CIRCLE HOMEOMORPHISMS AND SHEARS

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Abstract. We give parameterizations of homeomorphisms, quasisymmetric maps and symmetric maps of the unit circle in terms of shear coordinates for the Farey tesselation.

1. Introduction

The space $Homeo(S^1)$ of orientation preserving homeomorphisms of the unit circle S^1 is a classical topological group which is of interest in various fields of mathematics [13] and in the bosonic string theory in physics [21], [15], [16]. An important subgroup $QS(S^1)$ of quasisymmetric maps of S^1 plays a fundamental role in the Teichmüller theory of Riemann surfaces [2], [3], [9]. In fact, the *universal Teichmüller space* consists of all quasisymmetric maps which fix three distinguished points on S^1 namely it is isomorphic to $Möb(S^1)\backslash QS(S^1)$, where $Möb(S^1)$ is the group of (orientation preserving) Möbius maps which preserve S^1 [3]. The subgroup of symmetric maps $Sym(S^1)$ plays a prominent role in studying Teichmüller spaces of real dynamical systems [12], [6], [10].

The main results in this article are explicit parametrizations of the spaces $Möb(S^1)\backslash Homeo(S^1)$, $Möb(S^1)\backslash QS(S^1)$ and $Möb(S^1)\backslash Sym(S^1)$ in terms of shear coordinates for the Farey tesselation of the hyperbolic plane $\mathbf{H}$. To our best knowledge these are the only known explicit parametrizations of the above coadjoint orbit spaces. The unit circle S^1 is the boundary at infinity of $\mathbf{H}$.

The *shear* of the pair (Δ, Δ_1) of ideal hyperbolic triangles with disjoint interiors and a common boundary edge e is the signed hyperbolic distance between the orthogonal projections of the third vertices of Δ, Δ_1 onto e (see [19], [4], [17], or Section 3). The *Farey tesselation* $\mathcal{F}$ is a locally finite ideal geodesic triangulation of $\mathbf{H}$ which is preserved by the hyperbolic reflections in edges of $\mathcal{F}$ (see, for example, [16]). The set of edges of $\mathcal{F}$ is naturally partitioned into Farey generations (see [16] or Section 3). The shear of each pair of adjacent complementary triangles of $\mathcal{F}$ is zero.

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A homeomorphism $h : S^1 \to S^1$ induces a map $s_h : \mathcal{F} \to \mathbf{R}$, called the *shear map*, from the Farey tesselation $\mathcal{F}$ to the set $\mathbf{R}$ of real numbers as follows. Each $e \in \mathcal{F}$ is the common boundary side of a pair (Δ, Δ_1) of complementary triangles of $\mathcal{F}$. We define $s_h(e)$ to be the shear of the image pair $(h(\Delta), h(\Delta_1))$. It was known to be a challenging problem to characterize which maps $s : \mathcal{F} \to \mathbf{R}$ arise from homeomorphisms and which arise from quasisymmetric maps of S^1. We answer these questions below. (For a punctured surfaces S', the Teichmüller space $T(S')$ is parameterized using shears by Thurston [19] and Penner [17]; in the case of a closed surfaces S, Thurston [20] and Bonahon [4] gave a parameterization of $T(S)$ using shears on locally infinite tesselations.)

A *fan of geodesics* in $\mathcal{F}$ with *tip* $p \in S^1$ consists of all edges of $\mathcal{F}$ which have one endpoint p. Each fan in $\mathcal{F}$ has a natural ordering as follows. Fix a horocycle C with center at p whose orientation is such that the corresponding horoball is to the left of C. If e, e' are two geodesics with a common endpoint p, then we define $e < e'$ if point $e \cap C$ comes before point $e' \cap C$ on C, otherwise $e' < e$. The natural ordering on a fan induces a bijective correspondence of the geodesics of the fan with the integers $\mathbf{Z}$, and any two such correspondences differ by a translation in $\mathbf{Z}$. For each fan of $\mathcal{F}$ we fix one such correspondence.

Theorem A. *A shear map $s : \mathcal{F} \to \mathbf{R}$ is induced by a quasisymmetric map of S^1 if and only if there exists $M \geq 1$ such that for each fan of geodesics $e_n \in \mathcal{F}$, $n \in \mathbf{Z}$, and for all $m, k \in \mathbf{Z}$, we have*

$$\frac{1}{M} \leq \frac{e^{\frac{s_m}{2}} + e^{\frac{s_m}{2}+s_{m+1}} + \cdots + e^{\frac{s_m}{2}+s_{m+1}+\cdots+s_{m+k}}}{e^{-\frac{s_m}{2}} + e^{-\frac{s_m}{2}-s_{m-1}} + \cdots + e^{-\frac{s_m}{2}-s_{m-1}-\cdots-s_{m-k}}} \leq M,$$

where $s_n = s(e_n)$.
Moreover, $s : \mathcal{F} \to \mathbf{R}$ is induced by a symmetric map of S^1 if and only if

$$\frac{e^{\frac{s_m}{2}} + e^{\frac{s_m}{2}+s_{m+1}} + \cdots + e^{\frac{s_m}{2}+s_{m+1}+\cdots+s_{m+k}}}{e^{-\frac{s_m}{2}} + e^{-\frac{s_m}{2}-s_{m-1}} + \cdots + e^{-\frac{s_m}{2}-s_{m-1}-\cdots-s_{m-k}}} \rightrightarrows 1$$

as the Farey generations of e_{m-k} and e_{m+k} go to infinity.

For a fan of $\mathcal{F}$ with tip p, we define

$$s(p; m, k) = \frac{e^{\frac{s_m}{2}} + e^{\frac{s_m}{2}+s_{m+1}} + \cdots + e^{\frac{s_m}{2}+s_{m+1}+\cdots+s_{m+k}}}{e^{-\frac{s_m}{2}} + e^{-\frac{s_m}{2}-s_{m-1}} + \cdots + e^{-\frac{s_m}{2}-s_{m-1}-\cdots-s_{m-k}}}$$

for $m, k \in \mathbf{Z}$. Let C be a horoball with center at $h(p)$ where h is a quasisymmetric map which induces s. Then $s(p; m, k)$ represents the ratio of the length of the horocyclic arc on C between $h(e_{m+k})$ and

$h(e_m)$ to the length of the horocyclic arc on C between $h(e_{m-k})$ and $h(e_m)$. Define

$$M_s(p) = \sup_{m,k\in\mathbf{Z}} s(p;m,k).$$

If $M_s(p) < \infty$ then we say that s satisfies $M_s(p)$-*condition at the fan with tip* p. The above theorem states that a shear map $s : \mathcal{F} \to \mathbf{R}$ induces a quasisymmetric map if and only if

$$M_s = \sup_p M_s(p) < \infty \tag{1}$$

where the supremum is over all fans of $\mathcal{F}$.

It is quite surprising that the characterization of shears which give rise to quasisymmetric maps is so simple. The $M_s(p)$-condition is localized in a single fan of geodesics with tip p and the only additional information is that single $M_s = \sup_p M_s(p)$ works for all fans simultaneously. In particular, there is no information as how shears on close by geodesics not belonging to a single fan relate to each other.

We now interpret the Teichmüller topology of the universal Teichmüller space $M\ddot{o}b(S^1)\backslash QS(S^1)$ within the framework of Theorem A. That theorem parametrizes $M\ddot{o}b(S^1)\backslash QS(S^1)$ by the space $\mathcal{X}$ of all shear maps $s : \mathcal{F} \to \mathbf{R}$ which satisfy (1). We use $s(p;m,k)$ to introduce a natural topology on $\mathcal{X}$ such that the parametrization of $M\ddot{o}b(S^1)\backslash QS(S^1)$ by $\mathcal{X}$ is a homeomorphism. For $s_1, s_2 \in \mathcal{X}$ define

$$M_{s_1,s_2}(p) = \sup_{m,k}\Big(\max\Big\{\frac{s_1(p;m,k)}{s_2(p;m,k)}, \frac{s_2(p;m,k)}{s_1(p;m,k)}\Big\}\Big).$$

Theorem B. *Let $h_n, h \in M\ddot{o}b(S^1)\backslash QS(S^1)$. Then $h_n \to h$ as $n \to \infty$ in the Teichmüller topology if and only if $M_{s,s_n} = \sup_p M_{s,s_n}(p) \to 1$ as $n \to \infty$.*

Surprisingly enough the characterization of homeomorphisms involves more information then the parametrization of quasisymmetric homeomorphisms given by Theorem A. A *chain of geodesics* in $\mathcal{F}$ is a sequence $e_n \in \mathcal{F}$ of distinct edges such that e_n and e_{n+1} share a common endpoint for all $n \in \mathbf{N}$.

Theorem C. *A shear map $s : \mathcal{F} \to \mathbf{R}$ is induced by a homeomorphism of S^1 if and only if for each chain $e_n \in \mathcal{F}$, $n \in \mathbf{N}$, we have*

$$\sum_{n=1}^{\infty} e^{s_1^n + s_2^n + \cdots + s_n^n} = \infty$$

where $s_i^n = \pm s(e_i)$. *More precisely if* $e_n < e_{n+1}$ *then* $s_n^n = s(e_n)$; *otherwise* $s_n^n = -s(e_n)$. *For* $n > 1$ *and* $i < n$, $s_i^n = s(e_i)$ *if either* $e_i < e_{i+1}$ *and the number of times we change fans from* e_i *to* e_{n+1} *is even, or* $e_i > e_{i+1}$ *and the number of times we change fans is odd; otherwise* $s_i^n = -s(e_i)$.

A locally finite ideal triangulation of $\mathbf{H}$ with a distinguished oriented edge is called a *tesselation*. The space of all tesselations is isomorphic to the space $Homeo(S^1)$ by assigning to a tesselation τ a homeomorphism of S^1 (called the *characteristic map*) which maps the Farey tesselation $\mathcal{F}$ to the tesselation τ of $\mathbf{H}$ such that a distinguished oriented edge of $\mathcal{F}$ is mapped onto the distinguished oriented edge of τ (see Penner [16]). A *decorated tesselation* is a tesselation together with an arbitrary assignment of a horocycle at each vertex of the tesselation (see [16]).

Let C_1 and C_2 be two horocycles with different centers and let g be the geodesic whose endpoints are at the centers of C_1 and C_2. Then the *lambda length* $\lambda(g)$ of g is defined by

$$\lambda(g) = e^{-2\delta(C_1,C_2)}$$

where $\delta(C_1, C_2)$ is the signed hyperbolic distance between $G_1 = g \cap C_1$ and $G_2 = g \cap C_2$. The sign of $\delta(C_1, C_2)$ is positive if the geodesic arc between G_1 and G_2 is outside C_1, otherwise the sign is negative. Let g, g_1 be a wedge of geodesics in $\mathbf{H}$ and let C be a horocycle with center at the common endpoint of g and g_1. The *horocyclic length* $\alpha(g, g_1)$ of the wedge g, g_1 is the length of the arc of C between g and g_1. A decorated tesselation $\tilde{\tau}$ determines an assignment of lambda lengths to the edges of τ and of horocyclic lengths to the wedges of τ. This in turn defines an assignment of lambda lengths to the edges of the Farey tesselation $\mathcal{F}$ by the pull-back with the characteristic map as well as the assignment of horocyclic lengths to the wedges of $\mathcal{F}$ (see Penner [16], [17]).

Two decorated tesselations $\tilde{\tau}_1$ and $\tilde{\tau}_2$ induce the same lambda lengths on $\mathcal{F}$ if and only if $\tilde{\tau}_1$ is the image under an element of $M\ddot{o}b(S^1)$ of $\tilde{\tau}_2$. It is clear that not every assignment of lambda lengths on the Farey tesselation $\mathcal{F}$ will give a decorated tesselation such that the characteristic map is a homeomorphism of S^1. In fact the underlying tesselation is not in general an ideal triangulation of $\mathbf{H}$. Penner [16] posed the problem of determining which lambda lengths will give characteristic maps that are homeomorphisms or quasisymmetric maps of S^1. Penner and Sullivan [16, Theorem 6.4] showed that if lambda lengths are "pinched" namely if there is $K \geq 1$ such that $1/K \leq \lambda(e) \leq K$ for all $e \in \mathcal{F}$ then the characteristic map is quasisymmetric. We find necessary and sufficient conditions on the lambda lengths such that the characteristic maps are homeomorphisms, quasisymmetric or symmetric maps of S^1.

Theorem D. *A lambda length function* $\lambda : \mathcal{F} \to \mathbf{R}^+$ *induces a homeomorphism of* S^1 *if and only if for each chain of edges* $e_n \in \mathcal{F}$, $n \in \mathbf{N}$, *we have*

$$\sum_{n=1}^{\infty} \left(\lambda_n^{-\frac{1}{2}} \lambda_{n-1}^{\frac{1}{2}} \cdots \lambda_1^{\frac{(-1)^n}{2}} \right) \alpha_n = \infty$$

where $\lambda_i = \lambda(e_i)$ *and* α_n *is the horocyclic length of the wedge bounded by* e_n *and* e_{n+1}.

In the above theorem we used horocyclic length α_n together with the lambda lengths. We note that horocyclic lengths are expressed as rational functions of lambda lengths (see Penner [17], [16, Section 6]). Indeed, if g_1, g_2, g_3 are edges of an ideal triangle with decorations then by [17] we have

$$\alpha(g_1, g_2) = \frac{2\lambda(g_3)}{\lambda(g_1)\lambda(g_2)}.$$

Thus the series in the above theorem is completely determined in terms of lambda lengths.

The following theorem gives necessary and sufficient conditions on horocyclic lengths such that the characteristic maps are quasisymmetric and symmetric. We note that it is possible to express the same condition in terms of lambda lengths using the formula above.

Theorem E. *A lambda length function* $\lambda : \mathcal{F} \to \mathbf{R}$ *induces a quasisymmetric map of* S^1 *if and only if there exists* $K \geq 1$ *such that for each fan* $e_n \in \mathcal{F}$, $n \in \mathbf{Z}$, *and for all* $m \in \mathbf{Z}$ *and* $k \in \mathbf{N}$ *we have*

$$\frac{1}{K} \leq \frac{\alpha(e_m, e_{m+1}) + \alpha(e_{m+1}, e_{m+2}) + \cdots + \alpha(e_{m+k}, e_{m+k+1})}{\alpha(e_m, e_{m-1}) + \alpha(e_{m-1}, e_{m-2}) + \cdots + \alpha(e_{m-k}, e_{m-k-1})} \leq K.$$

Moreover, $\lambda : \mathcal{F} \to \mathbf{R}$ *induces a symmetric map of* S^1 *if and only if*

$$\frac{\alpha(e_m, e_{m+1}) + \alpha(e_{m+1}, e_{m+2}) + \cdots + \alpha(e_{m+k}, e_{m+k+1})}{\alpha(e_m, e_{m-1}) + \alpha(e_{m-1}, e_{m-2}) + \cdots + \alpha(e_{m-k}, e_{m-k-1})} \to 1$$

as the Farey generations of e_{m+k} *and* e_{m-k} *go to infinity independently of the fan.*

2. Quasisymmetric maps and barycentric extension

In the rest of the paper the hyperbolic plane is identified with the upper half-plane model $\mathbf{H} := \{z = x + iy |\ y > 0\}$ endowed with the metric $\rho(z) = \frac{|dz|}{y}$. The boundary at infinity $\partial_\infty \mathbf{H} = \hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is naturally identified with the unit circle S^1. Any two identifications of $\hat{\mathbf{R}}$ and S^1 differ by the postcomposition by a Möbius map of S^1. We choose 0, 1 and ∞ to be the three distinguished points on $\hat{\mathbf{R}}$.

Let $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ be a homeomorphism that fixes ∞ and let $M \geq 1$. Then $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ is said to be M-*quasisymmetric* if

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M$$

for all $x \in \mathbf{R}$ and $t > 0$ (see [2]).

The *universal Teichmüller space* $T(\mathbf{H})$ is the set of all quasisymmetric maps of $\hat{\mathbf{R}}$ that fix 0, 1 and ∞. A sequence $h_n \in T(\mathbf{H})$ converges to the basepoint $id \in T(\mathbf{H})$ in the *Teichmüller topology* if h_n are M_n-quasisymmetric with $M_n \to 1$ as $n \to \infty$. A sequence $h_n \in T(\mathbf{H})$ converges to $h \in T(\mathbf{H})$ in the *Teichmüller topology* if $h_n \circ h^{-1} \to id$ as $n \to \infty$ in the above sense.

A quasisymmetric map $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ extends to a quasiconformal map $f : \mathbf{H} \to \mathbf{H}$, and conversely a quasiconformal map $f : \mathbf{H} \to \mathbf{H}$ extends by continuity to a quasisymmetric map $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ (see [2]). The extension of $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ to a quasiconformal map of $\mathbf{H}$ is not unique. Douady and Earle defined a particularly nice extension operator from quasisymmetric maps of $\hat{\mathbf{R}}$ into quasiconformal maps of $\mathbf{H}$ called the *barycentric extension* (see [5]).

For a homeomorphism $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$, denote by $ex(h) : \mathbf{H} \to \mathbf{H}$ its barycentric extension introduced in [5]. We recall several properties of $ex(h)$ that are obtained by Douady and Earle [5]. The barycentric extension $ex(h)$ is a real-analytic diffeomorphism of $\mathbf{H}$ which is quasiconformal if and only if h is quasisymmetric. Moreover, the extension is conformaly natural in the sense that $ex(A \circ h \circ B) = A \circ ex(h) \circ B$ for all $A, B \in PSL_2(\mathbf{R})$ and for all homeomorphisms $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$. In addition, if $h_n \to h$ as $n \to \infty$ pointwise on $\hat{\mathbf{R}}$ then $ex(h_n) \to ex(h)$ as $n \to \infty$ in the C^∞-topology on C^∞ maps of $\mathbf{H}$. In particular, Beltrami coefficients $\mu(ex(h_n))$ of $ex(h_n)$ converge uniformly on compact subsets of $\mathbf{H}$ to the Beltrami coefficient $\mu(ex(h))$ of $ex(h)$.

Remark 2.1. For our purposes the barycentric extension serves quite well. Kahn and Markovic [11] constructed another quasiconformal extension in the case when the quasisymmetric maps are invariant under co-finite Fuchsian group in order to be able to estimate the norm of the corresponding Beltrami coefficient.

The following lemma is obtained by Markovic [14] (see also Douady-Earle [5] and Abikoff-Earle-Mitra [1]).

Lemma 2.2. *Let* $h_n : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ *be a sequence of homeomorphisms which fix* 0, 1 *and* ∞, *and let* μ_n *be Beltrami coefficients of the barycentric extensions* $ex(h_n)$ *of* h_n. *If there exists* $c_0 \geq 1$ *such that*

$$-c_0 \leq h_n(-1) \leq -\frac{1}{c_0}$$

then there exists a neighborhood U *of the imaginary unit* $i \in \mathbf{H}$ *and a constant* $0 < c < 1$ *such that*

$$\|\mu_n|_U\|_\infty \leq c < 1$$

for all n.

Proof. We note that the angle distance with respect to $i \in \mathbf{H}$ between all pairs of consecutive points in $\{\infty, -1, 0, 1\} \subset \hat{\mathbf{R}}$ is bounded below by a constant less than π and bounded above by π. Then [14, Lemma 3.6] directly implies the desired conclusion. □

3. The Farey tesselation and the shear map

Let Δ_0 be an ideal geodesic triangle in $\mathbf{H}$ with vertices 0, 1 and ∞. Let Γ be the group generated by hyperbolic reflections in the sides of Δ_0. The Farey tesselation $\mathcal{F}$ is an ideal triangulation of $\mathbf{H}$ which is the Γ-orbit of the boundary sides of Δ_0. In other words, each edge in $\mathcal{F}$ is obtained by applying finitely many inversions in the sides Δ_0 to an edge of Δ_0 (see, for example, [16]). The set of endpoints of the edges in $\mathcal{F}$ is $\hat{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\}$.

We define *Farey generation* of edges of $\mathcal{F}$ as follows. A boundary edge of Δ_0 has Farey generation 0. If a boundary edge of $\mathcal{F}$ is obtained by n reflections of an edge of generation 0 (where n is the smallest such number) then its Farey generation is n.

Let (Δ_1, Δ_2) be a pair of ideal triangles in $\mathbf{H}$ with disjoint interiors and a common boundary side. Let $A \in PSL_2(\mathbf{R})$ be the unique Möbius map that sends Δ_1 onto the triangle with vertices -1, 0 and ∞, and that sends the common boundary side of (Δ_1, Δ_2) onto the geodesic with vertices 0 and ∞. Then $A(\Delta_2)$ has vertices 0, e^r and ∞ for some $r \in \mathbf{R}$. The *shear* of the pair of triangles (Δ_1, Δ_2) is by definition equal to r. Alternatively, the shear of a pair (Δ_1, Δ_2) of adjacent triangles is the signed distance of the projections onto common boundary side e of vertices of Δ_1 and Δ_2 opposite e, where e is oriented to the left as seen from Δ_1. Note that the shear of (Δ_1, Δ_2) is equal to the shear of

(Δ_2, Δ_1). For example, any two adjacent triangles in the complement of the Farey tesselation $\mathcal{F}$ have shear 0.

Let $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ be a homeomorphism. Every geodesic of $\mathbf{H}$ has exactly two distinct ideal endpoints on $\hat{\mathbf{R}}$ and, conversely every two points on $\hat{\mathbf{R}}$ determine a geodesic in $\mathbf{H}$. Thus, the space $\mathcal{G}$ of (oriented) geodesics in $\mathbf{H}$ is identified with the set of pairs of distinct points in $\hat{\mathbf{R}}$. Therefore, the homeomorphism $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ extends to a homeomorphism $h : \mathcal{G} \to \mathcal{G}$ of the space of geodesics $\mathcal{G}$. In particular, $h(\mathcal{F})$ is an ideal triangulation of $\mathbf{H}$ whose complementary triangles are $h(\Gamma(\Delta_0))$.

Definition 3.1. Let $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ be a homeomorphism. An edge $e \in \mathcal{F}$ is on the boundary of exactly two complementary triangles Δ_1, Δ_2. Then we assign to $e \in \mathcal{F}$ the shear of the pair $(h(\Delta_1), h(\Delta_2))$ of triangles in $h(\Gamma(\Delta_0))$. This determines a map

$$s_h : \mathcal{F} \to \mathbf{R}$$

which is called the *shear map* of h.

If we are given a shear between two adjacent triangles and the position of one of the triangles, the other triangle is uniquely determined. More generally, a pair of adjacent triangles with an assigned shear is determined up to a Möbius map because any ideal hyperbolic triangle can be mapped onto any other ideal hyperbolic triangle by a Möbius map.

If $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ fixes 0, 1 and ∞, then it is uniquely determined by the shear map $s_h : \mathcal{F} \to \mathbf{R}$. Given a shear map $s : \mathcal{F} \to \mathbf{R}$ there exists a unique injective map h_s from the vertices $\hat{\mathbf{Q}} \subset \hat{\mathbf{R}}$ of the Farey tesselation $\mathcal{F}$ into $\hat{\mathbf{R}}$ such that h_s fixes 0, 1 and ∞. The map h_s realizes the shear map s and it is called a *characteristic map* of s (see [16] or next section for its definition).

4. Homeomorphisms and shears

We characterize shear maps $s : \mathcal{F} \to \mathbf{R}$ whose characteristic maps continuously extend to homeomorphisms of $\hat{\mathbf{R}}$. An arbitrary map $s : \mathcal{F} \to \mathbf{R}$ induces a cocycle map $H_s : \mathbf{H} \to \mathbf{H}$ which is piecewise Möbius as follows. Let $H_s|_{\Delta_0} = id$. For any other complementary triangle $\Delta \in \Gamma(\Delta_0)$, let l be the geodesic arc connecting the center of Δ_0 to the center of Δ. Let $\{e_1, e_2, \dots, e_n\}$ be the edges in $\mathcal{F}$ which intersect l in the given order such that e_1 is a boundary side of Δ_0 and e_n is a boundary side of Δ. We orient e_i to the left as seen from Δ_0. Then we set $H_s|_{\Delta} =$

$T_{e_1}^{s(e_1)}\circ T_{e_2}^{s(e_2)}\circ\cdots\circ T_{e_n}^{s(e_n)}$, where $T_{e_i}^{s(e_i)}$ is the hyperbolic translation with the oriented axis e_i and the signed translation length $s(e_i)$. The map H_s is not well-defined on the edges $\mathcal{F}$ since each edge e is on the boundary of exactly two complementary triangles Δ_e^1 and Δ_e^2. We choose $H_s|_e$ to be either $H_s|_{\Delta_e^1}$ or $H_s|_{\Delta_e^2}$. The cocycle map H_s preserves separation properties of the triples of complementary triangles of $\mathcal{F}$. Therefore, H_s extends to a monotone map $h_s:\hat{\mathbf{Q}}\to\hat{\mathbf{R}}$ which is called *characteristic map* of $s:\mathcal{F}\to\mathbf{R}$ (see Penner [16]).

Proposition 4.1. *With the above notation, the characteristic map $h_s:\hat{\mathbf{Q}}\to\hat{\mathbf{R}}$ extends by continuity to a homeomorphism of $\hat{\mathbf{R}}$ if and only if $H_s:\mathbf{H}\to\mathbf{H}$ is surjective.*

Proof. Since $h_s:\hat{\mathbf{Q}}\to\hat{\mathbf{R}}$ is order preserving on the dense subset $\hat{\mathbf{Q}}$ of $\hat{\mathbf{R}}\equiv S^1$, it follows that if h_s can be extended to a continuous map on $\hat{\mathbf{R}}$ then the extension is a homeomorphism.

If $H_s:\mathbf{H}\to\mathbf{H}$ is not onto, then there exists a maximal half-plane P not contained in $H_s(\mathbf{H})$. It follows that the image $h_s(\hat{\mathbf{Q}})$ does not intersect the interior of the interval on $\hat{\mathbf{R}}$ which is the boundary at infinity of P. Therefore, the map $h_s:\hat{\mathbf{Q}}\to\hat{\mathbf{R}}$ cannot be extended to a homeomorphism of $\hat{\mathbf{R}}$.

Assume that $H_s:\mathbf{H}\to\mathbf{H}$ is onto. Let $x\in\hat{\mathbf{R}}\setminus\hat{\mathbf{Q}}$. We need to show that h_s extends to x. Let P_i be a decreasing sequence of half-planes with boundary sides $e_i\in\mathcal{F}$ that accumulate at x, namely $\bigcap_i\overline{P_i}=x$. Since H_s is order preserving on triples of complementary triangles of $\mathcal{F}$, it follows that $H_s(P_i)$ is a decreasing sequence of half-planes. If $\bigcap_i H_s(P_i)\neq\emptyset$ then $H_s(\mathbf{H})\neq\mathbf{H}$, namely $H_s(\mathbf{H})\cap(\bigcap_i H_s(P_i))=\emptyset$. Thus $\bigcap_i H_s(P_i)=\emptyset$ and $\bigcap_i\overline{H_s(P_i)}$ is a single point $y\in\hat{\mathbf{R}}$. Then h_s extends to x by continuity such that $h_s(x)=y$. □

Proof of Theorem C. Using the above proposition we determine which shear maps induce homeomorphisms of $\hat{\mathbf{R}}$. Assume that $H_s:\mathbf{H}\to\mathbf{H}$ is not onto. Then there exists a maximal half-plane P of $\mathbf{H}$ which is not in the image of H_s. Let l be the boundary geodesic of the half-plane P. Then there exists a chain $e_n\in\mathcal{F}$ such that $H_s(e_n)\to l$ as $n\to\infty$. There are two possibilities for the sequence e_n. Either all e_n's share a common endpoint $x\in\hat{\mathbf{Q}}\subset\hat{\mathbf{R}}$ for $n\geq n_0$ namely the sub-chain e_n, for $n\geq n_0$, is a part of a single fan, or e_n's accumulate to a point $x\in\hat{\mathbf{R}}\setminus\hat{\mathbf{Q}}$ (which is equivalent to saying that no infinite subsequence of e_n's shares a common endpoint i.e. no tail of e_n's is a part of a single fan). In both cases the existence of the half-plane P is equivalent to the statement that h_s does not extend to a continuous map at $x \in \hat{\mathbf{R}}$.

Assume that we are in the first case. By pre-composition with an element of $PSL_2(\mathbf{Z})$, we can assume that $x = \infty$. In addition, we can assume that H_s fixes 0, 1 and ∞ by post-composing with an element of $PSL_2(\mathbf{R})$. Then l has one endpoint $x = \infty$ and the other endpoint $\bar{y} \in \mathbf{R}$ with either $\bar{y} > 1$ or $\bar{y} < 0$. If $\bar{y} > 1$, then

$$\bar{y} = 1 + \sum_{n=1}^{\infty} e^{s(e_1)+\cdots+s(e_n)}, \tag{2}$$

where $e_i \in \mathcal{F}$ is a geodesic with endpoints i and ∞ for $i \in \mathbf{N}$. If $\bar{y} < 0$ then

$$\bar{y} = -\sum_{n=0}^{-\infty} e^{-s(e_0)-\cdots-s(e_n)}, \tag{3}$$

where $e_i \in \mathcal{F}$ is a geodesic with endpoints i and ∞ for $i \in \mathbf{Z}^- \cup \{0\}$. Since e_i's belong to a single fan, the number of times we change fans from e_i to e_{n+1} is zero. Thus $s_i^n = s(e_i)$ for $i > 0$ and $s_i^n = -s(e_i)$ for $i \leq 0$. Therefore, h_s is continuous at $x \in \hat{\mathbf{R}}$ if and only if the series in (2) and the series in (3) diverge.

Assume now that we are in the second case. Namely, the chain e_n does not have a subsequence which shares a common endpoint and e_n's accumulate at $x \in \hat{\mathbf{R}} \setminus \hat{\mathbf{Q}}$. In other words, no tail of e_n's is in a single fan. The part of $\mathbf{H}$ bounded by e_n and e_{n+1} is called a *hyperbolic wedge*.

Given a hyperbolic wedge, there is a unique foliation of the wedge by horocyclic arcs which lie on horocycles with centers at the common endpoint of the two boundary geodesics of the wedge. Consider the wedges whose boundaries are the adjacent geodesics in the chain $h_s(e_n)$ and foliate each wedge by horocyclic arcs as above. Fix a point $P_1 \in h_s(e_1)$ and denote by $l(P_1)$ the leaf of the horocyclic foliation of the union of wedges that starts at P_1. Let W_n be the hyperbolic wedge bounded by $h_s(e_n)$ and $h_s(e_{n+1})$. We choose P_1 such that the length of $l(P_1) \cap W_1$ is $e^{s_1^1}$, where $s_1^1 = s(e_1)$ if $e_1 < e_2$, otherwise $s_1^1 = -s(e_1)$ (see Figure 1).

Proposition 4.2. *Under the above notation, the map h_s continuously extends to $x \in \hat{\mathbf{R}} \setminus \hat{\mathbf{Q}}$ if and only if the leaf $l(P_1)$ is of infinite length.*

Proof. Note that h_s extends by continuity to $x \in \hat{\mathbf{R}}$ if and only if $h_s(e_n)$ do not accumulate in $\mathbf{H}$.

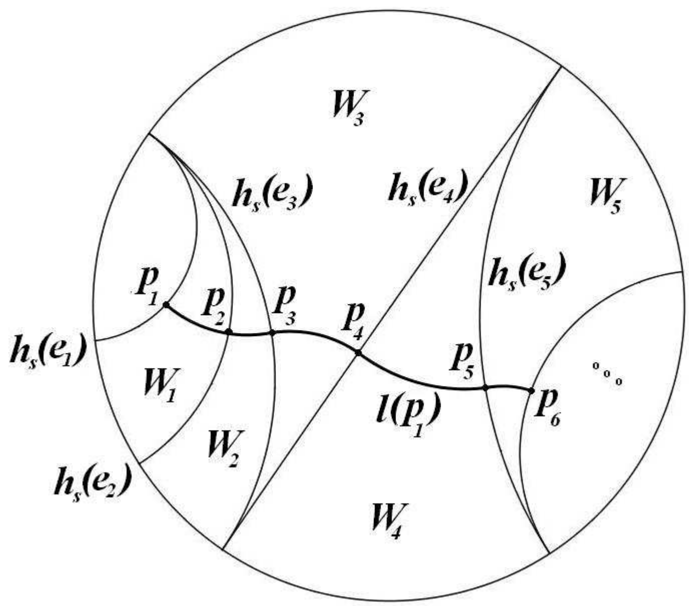

FIGURE 1. The leaf $l(P_1)$ of the foliation of $\cup_n W_n$ by horocycles.

Assume that h_s extends continuously to $x \in \hat{\mathbf{R}}$. Then $h_s(e_n)$ do not accumulate in $\mathbf{H}$. Therefore, the arc $l(P_1)$ accumulates at $\partial\mathbf{H}$ and it is necessarily of infinite length.

It remains to show that if $l(P_1)$ is of infinite length then h_s extends to $x \in \hat{\mathbf{R}}$ by continuity. Assume on the contrary that h_s does not extend to $x \in \hat{\mathbf{R}}$. This implies that $h_s(e_n)$ accumulate at a geodesic $g \subset \mathbf{H}$. We need to show that $l(P_1)$ has finite length.

Let a be the geodesic arc which connects $h_s(e_1)$ with g and that is orthogonal to both $h_s(e_1)$ and g. All the geodesics of the sequence $h_s(e_n)$ for $n \geq 2$ lie between $h_s(e_1)$ and g, and they intersect a. The angle of the intersection between $h_s(e_n)$ and a is necessarily bounded away from 0. We show that the length of $l(P_1)$ is comparable to the length of a which finishes the proof.

Consider a hyperbolic wedge W_n bounded with $h_s(e_n)$ and $h_s(e_{n+1})$. Let $a_n = a \cap W_n$, and let $P_n' = a \cap h_s(e_n)$. Then P_n' and P_{n+1}' are the endpoints of a_n. Let $P_n = l(P_1) \cap h_s(e_n)$ and let P_n'' be the endpoint of the horocyclic arc in the wedge W_{n-1} whose initial point is P_{n-1}' (see Figure 2). Let d_n be the geodesic arc with endpoints P_n and P_n', and

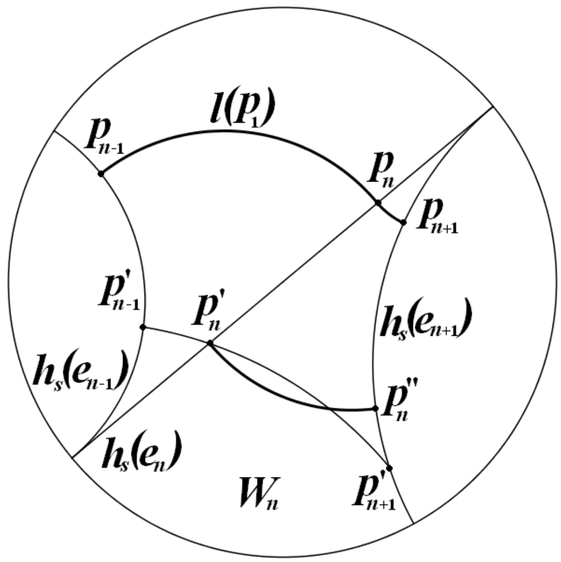

FIGURE 2. The points P_n, P'_n and P''_n.

let d'_n be the geodesic arc with endpoints P'_n and P''_n. Consider the hyperbolic triangle with vertices P'_{n-1}, P'_n and P''_n. Since the angle at P''_n is bounded away from 0 (by the uniform bound on the length of each a_n), it follows from the hyperbolic sine formula that there exists $C > 0$ such that $|d'_n| \leq C \cdot |a_{n-1}|$, where $|d'_n|, |a_n|$ are the lengths of d'_n, a_n, respectively. In addition, $|d_n| \leq |d_{n-1}| + |d'_n|$ follows by the definition of $l(P_1)$.

The above two estimates show that $\sum_{n\in\mathbf{N}} |d_n| \leq |d_1| + C \sum_{n\in\mathbf{N}} |a_n| = C_1|a| < \infty$. This implies that $l(P_1)$ stays a bounded distance from a. Thus the length of $l(P_1) \cap W_n$ and the length a_n are comparable to a multiplicative constant. Therefore $l(P_1)$ has finite length. $\square$

We use the above proposition to find a condition on the shear map s such that h_s has continuous extension to x. We compute the length of the above leaf $l(P_1)$ in terms of the shear map $s : \mathcal{F} \to \mathbf{R}$. Let l_n be the length of the horocyclic arc $l(P_1) \cap W_n$ in the wedge W_n between $H_s(e_n)$ and $H_s(e_{n+1})$. If e_n, e_{n+1} and e_{n+2} share a common endpoint, then an elementary hyperbolic geometry and the definition of H_s show that the length of $l(P_1) \cap W_{n+1}$ in the wedge W_{n+1} bounded by $H_s(e_{n+1})$ and $H_s(e_{n+2})$ is $l_n e^{s(e_{n+1})}$ if $e_{n+1} < e_{n+2}$, and the length is $l_n e^{-s(e_{n+1})}$

if $e_{n+2} < e_{n+1}$. If e_n, e_{n+1} and e_{n+2} do not share a common endpoint, then the length of $l(P_1) \cap W_{n+1}$ in the wedge W_{n+1} between $H_s(e_{n+1})$ and $H_s(e_{n+2})$ is $l_n^{-1}e^{s(e_{n+1})}$ if $e_{n+1} < e_{n+2}$, and the length is $l_n^{-1}e^{-s(e_{n+1})}$ if $e_{n+1} > e_{n+2}$. We choose $P_1 \in H_s(e_1)$ such that $l_1 = e^{s_1^1}$.

We show that $l_n = e^{s_1^n+s_2^n+\cdots+s_n^n}$ by induction which finishes the proof. Note that the choice of $P_1 \in H_s(e_1)$ is such that $l_1 = e^{s_1^1}$. Assume that $l_n = e^{s_1^n+s_2^n+\cdots+s_n^n}$ and we need to show that $l_{n+1} = e^{s_1^{n+1}+s_2^{n+1}+\cdots+s_{n+1}^{n+1}}$. We consider four possibilities and argue each separately. Assume first that e_n, e_{n+1} and e_{n+2} share a common endpoint and that $e_{n+1} < e_{n+2}$. Then $l_{n+1} = l_n e^{s(e_{n+1})} = e^{s_1^n+\cdots+s_n^n+s_{n+1}^{n+1}}$. Since there is no additional change of fans from e_{n+1} to e_{n+2}, we have $s_i^n = s_i^{n+1}$ for $i = 1, 2, \ldots, n$. This proves the formula in this case. The second case is when e_n, e_{n+1} and e_{n+2} share a common endpoint and $e_{n+2} < e_{n+1}$. Then we have $l_{n+1} = l_n e^{-s(e_{n+1})} = l_n e^{s_{n+1}^{n+1}}$ by the definition of s_{n+1}^{n+1}. The desired formula follows as in the previous case. In the third case we assume that e_n, e_{n+1} and e_{n+2} do not share a common endpoint and that $e_{n+1} < e_{n+2}$. Then $l_{n+1} = l_n^{-1}e^{s(e_{n+1})} = e^{-s_1^n-\cdots-s_n^n+s_{n+1}^{n+1}}$. Since we have one additional change of fan from e_{n+1} to e_{n+2}, we get that $s_i^{n+1} = -s_i^n$ for $i = 1, 2, \ldots, n$. This proves the formula in the third case. Finally, we assume that e_n, e_{n+1} and e_{n+2} do not share a common endpoint and that $e_{n+2} < e_{n+1}$. Then $l_{n+1} = l_n^{-1}e^{-s(e_{n+1})} = e^{-s_1^n-\cdots-s_n^n+s_{n+1}^{n+1}}$. As in the previous case this gives the desired formula. Therefore the series $\sum_{n=1}^{\infty} e^{s_1^n+\cdots+s_n^n}$ is the length of $l(P_1)$ and the proof of Theorem C is completed. □

5. Quasisymmetric maps and shears

In this section we characterize shear maps which give rise to quasisymmetric maps of $\hat{\mathbf{R}}$. This is the main result of the paper and, to our best knowledge, it gives the only known parametrization of the universal Teichmüller space $T(\mathbf{H})$.

Proof of the first part of Theorem A. We prove that the first condition in the theorem is necessary for $s : \mathcal{F} \to \mathbf{R}$ to be a shear map of a quasisymmetric map $h : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$.

Consider a fan of $\mathcal{F}$ with tip $p \in \hat{\mathbf{Q}}$. Let $A \in PSL_2(\mathbf{Z})$ be such that $A(p) = \infty$. Let $B \in PSL_2(\mathbf{R})$ be such that $B(h(p)) = \infty$. Then $B \circ h \circ A^{-1}$ fixes ∞ and the corresponding shear map is $s \circ A^{-1}$. Moreover, if h is M_1-quasisymmetric then $B \circ h \circ A^{-1}$ is M-quasisymmetric, where M is a function of M_1 and is independent of A and B. Therefore, we can study properties of a shear map on a single fan of $\mathcal{F}$ with tip p by studying properties on the fan of $\mathcal{F}$ with tip ∞.

Consider an M-quasisymmetric map h of $\hat{\mathbf{R}}$ which fixes ∞ and let $s:\mathcal{F}\to\mathbf{R}$ be the induced shear map. Then h satisfies

$$\frac{1}{M}\leq\frac{h(n+k)-h(n)}{h(n)-h(n-k)}\leq M \tag{4}$$

for all $n\in\mathbf{Z}$ and all $k\in\mathbf{N}$. This is the M-quasisymmetric condition taken at special symmetric triples in $\mathbf{Z}\subset\mathbf{R}$. We can further normalize h by post-composing with an affine map such that it fixes n, $n+1$ and ∞. The values at $\mathbf{Z}$ of such a normalized h are uniquely determined by shears on the fan of $\mathcal{F}$ with tip ∞ by the definition of the characteristic map.

Let e_n be the geodesic with endpoints n and ∞, and let $s_n=s(e_n)$ for the convenience of notation. The condition (4) is equivalent to

$$\frac{1}{M}\leq\frac{1+e^{s_{n+1}}+\cdots+e^{s_{n+1}+s_{n+2}+\cdots+s_{n+k-1}}}{e^{-s_n}+e^{-s_n-s_{n-1}}+\cdots+e^{-s_n-s_{n-1}-\cdots-s_{n-k+1}}}\leq M. \tag{5}$$

The condition (5) is equivalent to the first condition in Theorem A and this establishes the necessity of the first condition in Theorem A.

We assume that a shear map $s:\mathcal{F}\to\mathbf{R}$ satisfies property (5) at each fan of $\mathcal{F}$ and show that characteristic map $h_s:\hat{\mathbf{Q}}\to\hat{\mathbf{R}}$ extends to a quasisymmetric map of $\hat{\mathbf{R}}$.

We first show that $h_s:\hat{\mathbf{Q}}\to\hat{\mathbf{R}}$ extends to a homeomorphism of $\hat{\mathbf{R}}$. Since h_s is a strictly monotone map of $\hat{\mathbf{Q}}$ into $\hat{\mathbf{R}}$, it is enough to show that $h_s(\hat{\mathbf{Q}})$ is dense in $\hat{\mathbf{R}}$. Assume on the contrary that $\hat{\mathbf{R}}\setminus h_s(\hat{\mathbf{Q}})$ contains an interval I. Assume that I is a maximal such interval and let l be the geodesic in $\mathbf{H}$ with endpoints equal to the endpoints of I. There are two possibilities to consider. Either $h_s(\hat{\mathbf{Q}})$ contains exactly one endpoint of I or both endpoints of I do not lie in $h_s(\hat{\mathbf{Q}})$.

In the former case, the interval I has an endpoint $h_s(p)$ for some $p\in\hat{\mathbf{Q}}$. This implies that the image of the fan at p under h_s accumulates to the geodesic $l\in\mathbf{H}$. Let C be a horocycle based at p. Fix a single geodesic in the fan at $h_s(p)$. Then the sum of lengths of consecutive arcs of C cut out by the geodesics in the fan at $h_s(p)$ which accumulate to l starting from the fixed geodesic in the fan is finite. This implies that there exists a sequence of $2n$ consecutive arcs on C such that the ratio of the length of left n consecutive arcs to the length of the right

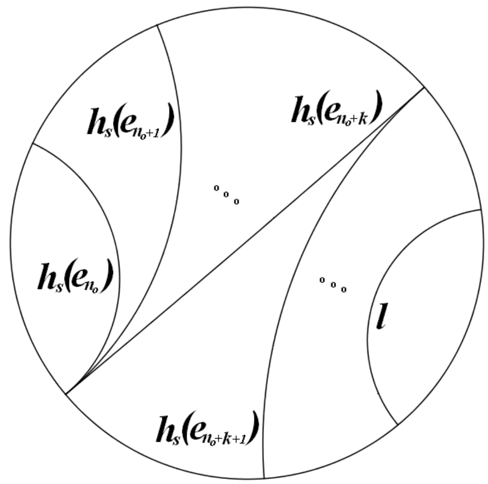

FIGURE 3. The accumulation to l.

n consecutive arcs is converging to ∞. Consequently, the condition (5) fails at the fan with tip p which is a contradiction.

In the later case, there is a sequence $e_n \in \mathcal{F}$ such that $h_s(e_n)$ converges to l and that no $h_s(e_n)$ shares an endpoint with l. Moreover, we can assume that each e_n shares one endpoint with e_{n+1} namely $\{e_n\}$ is a chain. We exhibit a sequence of pairs of adjacent triangles in $h(\mathcal{F})$ with shears converging to 0 or to ∞ which again contradicts condition (5). Let e_{n_0} be such that $h_s(e_{n_0})$ is close to l. Then e_{n0+1} shares an endpoint with e_{n_0}. Let e_{n0+k} be the edge in the sequence $\{e_n\}$ with largest index which shares an endpoint with e_{n_0}. Then e_{n0+k+1} does not share an endpoint with e_{n0+k-1} (see Figure 3). We consider the two adjacent triangle in $\mathcal{F}$ with common boundary edge e_{n0+k}. The image of the two triangles under h_s has sides $h_s(e_{n0+k-1})$, $h_s(e_{n0+k})$ and $h_s(e_{n0+k+1})$ close to the geodesic l. This implies that the other two sides are small in the Euclidean sense. Thus the shear is very large or very small which is a contradiction with condition (5). We proved that h_s extends to a homeomorphism of $\hat{\mathbf{R}}$.

It remains to show that $h_s : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ is a quasisymmetric map. Let $F_s = ex(h_s)$ be the barycentric extension of h_s (see Douady-Earle [5] for the definition). Then $F_s : \mathbf{H} \to \mathbf{H}$ is a real analytic diffeomorphism of $\mathbf{H}$. The map h_s is quasisymmetric if and only if F_s is quasiconformal. Let $\mu_{F_s} = \frac{\bar{\partial} F_s}{\partial F_s}$ be the Beltrami coefficient of F_s.

Assume on the contrary that F_s is not quasiconformal. Then there exists a sequence $z_n \in \mathbf{H}$ such that $|\mu_{F_s}(z_n)| \to 1$ as $n \to \infty$. Since F_s is a real analytic diffeomorphism (see [5]), it follows that z_n leaves every compact subset of $\mathbf{H}$. There are two possibilities for z_n. Either there exist a horoball D with center at ∞ and a subsequence z_{n_k} of z_n such that z_{n_k} lies outside the $PSL_2(\mathbf{Z})$ orbit of D, or sequence z_n enters the $PSL_2(\mathbf{Z})$ orbit of every horoball with center at ∞.

Suppose that we are in the former case. For simplicity, denote the subsequence z_{n_k} by z_n again. Let Δ_n be triangle in the complement of $\mathcal{F}$ which contains z_n. Let $A_n \in PSL_2(\mathbf{Z})$ be such that $A_n(\Delta_n) = \Delta_0$. Let $B_n \in PSL_2(\mathbf{R})$ be such that $B_n \circ h_s \circ A_n^{-1}$ fixes 0, 1 and ∞. By the conformal naturality of the barycentric extension, we have that $ex(B_n \circ h_s \circ A_n^{-1}) = B_n \circ F_s \circ A_n^{-1} = F_n$. Let $z_n' = A_n(z_n) \in \Delta_0$. Then z_n' belongs to a compact subset of $\mathbf{H}$ and $|\mu_{F_n}(z_n')| = |\mu_{F_s}(z_n)|$. Condition (5) implies that individual shears are bounded by $1/M$ from below and by M from above. This implies that the sequence of shear maps $s \circ A_n^{-1}$ corresponding to homeomorphisms $B_n \circ h_s \circ A_n^{-1}$ has a convergent subsequence in the sense that for each edge $e \in \mathcal{F}$ the sequence of real numbers $s \circ A_{n_k}^{-1}(e)$ converges as $k \to \infty$. The limiting map $s_\infty : \mathcal{F} \to \mathbf{R}$ satisfies property (5) in each fan with the constant M because each $s \circ A_n^{-1}$ does. By the normalization of $B_n \circ h_s \circ A_n^{-1}$, we get that $B_{n_k} \circ h_s \circ A_{n_k}^{-1}$ pointwise converges to a homeomorphism h_{s_∞} of $\hat{\mathbf{R}}$ with shear map s_∞. By the continuity of the barycentric extension, we get that $|\mu_{F_{n_k}}|$ converges to $|\mu_{ex(h_{s_\infty})}|$ uniformly on compact subsets of $\mathbf{H}$. This implies that for a compact subset K of $\mathbf{H}$ there exists $a < 1$ such that $|\mu_{F_{n_k}}| \leq a$ on K. On the other hand, we have that $|\mu_{F_{n_k}}(z_{n_k})| \to 1$ as $k \to \infty$ which gives a contradiction.

Suppose that we are in the later case. Namely, $|\mu_{F_s}(z_n)| \to 1$ as $n \to \infty$ with z_n entering the $PSL_2(\mathbf{Z})$ orbit of every horoball based at ∞. Let Δ_n be a complementary triangle of $\mathcal{F}$ which contains z_n. Let $A_n \in PSL_2(\mathbf{Z})$ be such that $A_n(\Delta_n) = \Delta_0$ and that $A_n(z_n) = z_n' \to \infty$ as $n \to \infty$. Let $B_n \in PSL_2(\mathbf{R})$ be such that $B_n \circ h_s \circ A_n^{-1} = h_n$ fixes 0, 1 and ∞. Then h_n satisfies property (5) with the same constant M as does h. By the conformal naturality of the barycentric extension, we have that $|\mu_{F_s}(z_n)| = |\mu_{ex(h_n)}(z_n')| \to 1$ as $n \to \infty$. Let $\lambda_n = Im(z_n')$ and let λ_n' be such that $\hat{h}_n(x) = \frac{1}{\lambda_n'} h_n(\lambda_n x)$ fixes 1. It is clear that $\hat{h}_n$ fixes 0 and ∞ as well. Let $w_n = \frac{1}{\lambda_n} z_n'$. Then $w_n \to i$ and $|\mu_{ex(\hat{h}_n)}(w_n)| = |\mu_{ex(h_n)}(z_n')| = |\mu_{F_s}(z_n)| \to 1$ as $n \to \infty$. We need the following lemma in order to finish the proof.

Lemma 5.1. *Under the above normalization, there exists a constant* $c_0 > 1$ *such that* $\frac{1}{c_0} \leq -\hat{h}_n(-1) \leq c_0$.

Proof. Let $k_n \in \mathbf{N}$ be such that $k_n \leq \lambda_n \leq k_n + 1$. Then $h_n(k_n) \leq h_n(\lambda_n) = \lambda'_n \leq h_n(k_n+1)$. By property (5), we have that $h_n(k_n+1) - h_n(k_n) \leq Mh_n(k_n)$. This implies that

$$h_n(k_n + 1) \leq (M+1)h_n(k_n) \leq (M+1)\lambda'_n. \tag{6}$$

By applying property (5) to h_n at points $-(k_n+1)$, 0 and k_n+1, we get that $\frac{1}{M}h_n(k_n+1) \leq -h_n(-k_n-1) \leq Mh_n(k_n+1)$. Similarly, we get that $\frac{1}{M}h_n(k_n) \leq -h_n(-k_n) \leq Mh_n(k_n)$. These two inequalities imply that

$$-Mh_n(k_n+1) \leq h_n(-k_n-1) \leq h_n(-\lambda_n) \leq h_n(-k_n) \leq -\frac{1}{M}h_n(k_n).$$

From (6), we get

$$h_n(k_n) \geq \frac{1}{M+1}h_n(k_n+1) \geq \frac{1}{M+1}\lambda'_n.$$

The above two inequalities and (6) give that

$$-M(M+1)\lambda'_n \leq h_n(-\lambda_n) \leq -\frac{1}{M(M+1)}\lambda'_n$$

which implies

$$-M(M+1) \leq \frac{1}{\lambda'_n}h_n(-\lambda_n) = \hat{h}_n(-1) \leq -\frac{1}{M(M+1)}.$$

Take $c_0 = M(M+1)$ and the above becomes $\frac{1}{c_0} \leq -\hat{h}_n(-1) \leq c_0$. □

We finish the proof using the above lemma. Note that $\hat{h}_n$ fixes 0, 1 and ∞, and $\hat{h}_n(-1)$ is bounded away from 0 and ∞ by the above lemma. Then Lemma 2.2 implies that $|\mu_{\hat{h}_n}| \leq c < 1$ in a neighborhood of $i \in \mathbf{H}$ and for all $n \in \mathbf{N}$ (see also [14, Lemma 3.6], [1], [5]). On the other hand, the assumption on w_n and conformal naturality of barycentric extension implies that $|\mu_{\hat{h}_n}(w_n)| \to 1$ as $n \to \infty$ which is a contradiction. This finishes the proof of the first statement in Theorem A. □

Proof of the second part of Theorem A. Consider a fan of geodesics of $\mathcal{F}$ with tip $p \in \hat{\mathbf{Q}}$ and assume that $e_n \in \mathcal{F}$, $n \in \mathbf{Z}$, is a fixed correspondence with $\mathbf{Z}$ induced by natural ordering as before. Let $a_n \in \hat{\mathbf{Q}}$ be the endpoint of e_n that is different from p. Then (a_k, a_m, a_n, p) are in the cyclic order of $\hat{\mathbf{R}}$ if $k < m < n$. The triple a_k, a_m, a_n is said to be *fan-symmetric* if $m-k=n-m$. The point a_m is said to be the *midpoint* of the triple.

Let $a_k,a_m,a_n\in\hat{\mathbf{Q}}$ be a fan-symmetric triple for the fan with tip $p\in\hat{\mathbf{Q}}$, where a_m is the mid-point of the triple. This implies that $cr(p,a_k,a_m,a_n)=\frac{(a_m-p)(a_n-a_k)}{(a_n-p)(a_m-a_k)}=2$. The *generation of a triple* e_k,e_m,e_n of geodesics is the minimum of the Farey generations of e_k and e_n . Let $h:\hat{\mathbf{R}}\to\hat{\mathbf{R}}$ be a symmetric map which fixes 0, 1 and ∞. If the generation of a triple e_k,e_m,e_n is large, it follows that the points a_k, p and a_n are close in the angle metric of $\hat{\mathbf{R}}$ with respect to $i\in\mathbf{H}$. The barycentric extension $ex(h)=F$ of h has Beltrami coefficient close to zero in a definite Euclidean neighborhood in $\mathbf{H}$ of the triple (a_k,p,a_n) (see [7]). A length-area argument implies that $cr(h(p),h(a_k),h(a_m),h(a_n))$ is close to 2 (see, for example, [12]). After post-composing h by $A\in PSL_2(\mathbf{R})$ such that $A\circ h(a_m)=\infty$, this is equivalent to the fact that the ratio $\frac{|A\circ h(p)-A\circ h(a_k)|}{|A\circ h(a_n)-A\circ h(p)}$ is close to 1. Let $s:\mathcal{F}\to\mathbf{R}$ be the shear map of h and let $s_i=s(e_i)$. Then for a given $\epsilon>0$, there exists $k=k(\epsilon)\in\mathbf{N}$ such that on any fan-symmetric triple of generation at least k the shear map $s:\mathcal{F}\to\mathbf{R}$ satisfies

$$\frac{1}{1+\epsilon}\leq\frac{1+e^{s_1}+\cdots+e^{s_1+s_2+\cdots+s_n}}{e^{-s_0}+e^{-s_0-s_{-1}}+\cdots+e^{-s_0-s_{-1}-\cdots-s_{-n}}}\leq 1+\epsilon. \tag{7}$$

Thus we established the necessity of the second condition in Theorem A.

We show that the second condition in Theorem A is also sufficient for a map to be symmetric. For any $k\in\mathbf{N}$, there are only finitely many geodesics in $\mathcal{F}$ whose generation is at most k. Together with (7), this implies that the shear map $s:\mathcal{F}\to\mathbf{R}$ is bounded and that $s(e)$ converges to 0 as the generation of e converges to ∞, where the speed of convergence depends only on the generation of $e\in\mathcal{F}$. The cocycle map h_s of the shear map s with property (7) extends to a homeomorphism of $\hat{\mathbf{R}}$. The proof follows the same lines as in the proof of the first part of Theorem A and we omit it here.

It remains to show that h_s is a symmetric map. We consider the barycentric extension $ex(h_s)=F_s$ of h_s. It is enough to show that F_s is an asymptotically conformal map of $\mathbf{H}$ (see [7]).

Assume on the contrary that there exists a sequence $z_n\in\mathbf{H}$ which leaves every compact subset of $\mathbf{H}$ such that $|\mu_{F_s}(z_n)|\geq c>0$. Let Δ_n be the ideal triangle in $\mathcal{F}$ which contains z_n. Let $A_n\in PSL_2(\mathbf{Z})$ be such that $A_n(\Delta_n)=\Delta_0$, where Δ_0 is the triangle in $\mathcal{F}$ with vertices 0, 1 and ∞. Let $B_n\in PSL_2(\mathbf{R})$ be such that $h_n=B_n\circ h_s\circ A_n^{-1}$ fixes 0, 1 and ∞. Let $z_n' = A_n(z_n) \in \Delta_0$ and let $F_n = ex(h_n)$ be the barycentric extension of h_n.

Assume first that a subsequence of z_n' stays in a compact part of Δ_0, and for simplicity we denote the subsequence by z_n' again. This implies that the sequence of Δ_n's contains infinitely many pairwise different triangles because z_n leave any compact subset of $\mathbf{H}$. In particular, the minimum of the generations of the edges of Δ_n converges to infinity as $n \to \infty$. Consequently, shear maps $s \circ A_n^{-1}$ converge to the zero map which implies that h_n converges pointwise on $\hat{\mathbf{R}}$ to the identity. On the other hand, $|\mu_{F_n}(z_n')| \geq c > 0$ by conformal naturality of the barycentric extension. This is a contradiction with the continuity properties of the barycentric extension (see [5] or Section 2).

In the other case, we assume that $z_n' \to \infty$ inside Δ_0 as $n \to \infty$. Let $\lambda_{n,1} = [Im(z_n')]$ be the greatest integer less than or equal to $Im(z_n')$. Clearly $\lambda_{n,1} \to \infty$ as $n \to \infty$. Let $\lambda_{n,1}' = h_n(\lambda_{n,1})$. Define $\frac{1}{\lambda_{n,1}'} h_n(\lambda_{n,1}x) = \tilde{h}_{n,1}(x)$ and note that $\tilde{h}_{n,1}(x)$ fixes 0, 1 and ∞.

Let $x, y, z \in \mathbf{Z}$ be symmetric points such that $\tilde{h}_{n,1}(x) \to x$ and $\tilde{h}_{n,1}(y) \to y$ as $n \to \infty$. If either $x > y > z$ and $x \neq 0$, or $x < y < z$ and $x \neq 0$, or $x < z < y$ and $x, y \neq 0$, then we claim that $\tilde{h}_{n,1}(z) \to z$ as $n \to \infty$. We prove this when $x > y > z$ and $x \neq 0$. Other cases are similar and they are left to the reader. For $z = 0$ we have immediately that $\tilde{h}_{n,1}(z) = z$. We assume now that $z \neq 0$. Since $x \neq 0$ we have that $\lambda_{n,1}x \to \pm\infty$ and $\lambda_{n,1}z \to \pm\infty$ as $n \to \infty$. This implies that the generation of the symmetric triples $e_{\lambda_{n,1}x}$, $e_{\lambda_{n,1}y}$ and $e_{\lambda_{n,1}z}$ goes to infinity as $n \to \infty$. Then the condition (7) implies that $\frac{\tilde{h}_{n,1}(x) - \tilde{h}_{n,1}(y)}{\tilde{h}_{n,1}(y) - \tilde{h}_{n,1}(z)} \to 1$ as $n \to \infty$ because $\lambda_{n,1}x, \lambda_{n,1}y, \lambda_{n,1}z \in \mathbf{Z}$ and $h_n(\lambda_{n,1}x), h_n(\lambda_{n,1}y), h_n(\lambda_{n,1}z)$ depend only on the shears at the fan with tip ∞. This gives $\tilde{h}_{n,1}(z) \to z$ as $n \to \infty$.

Recall that $\tilde{h}_{n,1}$ fixes 0, 1 and ∞. We use the statement in the above paragraph to show that $\lim_{n\to\infty} \tilde{h}_{n,1}(k) = k$ for all $k \in \mathbf{Z}$. Using the triple -1, 0 and 1, we get that $\lim_{n\to\infty} \tilde{h}_{n,1}(-1) = -1$. Then using the triple -1, 1 and 3 we get that $\lim_{n\to\infty} \tilde{h}_{n,1}(3) = 3$. The triple 1, 2 and 3 gives that $\lim_{n\to\infty} \tilde{h}_{n,1}(2) = 2$. Then using the triple 2, 3 and 4 gives the convergence for 4, and continuing like this we get the convergence $\lim_{n\to\infty} \tilde{h}_{n,1}(k) = k$ for all $k \in \mathbf{Z}^+$. Similarly, we get $\lim_{n\to\infty} \tilde{h}_{n,1}(k) = k$ for all $k \in \mathbf{Z}^-$.

Let $\lambda_{n,r}$ be the greatest integer multiple of 2^{r-1} which is less than or equal to $Im(z_n')$ for $r \in \mathbf{N}$. Clearly $\lambda_{n,r} \to \infty$ as $n \to \infty$. Let $\lambda_{n,r}' = h_n(\lambda_{n,r})$. Define $\frac{1}{\lambda_{n,r}'} h_n(\lambda_{n,r} x) = \tilde{h}_{n,r}(x)$ and note that $\tilde{h}_{n,r}(x)$ fixes 0, 1 and ∞. We claim that $\lim_{n\to\infty} \tilde{h}_{n,r}(k/2^i) = k/2^i$ for all $k \in \mathbf{Z}$ and $i = 0, \dots, r-1$. For a fixed r, the proof is by finite induction on i. The case $i = 0$ is proved in the above paragraph. Assume that the statement is true for i and we need to prove that it is true for $i+1$. The inductive hypothesis says that $\lim_{n\to\infty} \tilde{h}_{n,r}(k/2^i) = k/2^i$ for $k \in \mathbf{Z}$ because $\lambda_{n,r}\frac{k}{2^i} \in \mathbf{Z}$ for each $n \in \mathbf{N}$. Since each $k/2^{i+1}$, for $k \in \mathbf{Z}$ odd, is in the middle of $(k-1)/2^{i+1}$ and $(k+1)/2^{i+1}$ on which the convergence holds and since $\lambda_{n,r}\frac{k}{2^{i+1}} \in \mathbf{Z}$, it follows similar to the above that $\lim_{n\to\infty} \tilde{h}_{n,r}(k/2^{i+1}) = k/2^{i+1}$. This finishes the induction.

We use the Cantor diagonalization process to obtain a contradiction. The set $D = \{k/2^{r-1} : r \in \mathbf{N}, k \in \mathbf{Z}\}$ is a dense subset of $\hat{\mathbf{R}}$. We put D into a sequence $\{b_m\}$ such that if $b_m = k/2^{r-1}$ for minimal $r \in \mathbf{N}$ then $m \geq r$. Fix $m \in \mathbf{N}$. Then there exists n_m such that $|\tilde{h}_{n_m,m}(b_i) - b_i| < 1/m$ for $i = 1, 2, \dots, m$ and $|z_m'/\lambda_{n_m,m} - i| < 1/m$. This implies that $\tilde{h}_{n_m,m}$ converges pointwise to the identity on $\hat{\mathbf{R}}$. On the other hand, the Beltrami coefficient of $ex(\tilde{h}_{n_m,m})$ at $\frac{z_m'}{\lambda_{n_m,m}}$ is bounded away from 0 by conformal naturality of the barycentric extension. This is a contradiction. Therefore h_s is symmetric which finishes the proof of Theorem A. □

6. The topology on $\mathcal{X}$

Let $\mathcal{X}$ be the space of all shear maps $s : \mathcal{F} \to \mathbf{R}$ which satisfy condition (5) on each fan of geodesics in $\mathcal{F}$ with the same constant. Theorem A implies that the universal Teichmüller space $T(\mathbf{H})$ is parameterized by the space $\mathcal{X}$. We turn our attention to the topology on $\mathcal{X}$ which would make the map $T(\mathbf{H}) \to \mathcal{X}$ a homeomorphism.

Consider a shear map $s \in \mathcal{X}$ and a fan of geodesics in $\mathcal{F}$ with tip p. Let e_n, $n \in \mathbf{Z}$, be the enumeration of the fan. For a given horocycle C with center p, we denote by $s(p; n, k)$ the quotient of the length of arc of C between $h_s(e_{n+k})$ and $h_s(e_n)$ to the length of the arc of C between $h_s(e_n)$ and $h_s(e_{n-k})$, for $n, k \in \mathbf{Z}$. Note that $s(p; n, k)$ is the expression in the middle of (5) described in a coordinate independent fashion.

Let $M(s) \geq 1$ be the supremum of $s(p; n, k)$ over all $p \in \hat{\mathbf{Q}}$, $n, k \in \mathbf{Z}$. If $M(s) < \infty$, then we say that $s : \mathcal{F} \to \mathbf{R}$ satisfies $M(s)$-shear condition. For example, the shear map s_{id} of the basepoint $id \in T(\mathbf{H})$ is assigning 0 to each edge of $\mathcal{F}$ and $M(s_{id}) = 1$.

More generally, let $s_1, s_2 \in \mathcal{X}$. Define $M(s_1, s_2)$ to be the supremum of the maximum of $s_1(p; n, k)/s_2(p; n, k)$ and $s_2(p; n, k)/s_1(p; n, k)$ over all $p \in \hat{\mathbf{Q}}$, $n \in \mathbf{Z}$ and $k \in \mathbf{N}$. Note that $M(s_1, s_2) = M(s_2, s_1)$ and that $M(s_1, s_{id}) = M(s_1)$.

Let $h_n : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ be a sequence of quasisymmetric maps which fix 0, 1 and ∞, and which converge to the identity in the Teichmüller topology in $T(\mathbf{H})$. Then we immediately obtain that $M(s_{h_n}) \to 1$ as $n \to \infty$ from the quasisymmetric condition.

Proof of Theorem B. Recall that $h_n \to id$ in the Teichmüller topology if and only if $\sup \frac{cr(h_n(a), h_n(b), h_n(c), h_n(d))}{cr(a,b,c,d)} \to 1$ as $n \to \infty$, where the cross-ratio is $cr(a, b, c, d) = \frac{(c-a)(d-b)}{(d-a)(c-b)}$ and the supremum is over all quadruples $(a, b, c, d) \in (\hat{\mathbf{R}})^4$ with the cross-ratio between $1 + 1/M$ and $1 + M$ for some $M > 1$. By the definition, $h_n \to h$ as $n \to \infty$ in the Teichmüller topology if and only if $h_n \circ h^{-1} \to id$ in the Teichmüller topology. A quadruple of points in $\hat{\mathbf{Q}}$ with cross-ratio 2 where one point is the tip of the fan such that the other three points are endpoints of geodesics in the fan is *fan-symmetric* (see proof of Theorem A for equivalent definition). The cross-ratio of the image under h of a fan-symmetric quadruple is bounded away from 1 and ∞ because h is quasisymmetric. The above characterization of the Teichmüller topology when applied to $h_n \circ h^{-1} \to id$ at the images under h of all fan-symmetric quadruples gives that $M(s_{h_n}, s_h) \to 1$ as $n \to \infty$. This proves the necessity of the condition.

Given $h, h_n \in T(\mathbf{H})$ such that $M(s_{h_n}, s_h) \to 1$ as $n \to \infty$, we need to show that $h_n \to h$ as $n \to \infty$. Assume on the contrary that h_n does not converge to h in the Teichmüller topology. Let $F = ex(h)$ and $F_n = ex(h_n)$ be the barycentric extensions of h and h_n, respectively. The assumption implies that there exists $c > 0$ and a sequence $z_n \in \mathbf{H}$ such that $|\mu_F(z_n) - \mu_{F_n}(z_n)| \geq c$. There are two possibilities for the sequence z_n. Either there exists a horoball C with center ∞ and a subsequence z_{n_k} such that z_{n_k} is disjoint from the $PSL_2(\mathbf{Z})$ orbit of C, or for any horoball C with center ∞ only finitely many z_n's lie outside the $PSL_2(\mathbf{Z})$ orbit of C.

Assume we are in the former case. For the convenience of notation, replace z_{n_k} with z_n. Let $A_n \in PSL_2(\mathbf{Z})$ be such that $A_n(\Delta_n) = \Delta_0$, where Δ_n is a complementary triangle of $\mathcal{F}$ which contains z_n. Then $A_n(z_n)$ lies in a compact subset of $\mathbf{H}$. Let $B_n, B_n^* \in PSL_2(\mathbf{R})$ be such that $B_n \circ h \circ A_n^{-1}$ and $B_n^* \circ h_n \circ A_n^{-1}$ fix 0, 1 and ∞. Since $M(s_{h_n}, s_h) \to 1$ as $n \to \infty$, we get that $B_n \circ h \circ A_n^{-1}$ and $B_n^* \circ h_n \circ A_n^{-1}$ pointwise converge to the same quasisymmetric map. Therefore the Beltrami coefficients of their corresponding barycentric extensions converge uniformly on compact subsets of $\mathbf{H}$ to the same Beltrami coefficient (see [5]). This contradicts $|\mu_F(z_n) - \mu_{F_n}(z_n)| \geq c$.

Assume we are in the later case. Let $A_n \in PSL_2(\mathbf{Z})$ and $B_n, B_n^* \in PSL_2(\mathbf{R})$ be as above. Let $h_n' = B_n \circ h \circ A_n^{-1}$ and $h_n^* = B_n^* \circ h_n \circ A_n^{-1}$. In addition, we may assume that $A_n(z_n) \to \infty$ as $n \to \infty$. To find a contradiction in this case, we use the idea from the proof of the last part of Theorem A. Denote by $\lambda_{n,r}$ the largest integer multiple of 2^{r-1} which is less than or equal to $Im(z_n)$. Let $\lambda_{n,r}' = h_n'(\lambda_{n,r})$ and $\lambda_{n,r}^* = h_n^*(\lambda_{n,r})$. Then $\tilde{h}_{n,r}'(x) = \frac{1}{\lambda_{n,r}'} h_n'(\lambda_{n,r} x)$ and $\tilde{h}_{n,r}^*(x) = \frac{1}{\lambda_{n,r}^*} h_n^*(\lambda_{n,r} x)$ fix 0, 1 and ∞. For each $r \in \mathbf{N}$, sequences $\tilde{h}_{n,r}'(x)$ and $\tilde{h}_{n,r}^*(x)$ have convergent subsequences (in the pointwise sense) whose limits h_r^1 and h_r^2 agree on the set $\{k/2^{r-1} : k \in \mathbf{Z}\}$ because $M(s_{h_n}, s_h) \to 1$ as $n \to \infty$. The values of maps h_r^1 and h_r^2 on $\{k/2^{r-1} : k \in \mathbf{Z}\}$ depend only on the shears of h_n' and h_n^* on the fan with tip ∞. Using the Cantor diagonalization process, we find sequences $\tilde{h}_{n_m,m}'(x)$ and $\tilde{h}_{n_m,m}^*(x)$ whose pointwise limits h^1 and h^2 satisfy $h^1 = h^2$ and $\frac{z_m'}{\lambda_{n_m,m}} \to i \in \mathbf{H}$ as $m \to \infty$. This again gives a contradiction with $|\mu_F(z_m) - \mu_{F_m}(z_m)| \geq c$ by conformal naturality of the barycentric extension. $\square$

7. Decorated tesselations and lambda lengths

A *tesselation* τ of $\mathbf{H}$ is a locally finite countable geodesic lamination of $\mathbf{H}$ such that the components in $\mathbf{H} \setminus \tau$ are ideal hyperbolic triangles. A *decorated tesselation* $\tilde{\tau}$ is a tesselation τ of $\mathbf{H}$ together with an assignment of a horocycle to each vertex of τ whose center is that vertex (see [16]).

Let τ be a tesselation with a distinguished oriented edge $e = (x_i, x_t)$, where x_i is the initial point and x_t is the terminal point of e. Recall that $\mathcal{F}$ is the Farey tesselation and let $(-1, 1)$ be a distinguished oriented edge of $\mathcal{F}$. Denote by τ^0 the set of vertices of τ. Recall that $\hat{\mathbf{Q}} \subset \hat{\mathbf{R}}$ is the set of vertices of $\mathcal{F}$. There exists a unique map $h_\tau : \hat{\mathbf{Q}} \to \tau^0$ such that $h_\tau(x_i) = -1$, $h_\tau(x_t) = 1$ and that if $x, y, x \in \hat{\mathbf{Q}}$ are vertices of a complementary triangle of $\mathcal{F}$ then $h_\tau(x), h_\tau(y), h_\tau(z) \in \tau^0$ are the vertices of a complementary triangle of τ (see [16]). We call h_τ the *characteristic map* of τ. It is clear that $h_\tau : \hat{\mathbf{Q}} \to \hat{\mathbf{R}}$ extends by continuity to a homeomorphism of $\hat{\mathbf{R}}$ because $\hat{\mathbf{Q}}, \tau^0$ are dense in $\hat{\mathbf{R}}$ and h_τ is monotone on $\hat{\mathbf{Q}}$.

Given a decorated tesselation $\tilde{\tau}$ together with a distinguished oriented edge $e \in \tau$, Penner [17] assigns to each edge $f \in \mathcal{F}$ a positive number as follows. Let C_1 and C_2 be horocycles of the decoration $\tilde{\tau}$ based at the endpoints of $h_\tau(f) \in \tau$. Let $\delta(f)$ be a signed hyperbolic distance between $M_1 = h_\tau(f) \cap C_1$ and $M_2 = h_\tau(f) \cap C_2$, where the sign is positive if the arc of $h_\tau(f)$ between M_1 and M_2 is outside C_1 and C_2, otherwise the sign is negative (see [17]). The *lambda length* of $f \in \mathcal{F}$ is given by

$$\lambda(f) = e^{-2\delta(f)}.$$

This introduces the *lambda length* function $\lambda : \mathcal{F} \to \mathbf{R}^+$ for any decorated tesselation $\tilde{\tau}$ (see Penner [16]). Let $e, e' \in \tau$ be adjacent edges. Then $h_\tau : \mathcal{F} \to \tau$ maps adjacent edges $f, f' \in \mathcal{F}$ onto e, e', respectively. We define *horocyclic length* $\alpha(f, f')$ to be the length of the arc of the horocycle from $\tilde{\tau}$ with center the common endpoint of e and e' that lies inside the hyperbolic wedge with boundary sides e, e'.

Conversely, given a map $\lambda : \mathcal{F} \to \mathbf{R}$ there exists a monotone map $h_\lambda : \hat{\mathbf{Q}} \to \hat{\mathbf{R}}$, called the *characteristic map* of λ, and a decoration (i.e. choice of horocycles) on $h_\lambda(\hat{\mathbf{Q}})$ such that the lambda length of $h_\lambda(f)$ with respect to the decoration is equal to $\lambda(f)$. The characteristic map $h_\lambda : \hat{\mathbf{Q}} \to \hat{\mathbf{R}}$ does not always extend to a homeomorphism similar to the case of shears. It is a fundamental question in this theory to give necessary and sufficient condition on the map $\lambda : \mathcal{F} \to \mathbf{R}$ such that h_λ extends by continuity to a homeomorphism or perhaps to a quasisymmetric map. Penner and Sullivan [16, Theorem 6.4] gave a sufficient condition on the lambda lengths to induce a quasisymmetric map as follows. A lambda length function $\lambda : \mathcal{F} \to \mathbf{R}$ is said to be *pinched* if there exists $K > 1$ such that

$$\frac{1}{K} \leq \lambda(f) \leq K,$$

for all $f \in \mathcal{F}$. Penner and Sullivan showed that if $\lambda : \mathcal{F} \to \mathbf{R}$ is pinched then the characteristic map h_λ extends to a quasisymmetric map of $\hat{\mathbf{R}}$ [16, Theorem 6.4].

In Theorem E we give a necessary and sufficient condition such that h_λ is a quasisymmetric (as well as a symmetric) map of $\hat{\mathbf{R}}$.

Proof of Theorem E. Let $\tau = h_\lambda(\mathcal{F})$ be the geodesic lamination corresponding to the lambda lengths λ and let $\tilde{\tau}$ be the decorations at the vertices of τ corresponding to λ (see Penner [16] for the construction). Let $s : \mathcal{F} \to \mathbf{R}$ be shear map corresponding to h_λ. Let $e_n \in \mathcal{F}$, $n \in \mathbf{Z}$, be a fan of geodesics with tip p. Then we have

$$s(p;n,k)=\frac{\alpha(e_m,e_{m+1})+\alpha(e_{m+1},e_{m+2})+\cdots+\alpha(e_{m+k},e_{m+k+1})}{\alpha(e_m,e_{m-1})+\alpha(e_{m-1},e_{m-2})+\cdots+\alpha(e_{m-k},e_{m-k-1})}.$$

Theorem E immediately follows from Theorem A. □

In Theorem D, we find a necessary and sufficient condition such that h_λ extends to a homeomorphism of $\hat{\mathbf{R}}$. The criteria follows from the proof of Theorem C and it is obtained by calculating the length of $l(P_1)$ in terms of horocyclic and lambda lengths. Since the horocyclic lengths are expressed in terms of the lambda lengths, the formula can be written only in terms of the lambda lengths although we do not do this.

Proof of Theorem D. Let $\lambda:\mathcal{F}\to\mathbf{R}$ be an assignment of lambda lengths and let $h_\lambda:\hat{\mathbf{Q}}\to\hat{\mathbf{R}}$ be the characteristic map. Denote by τ the image tesselation $h_\lambda(\mathcal{F})$ and by $\tilde{\tau}$ the decoration which realizes the lambda lengths λ.

Let e_n, for $n\in\mathbf{N}$, be an arbitrary chain in $\mathcal{F}$. Denote by $\lambda_n=\lambda(e_n)$ the lambda length of e_n. Then $\lambda_n=e^{-2\delta_n}$, where δ_n is the signed hyperbolic distance between the horocycles of $\tilde{\tau}$ with centers at the endpoints of e_n. Thus $\lambda_n^{-1/2}=e^{\delta_n}$. Let W_n be the wedge with boundary sides $h_\lambda(e_n)$ and $h_\lambda(e_{n+1})$ and let C_n be the horocycle of the decoration $\tilde{\tau}$ with center at the common endpoint of $h_\lambda(e_n)$ and $h_\lambda(e_{n+1})$. Let α_n be the horocyclic length for the wedge with boundaries e_n and e_{n+1} namely the length of $C_n\cap W_n$. Let l_n be the length of $l(P_1)\cap W_n$, where P_1 is chosen such that $l_1=\lambda_1^{-\frac{1}{2}}\alpha_1=e^{\delta_1}\alpha_1$.

We need to show that $l_n=(\lambda_n^{-\frac{1}{2}}\lambda_{n-1}^{\frac{1}{2}}\cdots\lambda_1^{\frac{(-1)^n}{2}})\alpha_n$. An elementary hyperbolic considerations shows that $l_n=e^{d_n}\alpha_n$ where d_n is the signed distance from $l(P_1)\cap W_n$ to the horocycle C_n such that $d_n>0$ if $l(P_1)\cap W_n$ is outside C_n and that otherwise $d_n<0$. Therefore it remains to show that $e^{d_n}=\lambda_n^{-\frac{1}{2}}\lambda_{n-1}^{\frac{1}{2}}\cdots\lambda_1^{\frac{(-1)^n}{2}}$.

We finish the argument by induction on n. By our choice of P_1, we have immediately that $e^{d_1}=e^{\delta_1}=\lambda_1^{-\frac{1}{2}}$. Assume that $n>1$ and that $e^{d_n}=\lambda_n^{-\frac{1}{2}}\lambda_{n-1}^{\frac{1}{2}}\cdots\lambda_1^{\frac{(-1)^n}{2}}$. We calculate $e^{d_{n+1}}$. Since d_n is the signed distance from $l(P_1)\cap W_n$ to C_n, it follows that the signed distance of $l(P_1)\cap H_s(e_{n+1})$ to C_n is d_n. Since δ_{n+1} is the signed distance between C_n and C_{n+1}, it follows that $d_{n+1}=\delta_{n+1}-d_n$. This gives the desired formula. □

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