

INFINITESIMAL LIOUVILLE CURRENTS, CROSS-RATIOS AND INTERSECTION NUMBERS

FRANCIS BONAHOON AND DRAGOMIR ŠARIĆ

ABSTRACT. Many classical objects on a surface S can be interpreted as cross-ratio functions on the circle at infinity of the universal covering \tilde{S} . This includes closed curves considered up to homotopy, metrics of negative curvature considered up to isotopy and, in the case of interest here, tangent vectors to the Teichmüller space of complex structures on S . When two cross-ratio functions are sufficiently regular, they have a geometric intersection number, which generalizes the intersection number of two closed curves. In the case of the cross-ratio functions associated to tangent vectors to the Teichmüller space, we show that two such cross-ratio functions have a well-defined geometric intersection number, and that this intersection number is equal to the Weil-Petersson scalar product of the corresponding vectors.

Let S be a compact orientable surface of negative Euler characteristic. Its universal cover \tilde{S} has a well-defined *circle at infinity* $\partial_\infty \tilde{S}$, which can be described uniquely in terms of the topology of S , or even in terms of the fundamental group $\pi_1(S)$ [Gr, GhHa, CDP, BrHae]. The action of $\pi_1(S)$ on the universal cover continuously extends to $\tilde{S} \cup \partial_\infty \tilde{S}$.

A *cross-ratio function* (or *cross-ratio* for short) is a function α that associates a real number $\alpha(I_1, I_2) \in \mathbb{R}$ to each pair of two intervals $I, J \subset \partial_\infty S$ with disjoint closures, and that satisfies the following two conditions:

- (1) (Finite Additivity) $\alpha(I, J) = \alpha(I_1, J) + \alpha(I_2, J)$ whenever the interval I is split as the union of two disjoint intervals I_1 and I_2 ; similarly, $\alpha(I, J) = \alpha(I, J_1) + \alpha(I, J_2)$ whenever the interval J is split as the union of two disjoint intervals J_1 and J_2 .
- (2) (Invariance) α is invariant under the action of the fundamental group, in the sense that $\alpha(\gamma(I), \gamma(J)) = \alpha(I, J)$ for every $\gamma \in \pi_1(S)$ and every intervals $I, J \subset \partial_\infty \tilde{S}$.

Here an interval is allowed to be closed, semi-open or open according to whether it includes all, some, or none of its end points, respectively. We let $\mathcal{X}(S)$ denote the space of all cross-ratio functions.

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Most of the cross-ratio functions that we will consider in this article will be, in addition, *symmetric* in the sense that $\alpha(I, J) = \alpha(J, I)$ for every I and J . However, this property is not crucial.

If, in addition, a cross-ratio $\alpha \in \mathcal{X}(S)$ takes only non-negative values, then it is countably additive and consequently defines a $\pi_1(S)$ -invariant Radon measure on the space $\partial_\infty \tilde{S} \times \partial_\infty \tilde{S} - \Delta$, where Δ denotes the diagonal of the product. Such a measure is a *measure geodesic current*.

A fundamental example of cross-ratio function is associated to a complex structure m on the surface S . Then, the universal covering \tilde{S} is biholomorphically equivalent to the open unit disk $\mathbb{H}^2 \subset \mathbb{C}$, which provides an identification of $\partial_\infty \tilde{S}$ with the circle \mathbb{S}^1 bounding this disk, well-defined up to a linear fractional map preserving \mathbb{H}^2 . We can then consider the *Liouville geodesic current* L_m , defined by the property that

$$L_m(I, J) = \left| \log \frac{(a-c)(b-d)}{(a-d)(b-c)} \right|$$

if a, b and c, d are the end points of I and J , respectively. This example explains the terminology, and the connection with the classical cross-ratios. See [Bo₂, Theorem 13] for a characterization of which cross-ratios occur in this way. See also [Bo₁, Ot₁, Ot₂, Bo₃, La₁, La₂, LaMc] for various incarnations of cross-ratio functions.

This article is devoted to infinitesimal versions of these Liouville cross-ratios. Let $\mathcal{T}(S)$ be the *Teichmüller space* of S , considered as the space of isotopy classes of complex structures on S , and let $V \in T_{m_0} \mathcal{T}(S)$ be a vector tangent to $\mathcal{T}(S)$ at m_0 . If $t \mapsto m_t$ is a curve in $\mathcal{T}(S)$ passing through m_0 and tangent to V at $t = 0$, we can consider the derivative

$$L_V(I, J) = \frac{\partial}{\partial t} L_{m_t}(I, J)|_{t=0}$$

for every pair $I, J \subset \partial_\infty \tilde{S}$ of intervals with disjoint closures. It is fairly well-known that the derivative exists, and depends only on $V \in T_{m_0} \mathcal{T}(S)$, and not on the curve $t \mapsto m_t$ tangent to V . This L_V is the *infinitesimal Liouville cross-ratio* associated to $V \in T_{m_0} \mathcal{T}(S)$.

Such an infinitesimal Liouville cross-ratio does not induce a measure on $\partial_\infty \tilde{S} \times \partial_\infty \tilde{S} - \Delta$ any more. It has a weaker regularity property, in the sense that it only defines a *Hölder geodesic current*, namely a $\pi_1(S)$ -invariant linear functional on the space of Hölder continuous functions with compact support on $\partial_\infty \tilde{S} \times \partial_\infty \tilde{S} - \Delta$. See [BoSö, Ša₁, Ot₃].

The space $\mathcal{C}(S)$ of measure geodesic currents on S was introduced in [Bo₁] to construct a completion of the space of homotopy classes of weighted closed curves in S . A fundamental feature of this space is a continuous function

$$i: \mathcal{C}(S) \times \mathcal{C}(S) \rightarrow \mathbb{R},$$

which extends the geometric intersection function for closed curves. The proof that this geometric intersection function i is finite and continuous heavily depends on the regularity of measure geodesic currents.

For general cross-ratio functions $\alpha, \beta \in \mathcal{X}(S)$, it is formally possible to copy the above construction and attempt to define a *geometric intersection number* $i(\alpha, \beta)$. However, making sense of this intersection number amounts to proving the convergence of a certain infinite sum, and requires additional regularity hypotheses on the cross-ratios.

The main contribution of this article is the rigorous construction of intersection numbers for a class of cross-ratios which includes infinitesimal Liouville cross-ratios.

Given $\nu > 0$, a cross-ratio function $\alpha \in \mathcal{X}(S)$ is said to be ν -Hölder regular with respect to a complex structure $m_0 \in \mathcal{T}(S)$ if there exist constants $c_0, c_1 > 0$ such that

$$|\alpha(I, J)| \leq c_0 L_{m_0}(I, J)^\nu$$

for all intervals $I, J \subset \partial_\infty \tilde{S}$ with disjoint closure and such that $L_{m_0}(I, J) \leq c_1$; recall that L_{m_0} denotes the Liouville geodesic current associated to the complex structure m_0 . A Hölder regular cross-ratio function defines a Hölder geodesic current (compare [Ša₁]), but the converse is not true; for instance, a measure geodesic current with an atom (such as the one associated to a homotopy class of closed curves) is a Hölder geodesic current, but is not Hölder regular in the above sense.

Theorem 1. *If the cross-ratios $\alpha, \beta \in \mathcal{X}(S)$ are ν -Hölder regular with respect to some complex structure $m_0 \in \mathcal{T}(S)$ and for some $\nu > \frac{3}{4}$, then it is possible to define a geometric intersection number $i(\alpha, \beta)$, in a sense made precise by Theorem 13 below.*

The construction is based on a certain bundle over S with fiber the open annulus, it uses a covering of this bundle by “double boxes”, and it heavily relies on a relatively subtle growth estimate on the sizes of these double boxes. See §2–3.

A much easier property is that Theorem 1 can be applied to infinitesimal Liouville currents:

Proposition 2. *If $V \in T_{m_0} \mathcal{T}(S)$ is a vector tangent to the Teichmüller space at $m_0 \in \mathcal{T}(S)$, then the associated infinitesimal Liouville cross-ratio $L_V \in \mathcal{X}(S)$ is ν -Hölder regular with respect to m_0 for every $\nu < 1$.*

As a consequence, given two such tangent vectors $V, W \in T_{m_0} \mathcal{T}(S)$, we can make sense of the geometric intersection number $i(L_V, L_W)$ of their infinitesimal Liouville cross-ratios.

Theorem 3. *Let $t \mapsto m_t$ and $u \mapsto n_u$ be two differentiable curves in $\mathcal{T}(S)$, respectively tangent to the vectors V and $W \in T_{m_0} \mathcal{T}(S)$ at $m_0 = n_0 \in \mathcal{T}(S)$. Then,*

$$\frac{\partial}{\partial t} \frac{\partial}{\partial u} i(L_{m_t}, L_{n_u})|_{(t,u)=(0,0)} = i(L_V, L_W),$$

where $i(L_V, L_W)$ is the geometric intersection number provided by Theorem 1 and Proposition 2, and where $i(L_{m_t}, L_{n_u})$ is the classical intersection number of measure geodesic currents as in [Bo₁].

If we combine Theorem 3 with earlier work of Thurston and Wolpert [Wo₂], we automatically obtain:

Theorem 4. *Under the hypotheses of Theorem 1, the geometric intersection number $i(L_V, L_W)$ is equal, up to multiplication by a constant, to the scalar product $\omega_{\text{WP}}(V, W)$ of the tangent vectors $V, W \in T_{m_0}\mathcal{T}(S)$ under the Weil-Petersson metric of $\mathcal{T}(S)$. \square*

The constant depends on the topological type of the surface and on the conventions in the definition of the Weil-Petersson metric; see [Wo₂].

To a large extent, Theorems 1 and 3 complete the analogy between geometric intersection numbers and Weil-Petersson metric that was proposed in [Bo₂] (see also [Wo₃]). In particular, they provide a much more satisfactory framework than the clumsy construction of [Bo₂, §4], which had been designed to bypass the analytic subtleties caused by the lack of regularity of infinitesimal Liouville cross-ratios.

1. GEOMETRIC INTERSECTION NUMBERS

This section is mostly heuristic, and summarizes the definition of geometric intersection numbers in the case of measure geodesic currents.

It is conceptually convenient to endow the surface S with a complex structure m_0 . Then, the space $\partial_\infty\tilde{S} \times \partial_\infty\tilde{S} - \Delta$ has a natural identification with the space $G(\tilde{S})$ of oriented complete geodesics for the Poincaré metric of \tilde{S} , since such a geodesic joins two distinct points of the circle at infinity $\partial_\infty\tilde{S}$.

Let $DG(\tilde{S})$ be the *double geodesic space*, consisting of all pairs (g, h) of geodesics $g, h \in G(\tilde{S})$ which transversely meet at some point. Considering this intersection point $g \cap h$ and the tangent vectors of g and h at this point, we see that $DG(\tilde{S})$ can also be identified to the set of triples (\tilde{x}, v, w) consisting of a point $\tilde{x} \in \tilde{S}$ and of two distinct unit tangent vectors $v, w \in T_{\tilde{x}}\tilde{S}$ at \tilde{x} . This description makes it clear that the action of $\pi_1(S)$ on $DG(\tilde{S})$ is free and discontinuous, so that we can consider the quotient $DG(S) = DG(\tilde{S})/\pi_1(S)$.

Again, $DG(S)$ can be interpreted as the set of triples (x, v, w) consisting of a point $x \in S$ and of two distinct unit tangent vectors $v, w \in T_x S$ at x . In particular, it is a manifold of dimension 4. Note that it is non-compact, which is the major cause of the analytic problems that we will encounter.

Let a *box* in $G(\tilde{S}) \subset \partial_\infty\tilde{S} \times \partial_\infty\tilde{S}$ be a subset of the form $I \times J$, where I and J are intervals with disjoint closures in $\partial_\infty\tilde{S}$. A *double box* in $DG(\tilde{S}) \subset G(\tilde{S}) \times G(\tilde{S})$ is a subset of the form $B = B_1 \times B_2$ where B_1 and B_2 are two boxes of $G(\tilde{S})$ such that every geodesic $g \in B_1$ crosses every geodesic $h \in B_2$.

Finally, let a *double box* in the quotient $DG(S) = DG(\tilde{S})/\pi_1(S)$ be a subset B which is the image of a double box \tilde{B} in $DG(\tilde{S})$ small enough that $DG(\tilde{S}) \rightarrow DG(S)$ is injective on the closure of \tilde{B} , and consequently restricts to a homeomorphism $\tilde{B} \rightarrow B$.

Although the consideration of geodesics is convenient and more intuitive, the reader will notice that the spaces $G(\tilde{S})$, $DG(\tilde{S})$ and $DG(S)$, as well as the notions of boxes and double boxes, can be described without any reference to a complex structure m_0 on S .

For future reference, we note the following immediate property.

Lemma 5. *If B_1 and B_2 are two double boxes in $DG(S)$, their intersection $B_1 \cap B_2$ is a double box, and the complement $B_1 - B_2$ can be decomposed as the union of finitely many disjoint double boxes.* \square

Lemma 6. *The space $DG(S)$ can be decomposed as the union of a locally finite family of disjoint double boxes $\{B_i\}_{i \in I}$.*

Proof. It should be clear from definitions that every element of $DG(S)$ is contained in the interior of some double box. We can therefore write $DG(S)$ as the union of a locally finite family of double boxes. We can then arrange that these double boxes are disjoint by successive applications of Lemma 5. \square

A cross-ratio function $\alpha \in \mathcal{X}(S)$ associates a number $\alpha(B) = \alpha(I_1, I_2)$ to each box $B = I_1 \times I_2$ in $G(\tilde{S})$.

Two cross-ratio functions $\alpha, \beta \in \mathcal{X}(S)$ associate a number $\alpha \times \beta(B) = \alpha(B_1)\beta(B_2)$ to each double box $B = B_1 \times B_2$ in $DG(\tilde{S})$. Finally, if B is a double box in $DG(S)$ image of a double box $\tilde{B} \subset DG(\tilde{S})$, define $\alpha \times \beta(B) = \alpha \times \beta(\tilde{B})$. The invariance of α and β under the action of $\pi_1(S)$ guarantees that $\alpha \times \beta(B)$ depends only on α, β and B , and not on the double box \tilde{B} lifting B to $DG(\tilde{S})$.

We would like to define the *geometric intersection number* $i(\alpha, \beta)$ of the cross-ratios $\alpha, \beta \in \mathcal{X}(S)$ as the infinite sum

$$i(\alpha, \beta) = \sum_{i \in I} \alpha \times \beta(B_i)$$

for some decomposition $DG(S) = \bigcup_{i \in I} B_i$ as in Lemma 6.

When α and β are measure geodesic currents, this sum is proved to be (absolutely) convergent in [Bo₁, §4.2], and this for any decomposition of $DG(S)$ into disjoint double boxes.

However, for cross-ratio functions $\alpha, \beta \in \mathcal{X}(S)$, we need to find a scheme that provides a decomposition $DG(S) = \bigcup_{i \in I} B_i$ for which the above sum converges, and is independent of the decomposition of $DG(S)$ into double boxes provided by that scheme.

We actually will not quite carry out this program, and our construction will only use a locally finite decomposition into double boxes of a suitable *open dense subset* of $DG(S)$.

2. GOOD COVERINGS BY DOUBLE BOXES

In many of the estimates of the article, we say that the quantity X is *of order at most* Y , and we write $X \prec Y$, if there exists a constant $c > 0$ such that $X \leq cY$. We say that X is *of the same order as* Y , and we write $X \asymp Y$, if $X \prec Y$ and $Y \prec X$.

Choose a complex structure $m_0 \in \mathcal{T}(S)$ and a base point $\tilde{x}_0 \in \tilde{S}$. This defines a riemannian metric on the circle at infinity $\partial_\infty \tilde{S}$, where the distance between $\eta, \xi \in \partial_\infty \tilde{S}$ is defined as the angle between the Poincaré geodesics joining \tilde{x}_0 to η and ξ , respectively. Taking a different base point $\tilde{x}_0 \in \tilde{S}$ modifies this metric only up to bi-Lipschitz equivalence. However, if we change the complex structure $m_0 \in \mathcal{T}(S)$, the new metric is usually only bi-Hölder equivalent to the original one. We state this property for future reference.

Lemma 7. *On the circle at infinity $\partial_\infty \tilde{S}$, let d_0 be the metric induced as above by the choice of a complex structure $m_0 \in \mathcal{T}(S)$ and of a base point $\tilde{x}_0 \in \tilde{S}$, and let d_1 be similarly associated to $m_1 \in \mathcal{T}(S)$ and $\tilde{x}_1 \in \tilde{S}$. Then there exists $\nu \leq 1$ such that*

$$d_0(\xi, \xi')^{\frac{1}{\nu}} \prec d_1(\xi, \xi') \prec d_0(\xi, \xi')^\nu$$

for all $\xi, \xi' \in \partial_\infty \tilde{S}$.

In addition, ν tends to 1 as m_1 tends to m_0 in $\mathcal{T}(S)$. □

This riemannian metric on the circle at infinity $\partial_\infty \tilde{S}$ gives a riemannian metric on the geodesic space $G(\tilde{S}) \subset \partial_\infty \tilde{S} \times \partial_\infty \tilde{S}$, and therefore on the product $G(\tilde{S}) \times G(\tilde{S})$.

With this data, a relatively compact subset $X \subset G(\tilde{S}) \times G(\tilde{S})$ has *Minkowski m_0 -dimension* $\leq d$ if the volume of the ε -neighborhood U_ε of X in $G(\tilde{S}) \times G(\tilde{S})$ is such that

$$\text{vol}(U_\varepsilon) \prec \varepsilon^{4-d}$$

as $\varepsilon > 0$ is bounded above. This definition is clearly independent of the choice of base point $\tilde{x}_0 \in \tilde{S}$, but does depend on the complex structure $m_0 \in \mathcal{T}(S)$. The Minkowski dimension is also often called the *box counting dimension*, but this terminology would be here clumsy since we are already dealing with many types of boxes. Note that $d \leq 4$, and that a separating subset necessarily has Minkowski m_0 -dimension ≥ 3 .

Recall that the double geodesic space $DG(\tilde{S})$ consists of all pairs of geodesics $(g, h) \in G(\tilde{S}) \times G(\tilde{S})$ such that g and h transversely meet in one point.

A *subdivision scheme* for a subset $\Omega \subset DG(\tilde{S})$ consists of two families $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ such that:

- (1) each $I \in \mathcal{I}_n$ is an interval in the circle at infinity $\partial_\infty \tilde{S}$;
- (2) the family \mathcal{I}_{n+1} is obtained from \mathcal{I}_n by subdividing each interval $I \in \mathcal{I}_n$ into two intervals;
- (3) each $B \in \mathcal{B}_n$ is a double box $I_1 \times I_2 \times I_3 \times I_4$ in $DG(\tilde{S}) \subset (\partial_\infty \tilde{S})^4$ where I_1, I_2, I_3, I_4 are intervals of \mathcal{I}_n ;
- (4) $\Omega = \bigcup_{n=1}^{\infty} \bigcup_{B \in \mathcal{B}_n} B$;
- (5) any two double boxes $B \in \mathcal{B}_n$ and $B' \in \mathcal{B}_{n'}$ are disjoint.

The intervals $I \in \mathcal{I}_n$ can be closed, open or semi-open. In particular, in Condition (5), the closures of two boxes are allowed to have a non-trivial intersection. We actually will not worry much about the box boundaries, as they are irrelevant for the type of cross-ratio functions considered in the rest of the article.

Lemma 8. *Let Ω be an open subset of the double geodesic space $DG(\tilde{S}) \subset G(\tilde{S}) \times G(\tilde{S})$ that is relatively compact in $G(\tilde{S}) \times G(\tilde{S})$ and whose topological frontier $\delta\Omega$ in $G(\tilde{S}) \times G(\tilde{S})$ has Minkowski m_0 -dimension $\leq d$. Pick two numbers $0 < r \leq \frac{1}{2} \leq R < 1$. Then there exists a subdivision scheme $\{\mathcal{I}_n\}_{n \in \mathbb{N}}, \{\mathcal{B}_n\}_{n \in \mathbb{N}}$, such that:*

- (1) *the length of each interval $I \in \mathcal{B}_n$ is of order between r^n and R^n , for the metric on $\partial_\infty \tilde{S}$ defined by the complex structure m_0 and by a choice of base point in \tilde{S}_0 ;*
- (2) *each double box $B \in \mathcal{B}_n$ is at distance $\prec R^n$ from the complement of Ω in $G(\tilde{S}) \times G(\tilde{S})$;*
- (3) *the number of boxes in \mathcal{B}_n is $\prec r^{-dn}$.*

Proof. We will prove the result in the case where $r = R = \frac{1}{2}$, which of course implies the general case. (The result is stated in the above form for future reference).

Since Ω is relatively compact in $G(\tilde{S}) \times G(\tilde{S})$, there exists a finite family of boxes A_1, A_2, \dots, A_s in $G(\tilde{S})$ such that Ω is contained in the union of the products $A_i \times A_j$.

Each box A_i is of the form $A_i = I_i \times J_i$, where I_i and J_i are two disjoint intervals in the circle at infinity $\partial_\infty \tilde{S}$. Let \mathcal{I}_1 be the family of these finitely many intervals. By subdivision of the boxes A_i , we can arrange that these intervals are disjoint.

We now define the sequence \mathcal{I}_n by induction, where \mathcal{I}_1 is the family of the above intervals I_i, J_i , and where \mathcal{I}_{n+1} is obtained from \mathcal{I}_n by subdividing each interval into two intervals of equal lengths. In particular, the length of each $i \in \mathcal{I}_n$ is of order 2^{-n} .

Define the family \mathcal{B}_n to consist of all double boxes $B = I_1 \times I_2 \times I_3 \times I_4$ with all four I_i in \mathcal{I}_n , such that:

- (1) the double box B is contained in Ω ;

- (2) if $I'_i \in \mathcal{I}_{n-1}$ is the level $n-1$ interval that contains I_i , the double box $B' = I'_1 \times I'_2 \times I'_3 \times I'_4$ is not contained in Ω (so that $B' \notin \mathcal{B}_{n-1}$).

It is immediate from the construction that any two double boxes $B \in \mathcal{B}_n$ and $B' \in \mathcal{B}_{n'}$ are disjoint.

Also, the union $\bigcup_{n=1}^{\infty} \bigcup_{B \in \mathcal{B}_n} B$ is equal to Ω . Indeed, for every n , each pair of geodesics $(g, f) \in \Omega$ is contained in some double box $B = I_1 \times I_2 \times I_3 \times I_4$ with all four I_i in \mathcal{I}_n . This double box B will be contained in Ω for n large enough since Ω is open; this box will belong to \mathcal{B}_n for the first such n .

Therefore, $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ provide a subdivision scheme for Ω , and the lengths of the intervals of \mathcal{I}_n are of order 2^{-n} .

The construction also makes it clear that every $\tilde{B} \in \mathcal{B}_n$ is at distance $\prec 2^{-n}$ from the complement of Ω in $G(\tilde{S}) \times G(\tilde{S})$, and therefore at distance $\prec 2^{-n}$ from the frontier $\delta\Omega$ of Ω . Indeed, the level $n-1$ double box that contains \tilde{B} is not contained in Ω .

In particular, each $B \in \mathcal{B}_n$ is contained in the ϵ_n -neighborhood of the frontier $\delta\Omega$, where $\epsilon_n \asymp 2^{-n}$. By definition of Minkowski m_0 -dimension, the volume of this neighborhood is $\prec 2^{-(4-d)n}$. On the other hand, the volume of each $B \in \mathcal{B}_n$ is of order 2^{-4n} . Since the double boxes have disjoint interior, we conclude that the number of $B \in \mathcal{B}_n$ is of order at most 2^{dn} .

This concludes the proof of Lemma 8 when $r = R = \frac{1}{2}$, and therefore in the general case. \square

3. INTERSECTION NUMBER OF HÖLDER REGULAR CROSS-RATIOS

Now, consider two cross-ratio functions $\alpha, \beta \in \mathcal{X}(S)$ that are ν -Hölder regular with respect to m_0 , as defined in the introduction.

For an open subset $\Omega \subset DG(\tilde{S})$ satisfying the hypotheses of Lemma 8, let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ be the family of double boxes covering Ω provided by that statement. As in §1, each double box $B \in \mathcal{B}$ is the product $B = B_1 \times B_2$ of two boxes in the geodesic space $G(\tilde{S})$, and we can define $\alpha \times \beta(B) = \alpha(B_1)\beta(B_2)$ and

$$i_{\Omega}(\alpha, \beta) = \sum_{B \in \mathcal{B}} \alpha \times \beta(B).$$

Lemma 9. *Under the hypotheses of Lemma 8, let $\alpha, \beta \in \mathcal{X}(S)$ be ν -Hölder regular with respect to m_0 . If $\nu > \frac{d \log r}{4 \log R}$ and if $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is the family of double boxes provided by Lemma 8, the sum*

$$i_{\Omega}(\alpha, \beta) = \sum_{B \in \mathcal{B}} \alpha \times \beta(B).$$

is (absolutely) convergent.

Proof. For a box $B \subset G(\tilde{S})$ containing at least one geodesic of Ω (so that it is at bounded distance from the base point of \tilde{S}), the Liouville mass $L_{m_0}(B)$ is of the same order as the product of its side lengths. Therefore, if the double

box $B = B_1 \times B_2$ is in \mathcal{B}_n ,

$$\alpha \times \beta(B) = \alpha(B_1)\beta(B_2) \prec L_{m_0}(B_1)^\nu L_{m_0}(B_2)^\nu \prec R^{4\nu n}$$

because B_1 and B_2 have side lengths of order $\prec R^n$ and meet a fixed compact subset of $G(\tilde{S})$, namely the union of the images of Ω under the two projections $G(\tilde{S}) \times G(\tilde{S}) \rightarrow G(\tilde{S})$.

As a consequence, since the number of elements of \mathcal{B}_n is of order at most r^{-dn} ,

$$\sum_{B \in \mathcal{B}} \alpha \times \beta(B) = \sum_{n=1}^{\infty} \sum_{B \in \mathcal{B}_n} \alpha \times \beta(B) \prec \sum_{n=1}^{\infty} r^{-dn} R^{4\nu n} < \infty$$

if $\nu > \frac{d}{4} \frac{\log r}{\log R}$. □

Lemma 10. *Under the hypotheses of Lemma 9, suppose in addition that $\nu > 2 \left(\frac{\log r}{\log R} - 1 \right) + \frac{d}{4}$. Then the sum*

$$i_\Omega(\alpha, \beta) = \sum_{B \in \mathcal{B}} \alpha \times \beta(B).$$

is independent of the subdivision scheme providing the family of double boxes \mathcal{B} .

Proof. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ and $\mathcal{B}' = \bigcup_{n=1}^{\infty} \mathcal{B}'_n$ be two families of double boxes as in Lemma 8, respectively associated to families $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{I}'_n\}_{n \in \mathbb{N}}$ of intervals in $\partial_\infty \tilde{S}$.

For $k \leq n$, the subdivision scheme enables us to decompose each double box $B \in \mathcal{B}_k$ into 16^{n-k} double boxes $B' = I_1 \times I_2 \times I_3 \times I_4$ with all $I_i \in \mathcal{I}_n$. In particular, the side lengths of these new boxes are of order between r^n and R^n . Let $\mathcal{B}_{k,n}$ be the family of double boxes so obtained. Similarly define a family $\mathcal{B}'_{k,n}$ by subdividing each double box $B' \in \mathcal{B}'_k$ into 16^{n-k} double boxes whose side lengths are of order between r^n and R^n . Consider the families $\mathcal{C}_n = \bigcup_{k=1}^n \mathcal{B}_{k,n}$ and $\mathcal{C}'_n = \bigcup_{k=1}^n \mathcal{B}'_{k,n}$ of all double boxes so created.

Because their lengths are of order between r^n and R^n , each interval in \mathcal{I}_n meets at most $\prec \frac{R^n}{r^n}$ intervals of \mathcal{I}'_n , and conversely each interval in \mathcal{I}'_n meets $\prec \frac{R^n}{r^n}$ intervals of \mathcal{I}_n . We can therefore subdivide the double boxes of \mathcal{C}_n and \mathcal{C}'_n into a common family \mathcal{D}_n of disjoint double boxes, in such a way that each double box of \mathcal{C}_n and \mathcal{C}'_n is the union of a number $\prec \frac{R^{4n}}{r^{4n}}$ of double boxes of \mathcal{D}_n .

We split the family \mathcal{D}_n into three disjoint families $\mathcal{D}_n^{(0)}$, $\mathcal{D}_n^{(1)}$ and $\mathcal{D}_n^{(2)}$, where $\mathcal{D}_n^{(0)}$ consists of those $D \in \mathcal{D}_n$ that are contained in both $\bigcup_{C \in \mathcal{C}_n} C$ and $\bigcup_{C' \in \mathcal{C}'_n} C'$, where $\mathcal{D}_n^{(1)}$ consists of those D that are contained in $\bigcup_{C \in \mathcal{C}_n} C$ but not in $\bigcup_{C' \in \mathcal{C}'_n} C'$, and where $\mathcal{D}_n^{(2)}$ consists of those D that are contained in $\bigcup_{C' \in \mathcal{C}'_n} C'$ but not in $\bigcup_{C \in \mathcal{C}_n} C$.

We now use Condition (3) of Lemma 8, to show that the number of elements of $\mathcal{D}_n^{(1)}$ is smaller than one could have expected. Indeed, by definition, such a double box $D \in \mathcal{D}_n^{(1)}$ is contained in some double box $C \in \mathcal{C}_n$ and not in $\bigcup_{C' \in \mathcal{C}'_n} C' = \bigcup_{k=1}^n \bigcup_{B' \in \mathcal{B}'_k} B'$. It therefore meets some double box $B' \in \mathcal{B}'_m$ with $m > n$. By Condition (3) of Lemma 8, the double box C consequently is at distance $\prec R^n$ from the complement of Ω , and is therefore contained in an ϵ_n -neighborhood of the boundary $\delta\Omega$ with $\epsilon_n \asymp R^n$ since its diameter is $\prec R^n$. By a volume estimate, it follows that there can be at most $\prec \frac{R^{(4-d)n}}{r^{4n}}$ such double boxes $C \in \mathcal{C}_n$ containing a double box $D \in \mathcal{D}_n^{(1)}$. Since each $C \in \mathcal{C}_n$ contains at most $\prec \frac{R^{4n}}{r^{4n}}$ double boxes of \mathcal{D}_n , it follows that the number of elements of $\mathcal{D}_n^{(1)}$ is of order at most $\frac{R^{4n}}{r^{4n}} \frac{R^{(4-d)n}}{r^{4n}} = \frac{R^{(8-d)n}}{r^{8n}}$.

The same argument shows that $\mathcal{D}_n^{(2)}$ has $\prec \frac{R^{(8-d)n}}{r^{8n}}$ elements.

We are now ready to complete the proof of Lemma 10. To show that the two families of double boxes \mathcal{B} and \mathcal{B}' give the same value for $i_\Omega(\alpha, \beta)$, we need to prove that $\sum_{B \in \mathcal{B}} \alpha \times \beta(B) = \sum_{B' \in \mathcal{B}'} \alpha \times \beta(B')$, and therefore that

$$\sum_{n=1}^{\infty} \sum_{B \in \mathcal{B}_n} \alpha \times \beta(B) = \sum_{n=1}^{\infty} \sum_{B' \in \mathcal{B}'_n} \alpha \times \beta(B').$$

Consider a partial sum of the first series. By finite additivity of α and β ,

$$\begin{aligned} \sum_{k=1}^n \sum_{B \in \mathcal{B}_k} \alpha \times \beta(B) &= \sum_{C \in \mathcal{C}_n} \alpha \times \beta(C) \\ &= \sum_{D \in \mathcal{D}_n^{(0)}} \alpha \times \beta(D) + \sum_{D \in \mathcal{D}_n^{(1)}} \alpha \times \beta(D) \end{aligned}$$

since every double box of \mathcal{B}_k is the union of finitely many boxes of \mathcal{C}_n , and since every double box of \mathcal{C}_n is the union of finitely many boxes of $\mathcal{D}_n^{(0)}$ and $\mathcal{D}_n^{(1)}$.

Similarly,

$$\begin{aligned} \sum_{k=1}^n \sum_{B' \in \mathcal{B}'_k} \alpha \times \beta(B') &= \sum_{C' \in \mathcal{C}'_n} \alpha \times \beta(C') \\ &= \sum_{D \in \mathcal{D}_n^{(0)}} \alpha \times \beta(D) + \sum_{D \in \mathcal{D}_n^{(2)}} \alpha \times \beta(D). \end{aligned}$$

Taking the difference between these partial sums,

$$\begin{aligned} \left| \sum_{k=1}^n \sum_{B \in \mathcal{B}_k} \alpha \times \beta(B) - \sum_{k=1}^n \sum_{B' \in \mathcal{B}'_k} \alpha \times \beta(B') \right| \\ \leq \sum_{D \in \mathcal{D}_n^{(1)}} |\alpha \times \beta(D)| + \sum_{D \in \mathcal{D}_n^{(2)}} |\alpha \times \beta(D)| \\ \prec \frac{R^{(8-d)n}}{r^{8n}} R^{4\nu n} = \left(R^{8-d+4\nu} r^{-8} \right)^n \end{aligned}$$

since each $D = B_1 \times B_2 \in \mathcal{D}_n$ has side lengths $\prec R^n$, so that $\alpha(B_1) \prec L_{m_0}(B_1)^\nu \prec R^{2\nu n}$ and a similar estimate holds for $\beta(B_2)$.

Letting n go to infinity and using the hypothesis that $\nu > 2 \left(\frac{\log r}{\log R} - 1 \right) + \frac{d}{4}$, we conclude that

$$\sum_{k=1}^{\infty} \sum_{B \in \mathcal{B}_k} \alpha \times \beta(B) = \sum_{k=1}^{\infty} \sum_{B' \in \mathcal{B}'_k} \alpha \times \beta(B')$$

as required. \square

We are now going to apply this to a specific domain Ω .

Lemma 11. *For every complex structure m_0 on S and every $d > 3$, there exists an open subset $\Omega \subset DG(\tilde{S})$ such that*

- (1) Ω is relatively compact in $G(\tilde{S}) \times G(\tilde{S})$;
- (2) the quotient map $DG(\tilde{S}) \rightarrow DG(S) = DG(\tilde{S})/\pi_1(S)$ is injective on Ω , and sends Ω to an open dense set of $DG(S)$;
- (3) there exists finitely many $\gamma \in \pi_1(S)$ such that $\Omega \cap \gamma(\Omega) \neq \emptyset$;
- (4) the frontier $\delta\Omega$ of Ω in $G(\tilde{S}) \times G(\tilde{S})$ has Minkowski m_0 -dimension $< d$.

Namely, Ω is a fundamental domain for the action of $\pi_1(S)$ on $DG(\tilde{S})$, whose frontier has small dimension.

Proof. The complex structure m_0 defines a projection $p: DG(\tilde{S}) \rightarrow \tilde{S}$, which to a double geodesic $(g, h) \in DG(\tilde{S})$ associates the intersection point $g \cap h \in \tilde{S}$. (Note that this projection map depends on m_0).

Let ω be a compact fundamental polygon for the action of $\pi_1(S)$ on the universal cover \tilde{S} , bounded by a piecewise differentiable curve. Let Ω be the preimage of the interior of ω under p . The first three conclusions of the statement clearly hold.

The frontier $\delta\Omega$ of Ω in $G(\tilde{S}) \times G(\tilde{S})$ is the union of $p^{-1}(\delta\omega)$ and of the subset of the diagonal consisting of those $(h, h) \in G(\tilde{S}) \times G(\tilde{S})$ such that h meets ω . Since the boundary of ω is piecewise differentiable, it follows that the frontier $\delta\Omega$ has Minkowski m_0 -dimension 3, which is less than d by hypothesis. \square

Lemma 12. *Given numbers $d > 3$ and $0 < r \leq \frac{1}{2} \leq R < 1$, consider two cross-ratios $\alpha, \beta \in \mathcal{X}(S)$ that are ν -Hölder regular with respect to m_0 for $\nu > \max\{\frac{d}{4} \frac{\log r}{\log R}, 2\left(\frac{\log r}{\log R} - 1\right) + \frac{d}{4}\}$. Let $\Omega \subset DG(\tilde{S})$ be provided by Lemma 11, and consider the sum*

$$i_\Omega(\alpha, \beta) = \sum_{B \in \mathcal{B}} \alpha \times \beta(B)$$

as in Lemmas 9 and 10. This number $i_\Omega(\alpha, \beta)$ is independent of the choice of Ω .

Proof. Let Ω' be another domain as in Lemma 11. By Condition (3) of Lemma 11, there are finitely many elements $\gamma_1, \dots, \gamma_t \in \pi_1(S)$ such that Ω is covered by the union of the $\gamma_s(\Omega')$. Considering the domains $\Omega_s = \Omega \cap \gamma_s(\Omega')$, we first prove that

$$i_\Omega(\alpha, \beta) = \sum_{s=1}^t i_{\Omega_s}(\alpha, \beta).$$

For this, consider the subdivision scheme $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$, $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ for Ω provided by Lemma 8.

We then use the same interval family $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ to create a subdivision scheme $\{\mathcal{B}_n''\}_{n \in \mathbb{N}}$ for $\Omega'' = \bigcup_{s=1}^t \Omega_s$. More precisely, inductively define the family \mathcal{B}_n'' to consist of all double boxes $B'' = I_1'' \times I_2'' \times I_3'' \times I_4''$ with all four I_i'' in \mathcal{I}_n , such that:

- (1) the double box B'' is contained in Ω'' ;
- (2) if $I_i \in \mathcal{I}_{n-1}$ is the level $n-1$ interval that contains I_i'' , the double box $B = I_1 \times I_2 \times I_3 \times I_4$ is not contained in Ω'' .

As in the proof of Lemma 10, let \mathcal{C}_n be the family of all double boxes $B = I_1 \times I_2 \times I_3 \times I_4$, with all four I_i in \mathcal{I}_n , that are contained in Ω , and let $\mathcal{C}_n'' \subset \mathcal{C}_n$ be similarly associated to Ω'' . Then, by finite additivity,

$$\sum_{k=1}^n \sum_{B \in \mathcal{B}_k} \alpha \times \beta(B) = \sum_{C \in \mathcal{C}_n} \alpha \times \beta(C)$$

so that

$$i_\Omega(\alpha, \beta) = \lim_{n \rightarrow \infty} \sum_{C \in \mathcal{C}_n} \alpha \times \beta(C).$$

Similarly,

$$\sum_{s=1}^t i_{\Omega_s}(\alpha, \beta) = i_{\Omega''}(\alpha, \beta) = \lim_{n \rightarrow \infty} \sum_{C'' \in \mathcal{C}_n''} \alpha \times \beta(C'').$$

Each double box $C \in \mathcal{C}_n$ has side lengths of order between r^n and R^n , so that its contribution $\alpha \times \beta(C)$ to the above sums is bounded in absolute value by $R^{4\nu n}$. Also, the complement $\mathcal{C}_n - \mathcal{C}_n''$ consists of those $C \in \mathcal{C}_n$ which meet the union of the frontiers $\delta\Omega_s$. Since these frontier have Minkowski

m_0 -dimension $< d$, the now usual volume arguments show that the cardinal of $\mathcal{C}_n - \mathcal{C}_n''$ is bounded by $\frac{R^{(4-d)n}}{r^{4n}}$. Therefore,

$$\begin{aligned} \left| \sum_{C \in \mathcal{C}_n} \alpha \times \beta(C) - \sum_{C'' \in \mathcal{C}_n''} \alpha \times \beta(C'') \right| &\leq \sum_{C \in \mathcal{C}_n - \mathcal{C}_n''} |\alpha \times \beta(C)| \\ &\leq \frac{R^{(4-d)n}}{r^{4n}} R^{4\nu n}. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we conclude that

$$i_\Omega(\alpha, \beta) = i_{\Omega'}(\alpha, \beta) = \sum_{s=1}^t i_{\Omega_s}(\alpha, \beta)$$

since $R^{4-d+4\nu}r^{-4} < 1$ by our hypothesis on ν .

Considering the domains $\gamma_s^{-1}(\Omega_s)$ in Ω' , the same argument shows that

$$i_{\Omega'}(\alpha, \beta) = \sum_{s=1}^t i_{\gamma_s^{-1}(\Omega_s)}(\alpha, \beta).$$

In addition, because the cross-ratio functions α and β are invariant under the action of the fundamental group,

$$i_{\Omega_s}(\alpha, \beta) = i_{\gamma_s^{-1}(\Omega_s)}(\alpha, \beta)$$

for every s . (Note that γ_s distorts the metric induced by the complex structure m_0 on the circle at infinity $\partial_\infty \tilde{S}$ by a uniformly bounded Lipschitz factor, so that all estimates are preserved.)

It follows that $i_\Omega(\alpha, \beta) = i_{\Omega'}(\alpha, \beta)$. \square

This proves Theorem 1, in the following form.

Theorem 13. *The above construction provides a well-defined geometric intersection number $i(\alpha, \beta)$ for any two cross-ratio functions $\alpha, \beta \in \mathcal{X}(S)$ that are ν -Hölder regular with respect to the complex structure $m_0 \in \mathcal{T}(S)$ and for some $\nu > \frac{3}{4}$.*

A priori, *this intersection number may depend on the complex structure m_0 with respect to which α and β are ν -Hölder regular. However, it is a locally constant function of $m_0 \in \mathcal{T}(S)$.*

Proof. Pick numbers $d < 4$ and $0 < r < \frac{1}{2} < R < 1$ sufficiently close to 3 and $\frac{1}{2}$, respectively, that $\nu > \max\left\{\frac{d}{4} \frac{\log r}{\log R}, 2 \left(\frac{\log r}{\log R} - 1\right) + \frac{d}{4}\right\}$.

Choose a domain $\Omega \subset DG(\tilde{S})$ as in Lemma 11, and consider the family $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$ of double boxes in Ω provided by Lemma 8. Then define

$$i(\alpha, \beta) = i_\Omega(\alpha, \beta) = \sum_{B \in \mathcal{B}} \alpha \times \beta(B).$$

Lemmas 9, 10 and 12 show that this sum converges, and is independent of choices.

In addition, by Lemma 7, the estimates are stable under small perturbation of the complex structure m_0 . This guarantees the invariance under small perturbation of the complex structure m_0 . (The introduction of the numbers r , R and d in the various statements were specially designed to guarantee this.) \square

4. THE GEOMETRIC ESTIMATE

This section is devoted to the proof of Proposition 2, which says that infinitesimal Liouville cross-ratios are ν -Hölder regular for every $\nu < 1$. It will enable us to apply Theorem 13 to infinitesimal Liouville cross-ratios.

Let $V \in T_{m_0}\mathcal{T}(S)$ be a tangent vector to the Teichmüller space $\mathcal{T}(S)$ at the point m_0 . The infinitesimal deformation associated with V is described by an equivalence class $[\mu]$ of Beltrami coefficients, where the Beltrami coefficients μ and μ' are equivalent if and only if

$$\int_S \mu \varphi = \int_S \mu' \varphi$$

for all holomorphic quadratic differentials φ on S . We then define the *Teichmüller norm* $\|V\|$ of the tangent vector $V \in T_{m_0}\mathcal{T}(S)$ as

$$\|V\| = \min_{\mu} \|\mu\|_{\infty},$$

where the infimum is taken over all Beltrami differentials μ representing V .

Lemma 14. *There exists a universal constant $c_0 > 0$ such that*

$$|L_V(I, J)| \leq c_0 \|V\| L_{m_0}(I, J) |\log L_{m_0}(I, J)|$$

for all intervals $I, J \subset \partial_{\infty}\tilde{S}$ with disjoint closure such that $L_{m_0}(I, J) \leq \frac{1}{2}$.

To simplify the notation, set $L = L_{m_0}(I, J)$.

Proof. For the complex structure m_0 , biholomorphically identify the universal cover \tilde{S} to the upper half-plane $\mathbb{H}^2 \subset \mathbb{C}$. By invariance of the requested estimate under biholomorphic transforms of \mathbb{H}^2 , we can further arrange that $I = [-e^{\alpha}, -1]$ and $J = [1, e^{\alpha}]$ for some $\alpha > 0$. An easy computation then shows that $L_{m_0}(I, J) = 2 \log(\cosh \frac{\alpha}{2}) \asymp \alpha^2$.

The first variation formula for the solution of the Beltrami equation with coefficient $t\mu$ yields the formula (see [W₀₁])

$$L_V([a, b], [c, d]) = -\frac{2}{\pi} \operatorname{Re} \int_{\mathbb{H}^2} \mu(z) \frac{(a-b)(c-d)}{(z-a)(z-b)(z-c)(z-d)} dx dy$$

where $z = x + iy$.

In our situation, this provides a bound

$$|L_V(I, J)| \prec \|\mu\|_{\infty} \alpha^2 \int_{\mathbb{H}^2} \frac{dx dy}{|(z+1)(z+e^{\alpha})(z-1)(z-e^{\alpha})|},$$

since the intervals I and J have length $\asymp \alpha$.

An integration along the lines of [Ša₁, page 448] then gives

$$\int_{\mathbb{H}^2} \frac{dx dy}{|(z+1)(z+e^\alpha)(z-1)(z-e^\alpha)|} \prec |\log \alpha| \asymp |\log L_{m_0}(I, J)|$$

Combining these two inequalities, taking the infimum over all Beltrami coefficients μ representing V , and observing that the constants hidden in the symbols \prec and \asymp are universal completes the proof. \square

Lemma 14 proves the Hölder regularity property of Proposition 2.

5. PROOF OF THEOREM 3

We restate Theorem 3 for the convenience of the reader.

Theorem 15. *Let $t \mapsto m_t$ and $u \mapsto n_u$ be two differentiable curves in $\mathcal{T}(S)$, respectively tangent to the vectors V and $W \in T_{m_0}\mathcal{T}(S)$ at $m_0 = n_0$. Consider the associated Liouville geodesic currents L_{m_t} and L_{n_u} , and the infinitesimal Liouville currents $L_V = \frac{d}{dt}L_{m_t}|_{t=0}$ and $L_W = \frac{d}{du}L_{n_u}|_{u=0}$. Then,*

$$\frac{\partial}{\partial t} \frac{\partial}{\partial u} i(L_{m_t}, L_{n_u})|_{(t,u)=(0,0)} = i(L_V, L_W),$$

where $i(L_V, L_W)$ is the geometric intersection number provided by Theorem 1 and Proposition 2, and where $i(L_{m_t}, L_{n_u})$ is the classical intersection number of measure geodesic currents as in [Bo₁].

Proof. As in the construction of intersection numbers in the proof of Theorem 3, pick numbers $d > 3$, $0 < r < \frac{1}{2} < R < 1$ and $\nu < 1$ such that $\nu > \max\{\frac{d}{4} \frac{\log r}{\log R}, 2\left(\frac{\log r}{\log R} - 1\right) + \frac{d}{4}\}$. For every $\alpha, \beta \in \mathcal{X}(S)$ that are ν -Hölder regular with respect to m_0 , we can then define the intersection number

$$i(\alpha, \beta) = \sum_{B \in \mathcal{B}} \alpha \times \beta(B)$$

as in the proof of Theorem 3.

Lemma 7 and Proposition 2 show that, for t, t', u, u' sufficiently close to 0, the cross-ratio functions $L_{m_t}, L_{n_u}, \frac{\partial}{\partial t}L_{m_t}$ and $\frac{\partial}{\partial u}L_{n_u}$ are ν -Hölder regular with respect to the complex structures $m_{t'}$ and $n_{u'}$. Similarly, the family $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_n$ of double boxes satisfy the size estimates of Lemma 8 with respect to the complex structures $m_{t'}$ and $n_{u'}$.

For a given double box $B = B_1 \times B_2 \in \mathcal{B}$, with $B_1, B_2 \subset G(\tilde{S})$, write

$$\begin{aligned} \frac{\partial}{\partial u}(L_{m_t} \times L_{n_u}(B)) &= \frac{\partial}{\partial u}(L_{m_t}(B_1) \times L_{n_u}(B_2)) = L_{m_t}(B_1) \times \left(\frac{\partial}{\partial u}L_{n_u}(B_2)\right) \\ &= L_{m_t} \times \left(\frac{\partial}{\partial u}L_{n_u}\right)(B) \end{aligned}$$

Summing over all boxes $B \in \mathcal{B}$ and using the fact that all convergence estimates are uniform in t ,

$$\frac{\partial}{\partial u} i(L_{m_t}, L_{n_u}) = \sum_{B \in \mathcal{B}} L_{m_t} \times \left(\frac{\partial}{\partial u} L_{n_u} \right) (B) = i \left(L_{m_t}, \frac{\partial}{\partial u} L_{n_u} \right).$$

Iterating this argument (and again using the uniform convergence estimates) then gives

$$\frac{\partial}{\partial t} \frac{\partial}{\partial u} i(L_{m_t}, L_{n_u}) = \sum_{B \in \mathcal{B}} \left(\frac{\partial}{\partial t} L_{m_t} \right) \times \left(\frac{\partial}{\partial u} L_{n_u} \right) (B) = i \left(\frac{\partial}{\partial t} L_{m_t}, \frac{\partial}{\partial u} L_{n_u} \right)$$

for every (t, u) sufficiently close to $(0, 0)$, and in particular for $(t, u) = (0, 0)$. \square

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F.B.: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES CA 90089-2532, U.S.A.

E-mail address: `fbonahon@math.usc.edu`

D.S.: DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE, CITY UNIVERSITY OF NEW YORK, FLUSHING NY 11367, U.S.A.

D.S.: MATHEMATICS PH.D. PROGRAM, THE CUNY GRADUATE CENTER, 365 FIFTH AVENUE, ROOM 4208, NEW YORK NY 10016-4309, U.S.A.

E-mail address: `dragomir.saric@qc.cuny.edu`