmüller-type Beltrami coefficient and the new smooth Beltrami coefficient represent the same point in the universal Teichmüller space $T(\mathbf{D})$ and that the sequence of new smooth Beltrami coefficients (obtained by shrinking the neighborhoods of zeros of φ to a zero area set) converges to the Teichmüller-type Beltrami coefficient uniformly on the compact subsets of the complement of the set of zeros of each leaf. Moreover, the essential supremum norm of the approximating sequence approaches the norm of the Teichmüller-type Beltrami coefficient (see [41, Proposition 5.1]). Thus, Teichmüller-type Beltrami coefficients are "well" approximated by smooth Beltrami coefficients and we can consider them as elements of Teichmüller classes as well.

Remark 7.3. We noted that the union of the lifts of the Teichmüller spaces of all finite unbranched coverings of the base surface to the Teichmüller space $T(\mathcal{S}_G)$ of the compact solenoid $\mathcal{S}_G$ is dense in $T(\mathcal{S}_G)$. Moreover, if covering surface S_1 is covered by another covering surface S_2 then $T(S_1)$ embeds by isometry into $T(S_2)$ (a consequence of the Teichmüller's theorem for surfaces). One can consider a metric on $T(\mathcal{S}_G)$ to be the "limit" metric of the Teichmüller metrics on the union of the Teichmüller spaces of finite coverings. The above theorem says that the Teichmüller metric on $T(\mathcal{S}_G)$ (induced by taking the quasiconformal constants of the quasiconformal maps between the compact solenoids) agrees with the "limit" metric (because they agree on a dense subset). In particular, the extremal quasiconformal map between two TLC complex solenoids is given by the lift of the extremal maps between the surfaces (note that the Teichmüller class contains quasiconformal maps which are not lifts of maps between surfaces).

We note that Definition 5.8 is equivalent to the following definition of $T(\mathcal{S}_G)$ because each leaf is dense in $\mathcal{S}_G$. Let G be a Fuchsian group such that $\mathbf{D}/G$ is a closed surface and let G_n be the intersection of all subgroups of G of index at most n. Then G_n is a finite index characteristic subgroup of G.

Definition 7.4. The Teichmüller space $T(\mathcal{S}_G)$ of the compact solenoid $\mathcal{S}_G$ is the space of all smooth Beltrami coefficients μ on the unit disk $\mathbf{D}$ which are "almost invariant" under G, i.e. which satisfy

$$\sup_{A \in G_n} \|\mu - A^*(\mu)\|_\infty \to 0$$

as $n \to \infty$, up to the Teichmüller equivalence in the universal Teichmüller space $T(\mathbf{D})$.

Remark 7.5. The proof of Theorem 7.1 uses the Reich-Strebel inequality in an essential way. It is important that we have a transverse measure m on $\mathcal{S}_G$ in order to be able to integrate leafwise holomorphic quadratic differentials on $\mathcal{S}_G$. If we use Definition 7.4 for $T(\mathcal{S}_G)$, then the Teichmüller metric is defined in terms of the quasiconformal constants of the quasiconformal maps of the unit disk $\mathbf{D}$. If we consider a holomorphic quadratic differential φ on $\mathbf{D}$ such that the Teichmüller type Beltrami coefficient $k\frac{|\varphi|}{\varphi}$ is almost invariant, then it seems difficult to directly show that it is extremal among all equivalent almost invariant Beltrami coefficients. Thus, even though Definition 7.4 is simpler than Definition 5.8, it seems beneficial to work with the later definition when studying extremal maps.

Any TLC Beltrami coefficient $\tilde{\mu}$ on $\mathcal{S}_G$ is a lift of a Beltrami coefficient μ on a closed Riemann surface S_n in the tower of Riemann surfaces defining a fixed TLC complex structure of $\mathcal{S}$. By Teichmüller's theorem for closed surfaces, there exists $0 < k < 1$ and $\varphi \in A(S_n)$ such that $k\frac{|\varphi|}{\varphi} \in [\mu]$. Then φ lifts to a TLC holomorphic quadratic differential $\tilde{\varphi} \in A(\mathcal{S}_G)$ and $k\frac{|\tilde{\varphi}|}{\tilde{\varphi}} \in [\tilde{\mu}]$. By the above theorem, we get immediately that $d([\tilde{\mu}], [0]) = d([\mu], [0])$. In other words [41],

Corollary 7.6. *Let S be a closed Riemann surface such that the TLC complex structure on the compact solenoid $\mathcal{S}_G$ can be obtained by lifting the complex structure of S. Then the natural inclusion map*

$$i : T(S) \to T(\mathcal{S}_G)$$

obtained by mapping Beltrami coefficients on S to their lifts on $\mathcal{S}_G$ is an isometry for the Teichmüller metrics.

The Teichmüller space $T(S)$ of a closed surface S is a finite-dimensional complex manifold. Any two points $[f : S \to S_1]$ and $[g : S \to S_2]$ in $T(S)$ are connected by a unique Teichmüller-type geodesic path $t \mapsto [t\frac{|\varphi|}{\varphi}]$, $0 \leq t \leq k$, for $\varphi \in A(S_1)$ and some $0 < k < 1$.

On the other hand, the Teichmüller space $T(\mathbf{D})$ of the unit disk $\mathbf{D}$ is infinite-dimensional non-separable complex Banach manifold. There are points in $T(\mathbf{D})$ which are not connected by a Teichmüller-type geodesic path. However, Lakic [21] observed that an open, dense subset of $T(\mathbf{D})$ is connected by a Teichmüller-type geodesic to the basepoint $[0] \in T(\mathbf{D})$.

The Teichmüller space $T(\mathcal{S}_G)$ of the universal hyperbolic solenoid $\mathcal{S}_G$ is also an infinite-dimensional complex Banach manifold, but it is separable. This is the first example of a separable Teichmüller space which is the "smallest" possible infinite-dimensional space. Moreover, even though each leaf is non-compact, the solenoid $\mathcal{S}_G$ is a compact space. In addition, the union of lifts of Teichmüller spaces of all closed surfaces of genus at least two is dense in $T(\mathcal{S}_G)$ (see Nag-Sullivan [31] or [41]) and we showed in the above corollary that each such point is connected to the basepoint by a Teichmüller-type geodesic path. Based on the above remarks, one would hope that each point in $T(\mathcal{S}_G)$ is connected by a Teichmüller-type geodesic to the basepoint. If not, at least one would expect this to be true for a large subset of $T(\mathcal{S}_G)$. However, the situation for $T(\mathcal{S}_G)$ is unexpectedly different (see [14]).

Theorem 7.7. *The set of points in the Teichmüller space $T(\mathcal{S}_G)$ of the compact solenoid $\mathcal{S}_G$ which do not have a Teichmüller-type extremal representative is generic in $T(\mathcal{S}_G)$. That is, the set of points that do have a Teichmüller-type representative is of the first kind in the sense of Baire with respect to the Teichmüller metric.*

Proof. For the benefit of the reader, we give a short description of the ideas involved in the proof. The key idea is to exploit the difference between the L^1-norm and the Bers norm on the space of transversely continuous holomorphic quadratic differentials $A(\mathcal{S}_G)$ on $\mathcal{S}_G$. In particular, $A(\mathcal{S}_G)$ is complete for the Bers norm and incomplete for the L^1-norm.

We sketch the proof that there exist points in $T(\mathcal{S}_G)$ which do not have a Teichmüller-type Beltrami coefficient representatives. The proof that they are generic is just an easy modification.

Assume on the contrary that all points in $T(\mathcal{S}_G)$ have Teichmüller-type representatives. Let $A^1 = \{\varphi \in A(\mathcal{S}_G); \|\varphi\|_{L^1} = 1\}$ and let $A^1(N) = \{\varphi \in A^1; \|\varphi\|_{Bers} \leq N\}$, where $N \in \mathbb{N}$. Then $A^1 = \cup_{N=1}^{\infty} A^1(N)$. We define a map

$$\pi : T(\mathcal{S}) \to A^1 \cup \{0\}$$

by $\pi([\mu]) = \varphi$ if $[\mu] \neq [0]$, where $k\frac{|\varphi|}{\varphi} \in [\mu]$ and φ is normalized such that $\|\varphi\|_{L^1} = 1$, and by $\pi([0]) = 0$. Then

$$T(\mathcal{S}_G) = \cup_{N=1}^{\infty} \pi^{-1}(A^1(N)) \cup \{[0]\}.$$

We recall (see [14, Proposition 4.2]) that each $\pi^{-1}(A^1(N)) \cup [0]$ is closed in $T(\mathcal{S}_G)$ under our assumption above. To see this, note that if $[k_n\frac{|\varphi_n|}{\varphi_n}] \to [k\frac{|\varphi|}{\varphi}]$ then $k_n \to k$ and $\int_{\mathcal{S}} \frac{|\varphi_n|}{\varphi_n}\varphi dm \to 1$ as $n \to \infty$, by the Reich-Strebel inequality. We assume that $[k_n\frac{|\varphi_n|}{\varphi_n}] \in \pi^{-1}(A^1(N))$. Then $\int_{\mathcal{S}} \frac{|\varphi_n|}{\varphi_n}\varphi dm \to 1$ implies that $\varphi \in A^1(N)$, i.e. $[k\frac{|\varphi|}{\varphi}] \in \pi^{-1}(A^1(N))$.

This implies that at least one $\pi^{-1}(A^1(N))$ is of the second kind in the sense of Baire and hence it has an interior. We obtain a contradiction by showing that each $\pi^{-1}(A^1(N))$ is nowhere dense, hence of the first kind in the sense of Baire.

The rest of the proof depends on a geometric construction. Assume that $\pi^{-1}(A^1(N))$ has an interior and let $[\mu]$ be a TLC point in the interior, namely μ is equivalent to $k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}$ with $\tilde{\varphi}$ a lift of a holomorphic quadratic differential φ on a closed Riemann surface S. Denote by S_b a surface obtained by cutting S along a non-separating simple closed geodesic b. We consider a $\mathbb{Z}_n$-cover S_n of S obtained by cyclically gluing n-copies of the surface S_b. Let $0 < r < 1$ and denote by $S_{n,r}$ the $[rn]/n$ portion of S_n which is made out of $[rn]$ neighboring copies of S_b. The boundary of $S_{n,r}$ consists of two curves which are copies of b. Let φ_n be a quadratic differential on S_n obtained by lifting φ on the $S_{n,r}$ part and defining it to be zero on the $S_n - S_{n,r}$ part. Let $\tilde{\varphi}_n$ be the lifted quadratic differential to $\mathcal{S}_G$. Note that φ_n and $\tilde{\varphi}_n$ are piecewise holomorphic. It turns out that $\tilde{\varphi}_n$ can be approximated by holomorphic quadratic differential $\tilde{\psi}_n$ on $\mathcal{S}_G$ in the L^1-norm such that $\tilde{\psi}_n$ is a lift of a holomorphic quadratic differential ψ_n on S_n (see [14, Lemma 4.3]).

Let $\tilde{S}_{n,r}$ denote the pre-image of $S_{n,r}$ in the solenoid $\mathcal{S}_G$. Then $\alpha(\tilde{S}_{n,r}) = [nr]/n$, where α is the product measure of the leafwise hyperbolic area measure and the transverse Haar measure m multiplied by an appropriate constant such that $\alpha(\mathcal{S}_G) = 1$. We keep the notation $[\mu]$ for the fixed TLC point in the interior of $\pi^{-1}(A^1(N))$. Let ν_n on $\mathcal{S}_G$ be defined by $\nu_n = (1+r)k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}$ on $\tilde{S}_{n,r}$ and $\nu_n = k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}$ on $\mathcal{S}_G - \tilde{S}_{n,r}$. By the Reich-Strebel inequality, the Beltrami coefficient ν_n when considered as a functional on $A(\mathcal{S}_G)$ is close to achieving its norm on a holomorphic quadratic differential which is similar to $\tilde{\psi}_n$ in the L^1 sense, i.e. the integral of its absolute value when coupled with the Haar measure over $\mathcal{S}_G - \tilde{S}_{n,r}$ is converging to zero as $n \to \infty$.

If r is small enough then $[\nu_n] \in \pi^{-1}(A^1(N))$. This implies that $\tilde{\psi}_n$ is in $A^1(N)$. This is a contradiction with

$$1 = \int_{\mathcal{S}_G} |\tilde{\psi}_n| dm \leq \|\tilde{\psi}_n\|_{Bers} \alpha(\tilde{S}_{n,r}) + \int_{\mathcal{S}-\tilde{S}_{n,r}} |\tilde{\psi}_n| dm$$

because the right side can be made arbitrary small for n large and r small enough. Therefore, our starting assumption that all points have Teichmüller-type extremal representatives is not correct.

To show the stronger statement that the set of points which have Teichmüller-type Beltrami coefficient representatives is of the first kind, it is enough to assume that it is of second kind and use this set instead of the whole $T(\mathcal{S}_G)$ in the above argument. □

We recall that each point $[\mu] \in T(\mathcal{S}_G)$ is approximated by a sequence $[\mu_n] \in T(\mathcal{S}_G)$ of TLC points. Each μ_n is Teichmüller equivalent to a unique Teichmüller-type Beltrami coefficient $k_n \frac{|\varphi_n|}{\varphi_n}$, where φ_n is a TLC holomorphic quadratic differential on $\mathcal{S}_G$. We say that $[\mu]$ is *well-approximated* by the TLC sequence $[\mu_n]$ if

$$\sum_{n=1}^{\infty} \|k_n \varphi_n - k_{n+1} \varphi_{n+1}\|_{Bers} < \infty.$$

Theorem 7.8. *If a non locally transversely constant point in $T(\mathcal{S}_G)$ is well-approximated by transversely locally constant points then it contains a Teichmüller-type extremal Beltrami coefficient representative.*

Remark 7.9. We note that the above two theorems have counterparts in the infinitesimal setting. Namely, a generic vector in the tangent space at the basepoint $[0] \in T(\mathcal{S}_G)$ does not achieve its norm on $A(\mathcal{S}_G)$ (when considered as a linear functional on $A(\mathcal{S}_G)$), namely it cannot be represented by a Teichmüller-type Beltrami coefficient $k\frac{|\varphi|}{\varphi}$, for $k > 0$ and $\varphi \in A(\mathcal{S}_G)$ (see [14, Theorem 3]). A well-approximated non TLC vector in the tangent space at the basepoint $[0] \in T(\mathcal{S}_G)$ does achieve its norm on $A(\mathcal{S}_G)$ (see [14, Theorem 2']).

The set of real numbers which are not well-approximated by rational numbers is of full Lebesgue measure on the real line. From Theorem 7.7 and Theorem 7.8, we immediately obtain a similar statement for well-approximation with TLC points in $T(\mathcal{S}_G)$ (see [14]).

Corollary 7.10. *The set of points in $T(\mathcal{S}_G)$ which are not well-approximated by transversely locally constant marked complex structures is generic in $T(\mathcal{S}_G)$.*

8 The Modular group of the compact solenoid

The following definition was given by C. Odden [37]:

Definition 8.1. The *Modular group* $\mathrm{Mod}(\mathcal{S}_G)$ consists of all quasiconformal self-maps of $\mathcal{S}_G$ which preserve the baseleaf up to isotopy.

The Modular group $\mathrm{Mod}(\mathcal{S}_G)$ acts on the Teichmüller space $T(\mathcal{S}_G)$ by

$$[f : \mathcal{S}_G \to \mathcal{X}] \mapsto [f \circ g^{-1} : \mathcal{S}_G \to \mathcal{X}],$$

where $[g : \mathcal{S}_G \to \mathcal{S}] \in \mathrm{Mod}(\mathcal{S}_G)$ and $[f] \in T(\mathcal{S}_G)$.

Definition 8.2. (see [5], [37]) A *partial automorphism* of the fundamental group $G = \pi_1(S_0)$ is an isomorphism between two finite index subgroups of G. Two partial automorphisms $\psi_1 : K_1 \to H_1$ and $\psi_2 : K_2 \to H_2$ are said to be *equivalent* if they agree on the intersection of their domains. The *virtual automorphism group* $\mathrm{Vaut}(G)$ *of the surface group* G is by the definition the group of equivalence classes of partial automorphisms. Note that the virtual automorphism group is also called the (abstract) *commensurator group* $\mathrm{Comm}(G)$ of the surface group G and we use this notation in the rest of the chapter.

In [5], a natural group in which each element is given by two non-isomorphic pointed covers of the same degree of the base surface (S_0, x_0) is shown to act on the union of Teichmüller spaces of all closed surfaces of genus at least two, namely the subset of $T(\mathcal{S})_G$ consisting of all TLC points. The above group is naturally isomorphic to the commensurator group $\mathrm{Comm}(G)$ of the surface group $G = \pi_1(S_0)$. The action is isometric for the Teichmüller distance on the union of Teichmüller spaces of all closed surfaces of genus at least two and it extends by the continuity to the action on the Teichmüller space $T(\mathcal{S}_G)$. One should note that our definition of the Teichmüller metric on $T(\mathcal{S}_G)$ does not guarantee that the above union embeds isometrically in $T(\mathcal{S}_G)$; this is a consequence of the Reich-Strebel theorem for $\mathcal{S}_G$ (see Corollary 7.6). However, we do not need to use Corollary 7.6 to show that the continuous extension is possible; it is enough to note that the Teichmüller metric on the above union is bi-Lipschitz (with constant 1/3) to the Teichmüller metric on $T(\mathcal{S}_G)$ (which is a consequence of the standard result comparing Teichmüller metric on Teichmüller space of a Riemann surface with the restriction of Teichmüller metric of the universal Teichmüller space $T(\mathbf{D})$ to its embedding into $T(\mathbf{D})$ due to McMullen [29], [18]).

The following theorem (see [37]) gives a natural interpretation of the commensurator group $\mathrm{Comm}(G)$ in terms of the solenoid.

Theorem 8.3. *Let $\mathcal{S}_G$ be the G-tagged compact solenoid. Fix an identification of the baseleaf of $\mathcal{S}_G$ with $\mathbf{D}$. Then the Modular group* $\mathrm{Mod}(\mathcal{S})$ *is isomorphic* *to the commensurator group* Comm(G) *of the base surface group* G*. The isomorphism is given by the restriction of* Mod$(\mathcal{S}_G)$ *to the baseleaf.*

The group of baseleaf preserving conformal maps of $\mathcal{S}_G$ (which is a subgroup of the Modular group Mod(G)) is identified with the commensurator group $\mathrm{Comm}_{PSL_2(\mathbb{R})}(G)$ of G in $PSL_2(\mathbb{R})$ [37], where $\mathrm{Comm}_{PSL_2(\mathbb{R})}(G)$ consists of all $M \in PSL_2(\mathbb{R})$ for which there exist two finite index subgroups K and H of G such that $MKM^{-1} = H$. There are two cases, either G is an arithmetic group in which case $\mathrm{Comm}_{PSL_2(\mathbb{R})}(G)$ is dense in $PSL_2(\mathbb{R})$ or G is not arithmetic in which case $\mathrm{Comm}_{PSL_2(\mathbb{R})}(G)$ is a finite extension of G. In both cases, the group of conformal maps of the G-tagged solenoid $(\mathbf{D} \times \hat{G})/G$ is infinite (because it contains G in both cases), unlike for Riemann surfaces where it is finite. (Note that G acts non-trivially on $T(\mathcal{S}_G)$ even though it acts trivially on $T(\mathbf{D}/G)$.) Biswas and Nag [4] showed that the action of $\mathrm{Comm}_{PSL_2(\mathbb{R})}(G)$ on the G-tagged solenoid is ergodic (with respect to the product of the hyperbolic area measure on leaves and the transverse measure) if and only if G is arithmetic. For any Fuchsian uniformizing group G of a closed Riemann surface, a G-tagged solenoid represents the lift of the complex structure on $\mathbf{D}/G$ to $\mathcal{S}$. Thus, the isotropy group (in Mod$(\mathcal{S}_G)$) of a marked TLC point in $T(\mathcal{S}_G)$ is always infinite. We showed [27] that the isotropy group of any non-TLC point in $T(\mathcal{S}_G)$ is infinite as well. The basic idea was to show that the right action of the conformal covering group G_X for a non-TLC solenoid $\mathcal{X}$ commutes with the left action of G_X.

If a sequence of homeomorphisms of a closed surface converges uniformly on compact subsets to the identity, then the elements of its tail are isotopic to the identity. We showed [27] a corresponding statement for the solenoid $\mathcal{S}$.

Theorem 8.4. *Let $\mathcal{S}_G$ be a TLC complex solenoid and let $f_n : \mathcal{S}_G \to \mathcal{S}_G$ be a sequence of baseleaf preserving quasiconformal self maps of $\mathcal{S}_G$ that uniformly converges to the identity map. Then there exists n_0 such that f_n is homotopic to a baseleaf preserving conformal self map $c_n : \mathcal{S}_G \to \mathcal{S}_G$, for all $n > n_0$.*

A classical result on closed surfaces states that any two homeomorphisms which are homotopic are isotopic. Moreover, any two quasiconformal maps of two Riemann surfaces (possibly geometrically infinite) which are homotopic through bounded homotopy are isotopic through bounded quasiconformal isotopy, namely the quasiconformal constants of maps in the isotopy are uniformly bounded. We showed [27] similar result for the solenoid.

Theorem 8.5. *Let $f : X \to Y$ and $g : X \to Y$ be two homotopic quasiconformal maps of complex solenoids X and Y. Then f and g are isotopic by a uniformly quasiconformal isotopy.*

We also considered the orbits of $\mathrm{Mod}(\mathcal{S}_G)$ in $T(\mathcal{S}_G)$. It is an observation of Sullivan that the Ehrenpreis conjecture is equivalent to the statement that orbits of $\mathrm{Mod}(\mathcal{S}_G)$ are dense. We showed a weaker statement that orbits have accumulation points [27].

Theorem 8.6. *There exists a dense subset of $T(\mathcal{S}_G)$ such that the orbit of the Modular group $\mathrm{Mod}(\mathcal{S}_G)$ of any point in this subset has accumulation points in $T(\mathcal{S}_G)$. This subset contains only non-TLC points.*

An element h of $\mathrm{Mod}(\mathcal{S}_G)$ is called *mapping class like* if h conjugates a finite index subgroup K of the base surface group G onto itself, i.e. $hKh^{-1} = K$. C. Odden [37] showed that if a power h^n, $n \neq 0$, is mapping class like then h is mapping class like.

The Nielsen realization problem states that any finite subgroup of the Modular group of a closed surface is realized as a conformal group of a homeomorphic Riemann surface. We showed [27] a version of the Nielsen realization problem for the solenoid $\mathcal{S}_G$.

Theorem 8.7. *Any finite subgroup of $\mathrm{Mod}(\mathcal{S}_G)$ is cyclic and mapping class like. Consequently, elements of $\mathrm{Mod}(\mathcal{S}_G)$ which are not mapping class like are of infinite order.*

9 The Teichmüller space of the non-compact solenoid

Let $G < PSL_2(\mathbb{Z})$ be such that $\mathbf{D}/G$ is the once punctured Modular torus.

Definition 9.1. The *G-tagged non-compact solenoid* $\mathcal{S}_{nc}$ is the quotient of $\mathbf{D} \times \hat{G}$ by the action of G, where $A(z,t) := (Az, tA^{-1})$ for $(z,t) \in \mathbf{D} \times \hat{G}$ and $A \in G$.

Definition 9.2. An arbitrary *non-compact marked complex solenoid* is a complex solenoid $\mathcal{X}$ together with a differentiable, quasiconformal map $f : \mathcal{S}_{nc} \to \mathcal{X}$ which is continuous in the transverse direction in the local charts for the C^1-topology, and whose leafwise Beltrami coefficients are continuous in the transverse direction for the essential supremum norm when nearby leaves are identified using the canonical identifications coming from the G-tagged TLC complex structure of $\mathcal{S}_{nc}$.

The requirement that Beltrami coefficients are close on the whole leaves as opposed to being close in local charts is necessary because $\mathcal{S}_{nc}$ is non-compact. For marked compact solenoids we obtain the same property from the continuity in local charts because of the compactness. We introduced [36] the Teichmüller space $T(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$ as follows.

Definition 9.3. The Teichmüller space $T(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$ is the space of all differentiable, quasiconformal maps $f : \mathcal{S}_{nc} \to \mathcal{X}$ from the G-tagged solenoid to an arbitrary non-compact complex solenoid $\mathcal{X}$ up to conformal maps of the range and up to homotopy, where f is required to be continuous in the transverse direction in the local charts in the C^1-topology and the leafwise Beltrami coefficients of f are required to vary continuously on the global leaves in the essential supremum norm when leaves are canonically identified using the G-tagged complex structure of $\mathcal{S}_{nc}$.

The definition of $T(\mathcal{S}_{nc})$ is justified by the following density theorem analogous to the compact case (see [36]).

Theorem 9.4. *The union of the lifts of the Teichmüller spaces of all finite punctured hyperbolic surfaces covering the Modular torus is dense in the Teichmüller space $T(\mathcal{S}_{nc})$ of the punctured solenoid $\mathcal{S}_{nc}$.*

We introduced a representation definition of the Teichmüller space $T(\mathcal{S}_{nc})$ as follows [36]. Consider the space $\mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$ of all functions $\rho : G \times \hat{G} \to PSL_2(\mathbb{R})$ satisfying the following three properties:

Property 1: ρ is continuous;

Property 2 [G-equivariance]: for each $\gamma_1, \gamma_2 \in G$ and $t \in \hat{G}$, we have

$$\rho(\gamma_1 \circ \gamma_2, t) = \rho(\gamma_1, t\gamma_2^{-1}) \circ \rho(\gamma_2, t);$$

Property 3: for every $t \in \hat{G}$, there is a quasiconformal mapping $\phi_t : \mathbf{D} \to \mathbf{D}$ depending continuously on $t \in \hat{G}$ so that for every $\gamma \in G$, the following diagram commutes, where $\rho(\gamma, t) \circ \phi_t(z) = \phi_{t\gamma^{-1}} \circ \gamma(z)$:

$$\begin{array}{ccc}
\mathbf{D} \times \hat{G} & \xrightarrow{(z,t)\mapsto(\gamma z, t\gamma^{-1})} & \mathbf{D} \times \hat{G} \\
\phi_t \times \mathrm{id} \downarrow & & \downarrow \phi_{t\gamma^{-1}} \times \mathrm{id} \\
\mathbf{D} \times \hat{G} & \xrightarrow{(\phi_t(z),t)\mapsto(\rho(\gamma,t)\circ\phi_t(z)=\phi_{t\gamma^{-1}}\circ\gamma(z), t\gamma^{-1})} & \mathbf{D} \times \hat{G}
\end{array}$$

Since G is discrete, ρ is continuous if and only if it is continuous in its second variable. Therefore, it is enough to require continuity in the second variable in Property 1. Property 2 is a kind of homomorphism property of ρ mixing the leaves; notice in particular that taking $\gamma_2 = I$ gives $\rho(I, t) = I$

for all $t \in \hat{G}$. Property 3 mandates that for each $t \in \hat{G}$, ϕ_t conjugates the standard action of $\gamma \in G$ on $\mathbf{D} \times \hat{G}$ at the top of the diagram to the action

$$\gamma_\rho : (z, t) \mapsto (\rho(\gamma, t)z, t\gamma^{-1})$$

at the bottom, and we let $G_\rho = \{\gamma_\rho : \gamma \in G\} \approx G$. Notice that the action of G_ρ on $\mathbf{D} \times \hat{G}$ extends continuously to an action on $(\mathbf{D} \cup S^1) \times \hat{G}$. We finally define the solenoid (with marked hyperbolic structure)

$$\mathcal{S}_\rho = (\mathbf{D} \times_\rho \hat{G}) = (\mathbf{D} \times \hat{G})/G_\rho.$$

Define the group $\mathrm{Cont}(\hat{G}, PSL_2(\mathbb{R}))$ to be the collection of all continuous maps $\alpha : \hat{G} \to PSL_2(\mathbb{R})$, where the product of two $\alpha, \beta \in \mathrm{Cont}(\hat{G}, PSL_2(\mathbb{R}))$ is taken pointwise $(\alpha\beta)(t) = \alpha(t) \circ \beta(t)$ in $PSL_2(\mathbb{R})$. $\alpha \in \mathrm{Cont}(\hat{G}, PSL_2(\mathbb{R}))$ acts on $\rho \in \mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$ according to

$$(\alpha\rho)(\gamma, t) = \alpha(t\gamma^{-1}) \circ \rho(\gamma, t) \circ \alpha^{-1}(t).$$

We introduced the topology on $\mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$ as follows. Consider the natural metric d on $PSL_2(\mathbb{R})$ induced by identifying it with the unit tangent bundle of the unit disk $\mathbf{D}$. Let $\rho_1, \rho_2 \in \mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$ and let $\gamma_1, \dots, \gamma_j \in G$ be a generating set of G. The distance between ρ_1 and ρ_2 is given by

$$\max_{1 \le i \le j,\ t \in \hat{G}} d(\rho_1(\gamma_i, t), \rho_2(\gamma_i, t)). \tag{9.1}$$

This metric is not canonical, but any such two metrics induce the same topology.

$\mathrm{Hom}'(G \times \hat{G}, PSL_2(\mathbb{R})) := \mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))/\mathrm{Cont}(\hat{G}, PSL_2(\mathbb{R}))$ is equipped with the quotient topology of the above topology on $\mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$. We showed that $\mathrm{Hom}'(G \times \hat{G}, PSL_2(\mathbb{R}))$ is naturally homeomorphic to $T(\mathcal{S}_{nc})$ (see [36]).

Theorem 9.5. *There is a natural homeomorphism of the Teichmüller space $T(\mathcal{S}_{nc})$ of the solenoid $\mathcal{S}_{nc}$ with*

$$\mathrm{Hom}'(G \times \hat{G}, PSL_2(\mathbb{R})),$$

given by assigning to each $\rho \in \mathrm{Hom}'(G \times \hat{G}, PSL_2(\mathbb{R}))$ the corresponding marked hyperbolic solenoid $\mathcal{S}_\rho$.

10 The decorated Teichmüller space of the non-compact solenoid

We introduced [36] the decorated Teichmüller space $\tilde{T}(\mathcal{S}_{nc})$ of the punctured solenoid $\mathcal{S}_{nc}$. Points in $\tilde{T}(\mathcal{S}_{nc})$ are decorations of (homotopy classes of) marked hyperbolic structures up to isometries. It is convenient to use the presentation definition of the Teichmüller space $T(\mathcal{S}_{nc})$ for assigning decorations to hyperbolic metrics.

Recall that a puncture on $\mathcal{S}_{nc}$ is an end of a single leaf of $\mathcal{S}_{nc}$. Since $\mathcal{S}_{nc}$ is a G-tagged solenoid with $G < PSL_2(\mathbb{Z})$ the punctured torus group, an end has an explicit description in the universal cover $\mathbf{D} \times \hat{G}$. Denote by $\bar{\mathbb{Q}} \subset S^1$ the set of fixed points of the parabolic elements of G. Then the set of lifts to $\mathbf{D} \times \hat{G}$ of ends of $\mathcal{S}_{nc}$ is identified with $\bar{\mathbf{Q}} \times \hat{G}$.

Given a quasiconformal map $f : \mathcal{S}_{nc} \to \mathcal{X}$, the images of the ends in $\mathcal{S}_{nc}$ are the ends of $\mathcal{X}$. A decoration on $\mathcal{X}$ is easiest to understand in terms of a presentation description $\rho \in \mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$. Let $\mathcal{S}_\rho$ be a hyperbolic solenoid obtained from the representation ρ with corresponding quasiconformal map $\phi : \mathcal{S}_{nc} \to \mathcal{S}_\rho$.

We described [36] the punctures of $\mathcal{S}_\rho$ using the representation ρ. The quasiconformal map $\phi : \mathbf{D} \times \hat{G} \to \mathbf{D} \times \hat{G}$ extends continuously to a leafwise quasi-symmetric map $\phi : S^1 \times \hat{G} \to S^1 \times \hat{G}$. Recall that $\bar{\mathbf{Q}} \subset S^1$ parametrizes the endpoints of the standard triangulation of $\mathbf{D}$ invariant under $PSL_2(\mathbb{Z})$. We say that a point $(p,t) \in S^1 \times \hat{G}$ is a *ρ-puncture* if $\phi^{-1}(p,t) \in \bar{\mathbf{Q}}$, and a *puncture* of $\mathcal{S}_\rho$ itself is the G_ρ-orbit of a ρ-puncture. A *ρ-horocycle* at a ρ-puncture (p,t) is the horocycle in $\mathbf{D} \times \{t\}$ centered at (p,t) and a *horocycle* on $\mathcal{S}_\rho$ is the G_ρ-orbit of a ρ-horocycle.

We introduce an identification of horocycles with points in the light cone in Minkowski three space. Recall that Minkowski three space is $\mathbb{R}^3$ with the indefinite pairing $< \cdot, \cdot >$ whose quadratic form is $x^2+y^2-z^2$ for $(x,y,z) \in \mathbb{R}^3$. The upper sheet of the hyperboloid $\mathbb{H} := \{w = (x,y,z); < w,w >= -1,\ z > 0\}$ is a model for the hyperbolic plane and rays in the positive light cone $L^+ := \{u = (x,y,z) :< u,u >= 0,\ z > 0\}$ are identified with boundary points to the hyperbolic plane. The hyperbolic distance between $w_1, w_2 \in \mathbb{H}$ is equal to $\cosh < w_1, w_2 >$. The set of horocycles in $\mathbb{H}$ is identified with points of the positive light cone L^+ by the duality $w \mapsto \{u \in \mathbb{H}; < w,u >= -1\}$ (see [34]). A topology on the set of horocycles is induced by the correspondence with L^+ with its natural topology as a subset of $\mathbb{R}^3$.

Definition 10.1. A decoration on $\mathcal{S}_\rho$, or a "decorated hyperbolic structure" on $\mathcal{S}_\rho$, is a function $\tilde{\rho} : G \times \hat{G} \times \bar{\mathbf{Q}} \to PSL_2(\mathbb{R}) \times L^+$, where

$$\tilde{\rho}(\gamma, t, q) = \rho(\gamma, t) \times h(t, q)$$

with $\rho(\gamma, t) \in \mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$, which satisfies the following conditions:

Property 4: for each $t \in \hat{G}$, the image $h(t, \bar{\mathbf{Q}}) \subseteq L^+$ is discrete and the center of the horocycle $h(t, q)$ is $\phi_t(q)$, for all $(t, q) \in \hat{G} \times \bar{\mathbf{Q}}$ (using here the identification of L^+ with the space of horocycles);

Property 5: for each $q \in \bar{\mathbf{Q}}$, the restriction $h(\cdot, q) : \hat{G} \to L^+$ is a continuous function from $\hat{G}$ to L^+;

Property 6: $h(t, q)$ is ρ invariant in the sense that

$$\rho(\gamma, t)(h(t, q)) = h(t\gamma^{-1}, \rho(\gamma, t)q).$$

We introduced [36] the decorated Teichmüller space $\tilde{T}(\mathcal{S}_{nc})$ as follows. Let $\mathrm{Hom}(G \times \hat{G} \times \bar{\mathbf{Q}}, PSL_2(\mathbb{R}) \times L^+)$ denote the space of all decorated hyperbolic structures satisfying the properties above. We define a topology on $\mathrm{Hom}(G \times \hat{G} \times \bar{\mathbf{Q}}, PSL_2(\mathbb{R}) \times L^+)$. A neighborhood of $\tilde{\rho}(\gamma, t, q) = \rho(\gamma, t) \times h(t, q)$ consists of all $\tilde{\rho}_1(\gamma, t, q) = \rho_1(\gamma, t) \times h_1(t, q)$ such that ρ_1 belongs to a chosen neighborhood of ρ in $\mathrm{Hom}(G \times \hat{G}, PSL_2(\mathbb{R}))$, and the maps $h_1(\cdot, q) : \hat{G} \to L^+$ and $h(\cdot, q) : \hat{G} \to L^+$ are close in the supremum norm, for each $q \in \bar{\mathbf{Q}}$. The above condition and the invariance Property 6 implies that the set $h_1(t, \bar{\mathbf{Q}})$ is close to the set $h(t, \bar{\mathbf{Q}})$ in the Hausdorff metric, for each $t \in \hat{G}$.

Definition 10.2. The decorated Teichmüller space $\tilde{T}(\mathcal{S}_{nc})$ is the quotient

$$\tilde{T}(\mathcal{S}_{nc}) := \mathrm{Hom}(G \times \hat{G} \times \bar{\mathbf{Q}}, PSL_2(\mathbb{R}) \times L^+)/\mathrm{Cont}(\hat{G}, PSL_2(\mathbb{R})),$$

where $\alpha : \hat{G} \to PSL_2(\mathbb{R})$ acts on $\tilde{\rho}$ by

$$(\alpha\tilde{\rho})(\gamma, t, q) = \big(\alpha(t\gamma^{-1}) \circ \rho(\gamma, t) \circ \alpha^{-1}(t)\big) \times \big(\alpha(t)h(t, q)\big).$$

It is immediate that the forgetful map $\tilde{T}(\mathcal{S}_{nc}) \to T(\mathcal{S}_{nc})$ is a continuous surjection (see [36, Proposition 5.2]).

Given two horocycles in the hyperbolic plane, consider a geodesic connecting their centers. The horocycles intersect the geodesic at two points and the *lambda length* of the pair is defined as $\sqrt{2 \exp \delta}$, where δ is the signed length of the arc of the geodesic between the two points (see [34], [33]). The sign of δ is positive if the arc is outside the horoballs and it is negative if the arc is inside the horoballs. If $u, v \in L^+$ represent the horocycles then the lambda length is given by $\lambda(u, v) = \sqrt{- < u, v >}$.

Let τ_* be the Farey tesselation of the unit disk (see, for example, [33], [36], [35]). Then the vertices of τ_* are at $\bar{\mathbf{Q}}$ and $\tau_* \times \hat{G}$ is a tesselation of the universal cover $\mathbf{D} \times \hat{G}$ of $\mathcal{S}_{nc}$. Given a decoration $\tilde{\rho} = (\rho, h) \in \tilde{T}(\mathcal{S}_{nc})$, we consider the image tesselation $\phi(\tau_* \times \hat{G})$ of the universal cover $\mathbf{D} \times \hat{G}$ of $\mathcal{S}_\rho$ (where ϕ is the union of quasiconformal maps from Property 3). Then there is an assignment of lambda length $\lambda(e,t)$ to each edge (e,t), $e \in \tau_*$ and $t \in \hat{G}$, in the tesselation $\tau_* \times \hat{G}$ by

$$\lambda(e,t) = \lambda(h(p,t), h(q,t)),$$

where p, q are the endpoints of e. Thus we obtained a lambda length map $\lambda : \tilde{T}(\mathcal{S}_{nc}) \to (\mathbb{R}^{\tau_*}_{>0})^{\hat{G}}$, where $(\mathbb{R}^{\tau_*}_{>0})^{\hat{G}}$ are maps from $\hat{G}$ into the function space $\mathbb{R}^{\tau_*}_{>0}$ (see [36]). We consider the supremum norm over edges in τ_* on the function space $\mathbb{R}^{\tau_*}_{>0}$. Let $\mathrm{Cont}(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ be the space of continuous functions in the compact-open topology. In other words,

$$f \in \mathrm{Cont}(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$$

if

$$\sup_{e \in \tau_*} |f(t)(e) - f(t_1)(e)| \to 0$$

as $t \to t_1$, for all $t_1 \in \hat{G}$. Moreover, we define $\mathrm{Cont}^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ to be the set of G-invariant functions f in $\mathrm{Cont}(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$, i.e. $f \in \mathrm{Cont}^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ if $f \in \mathrm{Cont}(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ and

$$f(tA^{-1})(A(e)) = f(t)(e),$$

for $A \in G$ and $t \in \hat{G}$. We obtain [36]

Theorem 10.3. *The assignment of lambda lengths*

$$\lambda : \tilde{T}(\mathcal{S}_{nc}) \to \mathrm{Cont}^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$$

is a surjective homeomorphism. Namely, $\mathrm{Cont}^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ *parametrizes the decorated Teichmüller space* $\tilde{T}(\mathcal{S}_{nc})$.

A direct corollary to the above theorem is [36]:

Corollary 10.4. *The union of the lifts of the decorated Teichmüller spaces of all finite punctured surfaces covering the Modular torus is dense in the decorated Teichmüller space* $\tilde{T}(\mathcal{S}_{nc})$ *of the non-compact solenoid* $\mathcal{S}_{nc}$.

We consider the convex hull construction introduced in [15] and further utilized in [34] for punctured surfaces and in [33] for the universal Teichmüller space. The construction in [34] gives a decomposition of the decorated Teichmüller space of a punctured surface similar to [20]. Our approach is based on the universal Teichmüller space construction from [33] where the construction of [20] does not work.

A lambda length function $f \in \mathbb{R}_{>0}^{\tau_*}$ is said to be *pinched* if there exists $M > 1$ such that

$$1/M \leq f(e) \leq M$$

for all $e \in \tau_*$ (see [33]). Let $h \in (L^+)^{\bar{\mathbf{Q}}}$ and assume that the corresponding lambda length function $\lambda : e \mapsto \lambda(h(e))$, $e \in \tau_*$, is pinched. Consider the image $h(\bar{\mathbf{Q}}) \subset L^+$ of h and let $\mathcal{C}(h(\bar{\mathbf{Q}}))$ denote its convex hull as a subset of $\mathbb{R}^3$. Then the results from [33] give that $h(\bar{\mathbf{Q}})$ is a discrete and radially dense subset of L^+. Moreover, $h : \bar{\mathbf{Q}} \to L^+$ projects to a map $\bar{h} : \bar{\mathbf{Q}} \to S^1$ which extends to a quasisymmetric homeomorphism of S^1. In addition, the set of faces of the boundary $\partial\mathcal{C}(h(\bar{\mathbf{Q}}))$ of the convex hull $\mathcal{C}(h(\bar{\mathbf{Q}}))$ consists of Euclidean polygons which meet along their boundary edges, the set of faces is locally finite and boundary edges of faces of $\partial\mathcal{C}(h(\bar{\mathbf{Q}}))$ project to a locally finite geodesic lamination on the hyperbolic plane $\mathbb{H}$ whose geodesics have endpoints in $\bar{\mathbf{Q}}$ (see [33] for more details and proofs).

A decoration $\tilde{\rho} \in \tilde{T}(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$ gives a lambda length function $\lambda(\tilde{\rho}) \in \mathrm{Cont}^G(\hat{G}, \mathbb{R}_{>0}^{\tau_*})$. Namely, we obtain a Cantor set of lambda lengths $\lambda(\tilde{\rho})(t) : \tau_* \to \mathbb{R}_{>0}$, for $t \in \hat{G}$, and note that the lambda lengths are pinched uniformly in $t \in \hat{G}$ by the compactness of $\hat{G}$ and the transverse continuity of $\tilde{\rho}$ (see [33, Lemma 6.1]). The above convex hull construction applied to each leaf $\mathbf{D} \times \{t\}$ of the universal cover $\mathbf{D} \times \hat{G}$ gives a Cantor set of convex hulls which in turn produce a Cantor set of geodesic laminations on $\mathbf{D} \times \hat{G}$ which are invariant under the action of G. Denote by $\tau_{\tilde{\rho}}$ such obtained leafwise geodesic lamination on $\mathbf{D} \times \hat{G}$. The endpoints of geodesics in $\tau_{\tilde{\rho}}$ lie in $\bar{\mathbf{Q}} \times \hat{G}$ and we call such geodesic lamination a *tesselation* if all complementary regions are ideal triangles. In general, the complementary regions of $\tau_{\tilde{\rho}}$ on leaves can be arbitrary ideal hyperbolic polygons. A tesselation τ of $\mathbf{D} \times \hat{G}$ such that the restriction to each leaf $\tau(t) \subset \mathbf{D}$, $t \in \hat{G}$, is invariant under some finite index subgroup K of G is called a *TLC tesselation.* Equivalently, τ is a TLC tesselation of $\mathbf{D} \times \hat{G}$ if it is a lift of a tesselation on a Riemann surface $\mathbf{D}/K$, for some finite index subgroup $K < G$.

Definition 10.5. Let τ be a leafwise geodesic lamination on $\mathbf{D} \times \hat{G}$. Denote by $C(\tau)$ the set of all decorations for which the convex hull construction produces λ, i.e.

$$C(\tau) := \{\tilde{\rho} \in \tilde{T}(\mathcal{S}_{nc}); \partial\mathcal{C}(\lambda(\tilde{\rho})) = \tau\}.$$

We showed [36] that generically in $\tilde{T}(\mathcal{S}_{nc})$ convex hull constructions yield TLC tesselations. In more details,

Theorem 10.6. *The subset $C(\tau)$ of $\tilde{T}(\mathcal{S}_{nc})$ is open for each TLC tesselation τ, and $\cup_\tau C(\tau)$ is a dense open subset of $\tilde{T}(\mathcal{S}_{nc})$, where the union is over all TLC tesselations τ.*

11 A presentation for the Modular group of the non-compact solenoid

We define the *Modular group* $\mathrm{Mod}(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$ to consist of (analogously to the compact solenoid) all quasiconformal differentiable baseleaf preserving self-maps of $\mathcal{S}_{nc}$ up to isotopy (see [36]). We showed [36] an appropriate version of the characterization of $\mathrm{Mod}(\mathcal{S}_{nc})$ similar to the compact solenoid [37]. As in Section 9, let $G < PSL_2(\mathbb{Z})$ be the once punctured Modular torus group.

Theorem 11.1. *The restriction to the baseleaf of* $\mathrm{Mod}(\mathcal{S}_{nc})$ *gives an isomorphism of* $\mathrm{Mod}(\mathcal{S}_{nc})$ *with the subgroup of the commensurator group of G consisting of elements which map peripheral elements onto peripheral elements.*

The action of the Modular group $\mathrm{Mod}(\mathcal{S}_{nc})$ on the decorated Teichmüller space $\tilde{T}(\mathcal{S}_{nc})$ preserves the decomposition into sets $C(\tau)$, for τ a leafwise geodesic lamination on the solenoid, or equivalently a G-invariant geodesic lamination on the universal cover. It is convenient to consider TLC tesselation only. Then, as a consequence of the above theorem, the Modular group preserves the subspace of TLC tesselations. We showed [36] that an analogue of the Ehrenpreis conjecture in the decorated Teichmüller space $\tilde{T}(\mathcal{S}_{nc})$ is not correct.

Theorem 11.2. *The quotient $\cup_\tau C(\tau)/MCG_{BLP}(\mathcal{S}_{nc})$ is Hausdorff, where the union is over all TLC tesselations τ. Moreover, no orbit under* $\mathrm{Mod}(\mathcal{S}_{nc})$ *of a point in $\tilde{T}(\mathcal{S}_{nc})$ is dense.*

From now on, we restrict the action of $\mathrm{Mod}(\mathcal{S}_{nc})$ to the baseleaf. Then $\mathrm{Mod}(\mathcal{S}_{nc})$ preserves the space of all TLC tesselations on $\mathbf{D}$, i.e. it preserves the space of lifts of all ideal hyperbolic triangulations of all Riemann surfaces $\mathbf{D}/K$, where $K < G$ is of finite index. The Farey tesselation τ_* on $\mathbf{D}$ is a TLC tesselation which will be considered as a basepoint in our considerations.

We showed a transitivity statement for the family of TLC tesselations, or equivalently for the family $\{C(\tau)\}_\tau$, where τ belongs to all TLC tesselations (see [36]).

Theorem 11.3. $\mathrm{Mod}(\mathcal{S}_{nc})$ *acts transitively on* $\{C(\tau) : \tau \text{ is TLC}\}$.

Proof. We give a brief description of the proof. It is enough to show that any TLC tesselation of the unit disk $\mathbf{D}$ is mapped onto any other TLC tesselation of $\mathbf{D}$ by a homeomorphism of S^1 which conjugates one finite index subgroup of G onto another finite index subgroup of G. Such a homeomorphism of S^1 induces an element of the commensurator group of G which preserves parabolics, and conversely any element of the commensurator group of G which preserves parabolics is induced by a homeomorphism of S^1. Recall that a TLC tesselation of $\mathbf{D}$ is a lift of an ideal triangulation of $\mathbf{D}/K$, where $K < G$ is of finite index. In particular, the set of ideal vertices of the lifted TLC tesselation of $\mathbf{D}$ is $\bar{\mathbf{Q}}$ and the tesselation is K-invariant.

Moreover, it is enough to show that the Farey tesselation τ_* can be mapped by a homeomorphism inducing a parabolics preserving element of the commensurator group of G onto any other TLC tesselation of $\mathbf{D}$. Let τ be an arbitrary TLC tesselation of $\mathbf{D}$ which is invariant under a finite index subgroup K of G. We define a *characteristic map* for τ by giving an identification of the edges of τ_* and τ as follows (see [33]). We choose the edge e_0 of τ_* which joins -1 and 1 and orient it from -1 to 1. Such a distinguished oriented edge is called a DOE. We choose an arbitrary edge e of τ and give it an arbitrary orientation; e is a DOE of τ. The characteristic map is built by induction. We first identify DOEs e_0 and e with orientations. The construction of the map proceeds by identifying complementary ideal triangles of τ_* and τ according to their relative positions with respect to DOEs e_0 and e; in fact, the identifications of the triangles uniquely determine an identification of the edges of τ_* and τ. The DOEs $e_0 \in \tau_*$ and $e \in \tau$ separate $\mathbf{D}$ into left and right half-disks according to their orientations. We identify the immediate left triangle T_0 of τ_* with respect to e_0 to the immediate left triangle T of τ with respect to e. This forces the identification of boundary edges of T_0 and T such that the edges with endpoints at the initial points of DOEs get identified and the edges at terminal points of DOEs get identified. To proceed with the construction of the map, we give orientations to both edges of both triangles T_0 and T such that the triangles are on the right of the edges. Then we continue the identifications of the triangles on the immediate left of the two edges in τ_* with the triangles on the immediate left of the two edges in τ as above. This process continues indefinitely on the left side of DOEs and we do similar identifications on the right side of DOEs. It is not hard to see that the characteristic map between the edges of τ_* and τ extends to a order preserving map h from the ideal boundary points $\bar{\mathbf{Q}} \subset S^1$ onto itself. Then the characteristic map h extends to a homeomorphism of S^1 because the ideal points of the tesselations are dense in S^1 (see [36]).

We show that the characteristic map $h : S^1 \to S^1$ conjugates a finite index subgroup H of G onto K (see [36]). Let ω be an ideal fundamental polygon for K whose boundary edges are in τ. Then $h^{-1}(\omega)$ is an ideal polygon with boundary edges in τ_*. The boundary sides of ω are identified in pairs by elements of K and we consider the corresponding boundary sides pairs in $h^{-1}(\omega)$. Since $PSL_2(\mathbb{Z})$ acts freely and transitively on the oriented edges of τ_*, there exist unique maps in $PSL_2(\mathbb{Z})$ which identify corresponding boundary side pairs of $h^{-1}(\omega)$ with the correct orientation such that the quotient is homeomorphic to $\mathbf{D}/K$. Let H be the subgroup of $PSL_2(\mathbb{Z})$ generated by these elements. Then h conjugates H onto K (see [36]). □

We considered the isotropy group in $\mathrm{Mod}(\mathcal{S}_{nc})$ of a single TLC tesselation of $\mathbf{D}$. A basic result states that any orientation preserving homeomorphism of S^1 which fixes the Farey tesselation is necessarily an element of $PSL_2(\mathbb{Z})$ (see [36, Lemma 7.3]). Then

Theorem 11.4. *The isotropy subgroup in* $\mathrm{Mod}(\mathcal{S}_{nc})$ *of* τ*, for* τ *a TLC tesselation, is quasiconformally conjugate to* $PSL_2(\mathbb{Z})$*. The isotropy subgroup of* τ_* *is* $PSL_2(\mathbb{Z})$*.*

Let τ be a TLC tesselation of $\mathbf{D}$ which is invariant under $K < G$. Fix an edge e of τ. Then e is on the boundary of exactly two complementary ideal triangles of τ. The union of the two triangles is an ideal quadrilateral $P \subset \mathbf{D}$ whose one diagonal is e. If no two edges in the set $K\{e\}$ are immediate neighbors, then the operation of changing diagonals $K\{e\}$ along the orbit $K\{P\}$ of quadrilaterals is well-defined and produces a new TLC tesselation which is also invariant under K. Such an operation is called a *K-equivariant Whitehead move* (see [36]). This is a lift to the unit disk $\mathbf{D}$ of a classical Whitehead move on surface $\mathbf{D}/K$ considered by Hatcher and Thurston [19], Harer [20] and Penner [34]. Penner [33] also considered Whitehead moves on $\mathbf{D}$ without the equivariance property.

The above transitivity result implies that any TLC tesselation of $\mathbf{D}$ can be mapped by an element of $\mathrm{Mod}(\mathcal{S}_{nc})$ to its image under an equivariant Whitehead move. An element of $\mathrm{Mod}(\mathcal{S}_{nc})$ which achieves this is not unique; the ambiguity is up to pre-composition by an element of $\mathrm{Mod}(\mathcal{S}_{nc})$ which fixes the initial tesselation. If we are given a DOE e_1 on the initial TLC tesselation τ, then a DOE e_2 on the image tesselation τ_1 under a K-equivariant move on $K\{e\}$ is determined by $e_2 := e_1$ if $e_1 \notin K\{e\}$, or otherwise $e_2 := f_1$, where f_1 is the other diagonal in the quadrilateral containing e_1 oriented such that (e_1, f_1) is a positive basis at their intersection point. In this case the element of $\mathrm{Mod}(\mathcal{S}_{nc})$, called the *Whitehead homeomorphism*, is uniquely determined by mapping DOE onto DOE. Let h_τ and h_{τ_1} be the characteristic maps for τ and τ_1, namely $h_\tau(\tau_*) = \tau$, $h_\tau(e_0) = e_1$, $h_{\tau_1}(\tau_*) = \tau_1$ and $h_{\tau_1}(e_0) = e_2$, where $e_0 = (-1, 1)$ is DOE of τ_*. Then the above Whitehead homeomorphism is given by $h_{\tau_1} \circ h_\tau^{-1}$ (see [36]).

A basic fact due to Thurston, Hatcher, Harer and Penner is that any two ideal triangulations of a punctured surface are connected by a sequence of Whitehead moves. Therefore, if two TLC tesselations are invariant under K then they can be connected by a sequence of K-equivariant moves. If one TLC tesselation is invariant under K_1 and the other is invariant under K_2 then they can be connected by $(K_1 \cap K_2)$-equivariant Whitehead moves (because they are both invariant under $K_1 \cap K_2$). This transitivity of all equivariant Whitehead moves on the set of TLC tesselations allows us to give generators of $\mathrm{Mod}(\mathcal{S}_{nc})$ (see [36]). A composition of Whitehead homeomorphisms is called *geometric* if they are all K-equivariant, for a fixed subgroup K, and they are formed from a geometric sequence of Whitehead moves. We obtained [36]

Theorem 11.5. *Any element of the modular group* $\mathrm{Mod}(\mathcal{S}_{nc})$ *can be written as a composition* $w \circ \gamma$*, where* $\gamma \in PSL_2(\mathbb{Z})$ *and* w *is a geometric composition of* K*-equivariant Whitehead homeomorphisms for some fixed* K*.*

We describe a presentation of the Modular class group $\mathrm{Mod}(\mathcal{S}_{nc})$ from [6]. We first define a 2-complex called the *triangulation complex* $\mathcal{X}$ (see [6]). The vertices are all TLC tesselations of the unit disk $\mathbf{D}$. We already showed that $\mathrm{Mod}(\mathcal{S}_{nc})$ preserves the set of vertices $\mathcal{X}_0$ and it acts transitively on them. The Farey tesselation τ_* is the basepoint of $\mathcal{X}$.

The set of edges $\mathcal{X}_1$ is first defined at the base point τ_*. A vertex $\tau \in \mathcal{X}_0$ is connected to the basepoint τ_* if τ is obtained from τ_* by a single K-equivariant Whitehead move, for some finite index subgroup K of G. An edge at an arbitrary $\tau \in \mathcal{X}_0$ is the image under $h_\tau \in \mathrm{Mod}(\mathcal{S}_{nc})$ of an edge at the basepoint. Therefore, an edge connecting arbitrary $\tau, \tau_1 \in \mathcal{X}_0$ is obtained by a single "generalized" Whitehead move, namely the move is equivariant under a conjugate of K, $[G : K] < \infty$, by $h : S^1 \to S^1$ which induces an element of $\mathrm{Comm}(G)$. The difference from a (regular) Whitehead move is that h conjugates a proper subgroup K_1 of K onto another subgroup H_1 of G and the move is along the orbit of an edge for hKh^{-1} which is not a subgroup of G. However, the generalized hKh^{-1}-equivariant Whitehead move can be decomposed into finitely many H_1-equivariant Whitehead moves. On the other hand, the image at the basepoint of an edge at an arbitrary point is necessarily obtained by a (regular) Whitehead move [6]. Thus we do not introduce new edges at the basepoint τ_*. The set of edges $\mathcal{X}_1$ is invariant under $\mathrm{Mod}(\mathcal{S}_{nc})$ by definition.

The two cells $\mathcal{X}_2$ are introduced first at the basepoint τ_*. There are three kinds of two cells.

The *square two cells* are defined by adding a two cell to each cycle of four edges which are based at τ_* and have the following properties. The four edges are given by Whitehead moves equivariant with respect to the same finite index subgroup K of G. We assume that $e_1, e_2 \in \tau_*$ are two edges such that their corresponding orbits $K\{e_1\}$ and $K\{e_2\}$ have no pairs (whose one element is from $K\{e_1\}$ and the other is from $K\{e_2\}$) of adjacent edges in τ_*. (Since we take a K-equivariant Whitehead move for e_i, we implicitly assume that orbit $K\{e_i\} \subset \tau_*$ does not have adjacent edges in τ_*, for $i = 1, 2$.) Let f_1, f_2 be the other diagonals in the two quadrilaterals in $(\mathbf{D} - \tau_*) \cup \{e_1\}, (\mathbf{D} - \tau_*) \cup \{e_2\}$ containing e_1, e_2. Then we form a TLC tesselation τ_1 by performing a K-equivariant Whitehead move on τ_* along e_1; we form a TLC tesselation τ_2 by performing a K-equivariant Whitehead move on τ_1 along e_2; we form a TLC tesselation τ_3 by performing a K-equivariant Whitehead move on τ_2 along f_1; and we return to τ_* by performing a K-equivariant Whitehead move on τ_3 along f_2. The corresponding edges $E_1 = (\tau_*, \tau_1)$, $E_2 = (\tau_1, \tau_2)$, $E_3 = (\tau_2, \tau_3)$ and $E_4 = (\tau_3, \tau_*)$ make a closed path. We add a square two cell to $\mathcal{X}$ whose boundary is the above closed edge path.

The *pentagon two cells* are defined by adding a two cell whose boundary is a closed edge path of length five as follows. Let K be a finite index subgroup of G and let e_1, e_2 be two adjacent edges in τ_*. Assume that $e_1 \notin K\{e_2\}$. Let P be the pentagon in $(\mathbf{D} - \tau_*) \cup \{e_1, e_2\}$; the orbit of pentagons $K\{P\}$ has pairwise mutually disjoint interiors with possible identifications of their boundaries. We define a closed edge path of length five by Whitehead moves: E_1 is given by a Whitehead move along $K\{e_1\}$ where $K\{e_1\} \mapsto K\{f_1\}$; E_2 is given by a Whitehead move along $K\{e_2\}$ where $K\{e_2\} \mapsto K\{f_2\}$; E_3 is given by a Whitehead move along $K\{f_1\}$ where $K\{f_1\} \mapsto K\{f_3\}$; E_4 is given by a Whitehead move along $K\{f_2\}$ where $K\{f_2\} \mapsto K\{e_1\}$; and E_5 is given by a Whitehead move along $K\{f_3\}$ where $K\{f_3\} \mapsto K\{e_2\}$ (this is the classical pentagon relation on a surface lifted to $\mathbf{D}$; see, for example, [34],[33],[36]). We add a pentagon two cell whose boundary is such an edge path.

The *coset two cells* are defined by subdividing a single equivariant Whitehead move into several equivariant Whitehead moves as follows. Let K be a finite index subgroup of G and let K_1 be a finite index subgroup of K. Let $e \in \tau_*$ be such that no two edges in the orbit $K\{e\}$ are adjacent in τ_*. The *long edge* E is given by K-equivariant Whitehead move along $K\{e\}$. The *short edges* are given by K_1-equivariant Whitehead moves as follows. Since $k := [K : K_1] < \infty$, there exists finitely many $e_1, e_2, \ldots, e_k \in K\{e\}$ such that $e_i \notin K_1\{e_j\}$, for $i \neq j$, and $\cup_{i=1}^k K_1\{e_i\} = K\{e\}$. We define a sequence of short edges $E_1, \ldots E_k$ by $E_i = (\tau_{i-1}, \tau_i)$, where τ_i, for $i = 1, 2, \ldots, k$, is obtained from τ_{i-1} by performing a K_1-equivariant Whitehead move on τ_{i-1} along $K_1\{e_i\}$ and $\tau_0 = \tau_*$. The edge path $E_1, E_2, \ldots, E_k$ starts at τ_* and ends at the endpoint of E. Thus $E_1, \ldots, E_k, E$ is a closed edge path based at τ_* and we add a coset two cell whose boundary is the given path (see [6]).

A general two cell in $\mathcal{X}$ is the image under $\mathrm{Mod}(\mathcal{S}_{nc})$ of a two cell based at τ_*. It turns out that an image of a square or a pentagon two cell based at τ_* under $\mathrm{Mod}(\mathcal{S}_{nc})$ whose one vertex on its boundary is τ_* is of the same form as above. Namely, all the edges are Whitehead moves equivariant under a subgroup of $PSL_2(\mathbb{Z})$, while an image of a coset two cell under $\mathrm{Mod}(\mathcal{S}_{nc})$ has generalized Whitehead moves as edges whenever the long edge does not limit at τ_*. It may happen that two short edges limit at τ_*. The Modular group $\mathrm{Mod}(\mathcal{S}_{nc})$ preserves the set of two cells $\mathcal{X}_2$ by its definition (see [6]).

We showed that the triangulation complex is simply connected [6].

Theorem 11.6. *The Modular group* $\mathrm{Mod}(\mathcal{S}_{nc})$ *acts cellularly on the triangulation complex* $\mathcal{X}$. *The triangulation complex* $\mathcal{X}$ *is connected and simply connected.*

We gave [6] a presentation of the Modular group $\mathrm{Mod}(\mathcal{S}_{nc})$ using its action on $\mathcal{X}$. We already showed that $\mathrm{Mod}(\mathcal{S}_{nc})$ acts transitively on the vertices of $\mathcal{X}$ and that the isotropy group of τ_* is $PSL_2(\mathbb{Z})$. Therefore, each orbit of an edge contains an edge with one endpoint at τ_*. To give a presentation, it is necessary to find the isotropy groups of edges. There are two types of edges with one endpoint in τ_*, the set $\mathcal{E}^+$ of edges which are not inverted by the action of $\mathrm{Mod}(\mathcal{S}_{nc})$ and the set of edges $\mathcal{E}^-$ which are inverted by the action of $\mathrm{Mod}(\mathcal{S}_{nc})$. If $E \in \mathcal{E}^+$ is obtained by a K-equivariant Whitehead move then the isotropy group of E has to be contained in $PSL_2(\mathbb{Z})$ and it contains K. In fact, the isotropy group of E is a finite extension K' of K by elliptic elements in $PSL_2(\mathbb{Z})$ which preserve the other vertex τ of the edge $E = (\tau_*, \tau)$. Let $E_1 = (\tau_*, \tau_1) \in \mathcal{E}^-$ be an edge reversed by $\mathrm{Mod}(\mathcal{S}_{nc})$, where τ_1 is obtained by a K_1-equivariant Whitehead move and fixed by $K_1' > K$. Then the isotropy group of E_1 is generated by $K_1' < PSL_2(\mathbb{Z})$ which does not reverse the orientation of E_1 and by $k \in \mathrm{Mod}(\mathcal{S}_{nc})$ which reverses the orientation, where k is mapping class like (i.e. k conjugates a finite index subgroup of G onto itself) and $k^2 \in K_1' - K_1$ is elliptic fixing an edge in τ_* which implies $k^4 = id$ (see [6]).

We choose a single Whitehead move for each edge $E = (\tau_*, \tau)$ starting at τ_* by taking $e_0 = (-1, 1)$ to be a DOE of τ_* and defining a DOE of the other vertex τ as above. The set of these Whitehead moves together with $PSL_2(\mathbb{Z})$ generate $\mathrm{Mod}(\mathcal{S}_{nc})$. We describe relations coming from two cells in $\mathcal{X}$ for the chosen generating set.

Let Q be a square two cell based at τ_* with edges $E_i = (\tau_{i-1}, \tau_i)$, for $i = 1, \ldots, 4$, with $\tau_0 = \tau_4 = \tau_*$ such that τ_i are K-equivariant. Let $h_1 \in \mathrm{Mod}(\mathcal{S}_{nc})$ be the Whitehead homeomorphism corresponding to E_1, h_2 the Whitehead homeomorphism corresponding to the edge $E_2' = (\tau_*, h_1^{-1}(\tau_*))$, h_3

the Whitehead homeomorphism corresponding to the edge $E_3' = (\tau_*, (h_1 \circ h_2)^{-1}(\tau_*))$, and h_4 the Whitehead homeomorphism corresponding to the edge $E_4' = (\tau_*, (h_1 \circ h_2 \circ h_3)^{-1}(\tau_*))$. If $e_0 \notin K\{e_1, e_2\}$, where $K\{e_1\}$ and $K\{e_2\}$ are orbits which get changed in the definition of Q, then $h_1 \circ \cdots \circ h_4(e_0) = e_0$ which implies that

$$h_1 \circ \cdots \circ h_4 = id. \tag{11.1}$$

If $e_0 \in K\{e_1, e_2\}$ then $h_1 \circ \cdots \circ h_4(e_0) = \bar{e}_0$, where $\bar{e}_0$ is the opposite orientation of e_0, which implies that

$$h_1 \circ \cdots \circ h_4 = s_{e_0}, \tag{11.2}$$

where $s_{e_0} \in PSL_2(\mathbb{Z})$ is an elliptic element reversing the orientation of e_0.

Let P be a pentagon two cell based at τ_* with boundary edge path $E_i = (\tau_{i-1}, \tau_i)$, for $i = 1, \ldots, 5$, where $\tau_0 = \tau_5 = \tau_*$ and τ_i are K-equivariant. Let $K\{e_1, e_2\}$ be the orbits which get changed to obtain P. Let h_i be the Whitehead move corresponding to $(h_1 \circ \cdots \circ h_{i-1})^{-1}(E_i)$ as defined above. Then we have a pentagon relation

$$h_1 \circ \cdots \circ h_5 = id, \tag{11.3}$$

whenever $e_0 \notin K\{e_1, e_2\}$. We get

$$h_1 \circ \cdots \circ h_5 = \gamma_{e_0, \bar{e}_2}, \tag{11.4}$$

when $e_0 \in K\{e_1\}$, where $\gamma_{e_0, \bar{e}_2}$ maps e_0 onto $\bar{e}_2$. Finally, we get

$$h_1 \circ \cdots \circ h_5 = \gamma_{e_0, \bar{e}_1}, \tag{11.5}$$

when $e_0 \in K\{e_2\}$, where $\gamma_{e_0, \bar{e}_1}$ maps e_0 onto $\bar{e}_1$.

Let C be a coset two cell given by a long edge determined by Whitehead move along $K\{e\}$ and by short edges with respect to $K_1 < K$, where $k = [K : K_1] < \infty$. We note that given K_1, there are $k!$ paths of short edges connecting the two endpoints of the long edge. If $e_0 \notin K\{e\}$ then we obtain a coset relation

$$h \circ h_1 \circ \cdots \circ h_k = id, \tag{11.6}$$

where h is the Whitehead homeomorphism corresponding to the long edge and h_i is the Whitehead homeomorphism corresponding to the image $(h \circ h_1 \circ \cdots \circ h_{i-1})^{-1}(E_i)$ of the i-th short edge E_i. If $e_0 \in K\{e\}$ then we obtain coset relation

$$h \circ h_1 \circ \cdots \circ h_k = s_{e_0}, \tag{11.7}$$

where $s_{e_0} \in PSL_2(\mathbb{Z})$ reverses the orientation of e_0.

We obtained [6] a presentation for $\mathrm{Mod}(\mathcal{S}_{nc})$ as follows.

Theorem 11.7. *The modular group* $\mathrm{Mod}(\mathcal{S}_{nc})$ *is generated by the isotropy subgroup* $PSL_2(\mathbb{Z})$ *of the basepoint* $\tau_* \in \mathcal{X}$*, the isotropy subgroups* $\Gamma(E)$ *for* $E \in \mathcal{E}^{\pm}$*, and by the Whitehead homeomorphism* g_E *for* $E \in \mathcal{E}^+$ *chosen as above. The following relations on these generators give a complete presentation of* $\mathrm{Mod}(\mathcal{S}_{nc})$*:*

a) The inclusions of $\Gamma(E)$ *into* $PSL_2(\mathbb{Z})$*, for* $E \in \mathcal{E}^+$*, are given by* $\Gamma(E) = K'$*, where the terminal endpoint of* E *is invariant under the finite-index subgroup* $K' < PSL_2(\mathbb{Z})$*;*

b) The inclusions of $\Gamma^+(E)$ *into* $PSL_2(\mathbb{Z})$*, for* $E \in \mathcal{E}^-$*, are given by* $\Gamma(E) = K'$*, where the terminal endpoint of* E *is invariant under the finite-index subgroup* $K' < PSL_2(\mathbb{Z})$*;*

c) The relations introduced by the boundary edge-paths of two-cells in $\mathcal{F}$ *given by the equations (11.3), (11.4), (11.5), (11.1), (11.2), (11.6) and (11.7);*

d) The redundancy relations: for any two edges E *and* E' *in* $\mathcal{E}^{\pm}$ *and for any* $\gamma \in PSL_2(\mathbb{Z})$ *such that* $\gamma(E) = E'$*, we get the relation*

$$g_{E'} \circ \gamma' = \gamma \circ g_E,$$

where γ' *is the unique element of* $PSL_2(\mathbb{Z})$ *that satisfies* $\gamma'(e_0) = e_1'$ *with* $e_1' = g_{E'}^{-1}(\gamma(e_0))$*.*

Remark 11.8. The redundancy relations d) in the above theorem are introduced because we used more generators than necessary. We could have used Whitehead homeomorphisms of representatives of orbits of edges based at τ_* instead. Then we would not have to add relations d). However, it is not easy to give a proper enumeration of such orbits which would necessarily complicate the relations in c). Thus, for the sake of simplicity of relations, we used a larger set of generators in the above theorem.

We also showed [6] that $\mathrm{Mod}(\mathcal{S}_{nc})$ has no center.

Theorem 11.9. *The modular group* $\mathrm{Mod}(\mathcal{S}_{nc})$ *of the punctured solenoid* $\mathcal{S}_{nc}$ *has trivial center.*

12 Elements of $\mathrm{Mod}(\mathcal{S}_{nc})$ with small non-zero dilatations

In a recent joint work with V. Markovic [28], we showed the following

Theorem 12.1. *For every* $\epsilon > 0$ *there exist two finite index subgroups of* $PSL_2(\mathbb{Z})$ *which are conjugated by a* $(1+\epsilon)$*-quasisymmetric homeomorphism of the unit circle and this conjugation homeomorphism is not conformal.*

To construct the above groups and the quasisymmetric map, we use the generators of the modular group $\text{Mod}(\mathcal{S}_{nc})$ introduced in [36] (see also the proof of Theorem 11.3).

We obtained [28] the following corollary to the above theorem.

Corollary 12.2. *Let T_0 denote the Modular torus. Then for every $\epsilon > 0$ there are finite degree, regular coverings $\pi_1 : M_1 \to T_0$ and $\pi_2 : M_2 \to T_0$, and a $(1+\epsilon)$-quasiconformal homeomorphism $F : M_1 \to M_2$ that is not homotopic to a conformal map.*

The following corollary is an interpretation of Theorem 12.1 in terms of the Teichmüller space $T(\mathcal{S}_{nc})$ of the non-compact solenoid $\mathcal{S}_{nc}$. This is a significant progress in understanding the quotient $T(\mathcal{S}_{nc})/\text{Mod}(\mathcal{S}_{nc})$.

Corollary 12.3. *The closure in the Teichmüller metric of the orbit (under the Modular group $\text{Mod}(\mathcal{S}_{nc})$) of the basepoint in $T(\mathcal{S}_{nc})$ is strictly larger than the orbit. Moreover, the closure of this orbit is uncountable.*

13 Some open problems

We discuss some open question concerning the Teichmüller space $T(\mathcal{S}_G)$ and the Modular group $\text{Mod}(\mathcal{S}_G)$ of the solenoid.

As we already mentioned, a conjecture by L. Ehrenpreis states that given any two closed Riemann surfaces of genus at least two and given any $\epsilon > 0$ there exist unbranched, finite sheeted, holomorphic covers of these surfaces that are $(1+\epsilon)$-quasiconformal. D. Sullivan gave the following equivalent formulation in terms of the compact solenoid:

1. Is it true that the Modular group $\text{Mod}(\mathcal{S}_G)$ has dense orbits in the Teichmüller space $T(\mathcal{S}_G)$ of the compact solenoid $\mathcal{S}_G$?

We also considered the Teichmüller space $T(\mathcal{S}_{nc})$ and the Modular group $\text{Mod}(\mathcal{S}_{nc})$ of the noncompact solenoid $\mathcal{S}_{nc}$. Therefore we can ask the analogous question in this setup:

2. Is it true that the Modular group $\text{Mod}(\mathcal{S}_{nc})$ has dense orbits in the Teichmüller space $T(\mathcal{S}_{nc})$ of the noncompact solenoid $\mathcal{S}_{nc}$?

It is interesting to note that a positive answer to question 1 does not immediately give a positive answer to question 2. This is easiest to understand in terms of the original formulation by Ehrenpreis. To see this, assume for the moment that any two closed surfaces have unbranched, finite sheeted, holomorphic covers which are quasiconformal with arbitrary small dilatation. When considering two punctured surfaces, one is tempted to fill in the punctures and find unbranched holomorphic covers of the compactified surfaces which are quasiconformal with small dilatation. However, the problem is that the quasiconformal map does not necessarily send the lifts of the punctures on one surface to the lifts of the punctures on the other surface.

We considered the Teichmüller metric on $T(\mathcal{S}_G)$ and the existence of geodesics between points. If a map is of Teichmüller-type then we showed that it is extremal and that there is a unique geodesic between the point determined by the Teichmüller-type map and the base point of $T(\mathcal{S}_G)$. Moreover, we showed that only a small subset of $T(\mathcal{S}_G)$ has Teichmüller-type representative. We ask

3. Is it true that any point in the Teichmüller space $T(\mathcal{S}_G)$ has an (unique) extremal representative?

If the answer is positive, then any two points are connected by a (unique) geodesic. Even if the answer is negative, it is still possible to have geodesics connecting a point in $T(\mathcal{S}_G)$ without an extremal representative to the base point.

4. Is it possible to connect any two points in $T(\mathcal{S}_G)$ by a (unique) geodesic?

We also established a sufficient condition for a point in $T(\mathcal{S}_G)$ to have a Teichmüller-type representative. The condition is given in terms of the approximating sequence of TLC structures. We ask for additional sufficient conditions.

5. Is there a sufficient condition for a point $[f] \in T(\mathcal{S}_G)$ expressed only in terms of the geometry of the point $[f]$ to have a Teichmüller-type representative?

A classical statement about duality of the cotangent and tangent space for Teichmüller spaces of Riemann surfaces is false for $T(\mathcal{S}_G)$. We therefore ask

6. Does the tangent space $L^\infty_s(\mathcal{S}_G)/N(\mathcal{S}_G)$ at the basepoint of $T(\mathcal{S}_G)$ have a pre-dual?

It is a classical fact that any biholomorphic map of the Teichmüller space of a finite Riemann surface is given by the geometric action of an element of the extended mapping class group. This is recently proved for all infinite Riemann surfaces as well [26] (see also [16], [40], [11], [10], [22]). We ask analogous question of $T(\mathcal{S}_G)$.

7. Does every biholomorphism (isometry) of $T(\mathcal{S}_G)$ *arise by the geometric action of the full mapping class group* $\mathrm{Mod}_{full}(\mathcal{S}_G)$ *(i.e. homotopy classes of self maps of* $\mathcal{S}_G$ *not necessarily fixing the baseleaf and allowing orientation reversing elements)?*

We considered the Modular group $\mathrm{Mod}(\mathcal{S}_{nc})$ of the noncompact solenoid $\mathcal{S}_{nc}$ and found an explicit set of generators and a presentation. We ask the analogous question for the compact solenoid.

8. Find an explicit set of generators of $\mathrm{Mod}(\mathcal{S}_G)$ *of the compact solenoid* $\mathcal{S}_G$*.*

9. Find a presentation of $\mathrm{Mod}(\mathcal{S}_G)$*.*

We expect that these modular groups are infinitely generated.

10. Show that $\mathrm{Mod}(\mathcal{S}_G)$ *and* $\mathrm{Mod}(\mathcal{S}_{nc})$ *are infinitely generated.*

Recall that an element of $\mathrm{Mod}(\mathcal{S}_G)$ is called a mapping class like if it is a lift of a self map of a closed surface. C. Odden [37] asked the following question:

11. Is it true that $\mathrm{Mod}(\mathcal{S}_G)$ *and* $\mathrm{Mod}(\mathcal{S}_{nc})$ *are generated by mapping class like elements?*

Study properties of $\mathrm{Mod}(\mathcal{S}_G)$. In particular,

12. Is there a classification of the elements of $\mathrm{Mod}(\mathcal{S}_G)$ *according to their actions on* $\mathcal{S}_G$ *similar to the Thurston's classification of the mapping class group elements of a closed surface?*

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