UNIFORM WEAK* TOPOLOGY AND EARTHQUAKES IN THE HYPERBOLIC PLANE

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Abstract. We prove that the bijective correspondence between the space of bounded measured laminations $\mathcal{ML}_b(\mathbb{H})$ and the universal Teichmüller space $T(\mathbb{H})$ given by $\lambda \mapsto E^\lambda|_{S^1}$ is a homeomorphism for the uniform weak* topology on $\mathcal{ML}_b(\mathbb{H})$ and the Teichmüller topology on $T(\mathbb{H})$, where E^λ is an earthquake with earthquake measure λ. A corollary is that earthquakes with discrete earthquake measures are dense in $T(\mathbb{H})$. We also establish infinitesimal versions of the above results.

1. Introduction

A Riemann surface is said to be *hyperbolic* if its universal covering is the hyperbolic plane $\mathbb{H}$. [1] The ideal boundary $\partial\mathbb{H}$ of the hyperbolic plane $\mathbb{H}$ is homeomorphic to the unit circle S^1. The Teichmüller space $T(\mathbb{H})$ of the hyperbolic plane $\mathbb{H}$, called the *universal Teichmüller space*, is the space of all quasisymmetric maps of the unit circle S^1 modulo post-composition by Möbius maps which preserve $\mathbb{H}$. There is a natural complex analytic embedding of the Teichmüller space of any hyperbolic Riemann surface into the universal Teichmüller space $T(\mathbb{H})$ (see [7]).

Earthquake maps in the hyperbolic plane $\mathbb{H}$ (and on any hyperbolic Riemann surface) were introduced by Thurston [21]. An earthquake in the hyperbolic plane is a bijective map $E : \mathbb{H} \to \mathbb{H}$ which is *supported* on a geodesic lamination $\mathcal{L}$ in $\mathbb{H}$ in the sense that it is a hyperbolic isometry on each *stratum* (i.e. a leaf of $\mathcal{L}$ or a component of $\mathbb{H} \setminus \mathcal{L}$) of $\mathcal{L}$, and which (relatively) translates to the left points of different strata of $\mathcal{L}$. An earthquake $E : \mathbb{H} \to \mathbb{H}$ continuously extends to a homeomorphism of S^1 and it induces a transverse Borel measure to its support lamination $\mathcal{L}$, called the *earthquake measure*. In particular, the earthquake measure of E is a measured lamination whose support is $\mathcal{L}$ and it measures the amount of the relative movement to the left by E. An earthquake measure λ uniquely determines earthquake $E^\lambda : \mathbb{H} \to \mathbb{H}$ up to post-composition by Möbius maps.

Thurston [21] proved that any homeomorphism of the unit circle S^1 is obtained as the continuous extension of an earthquake in $\mathbb{H}$ to its boundary S^1. In other words, any homeomorphism of S^1 can be geometrically constructed as the continuous extension to the boundary S^1 of a piecewise isometry of $\mathbb{H}$ which moves strata of its support geodesic lamination to the left by the amount given by a transverse Borel measure to the lamination. However, the relationship between homeomorphisms and earthquake measures of the earthquakes inducing them is not a simple one.

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[1] We are particularly interested in the geometrically infinite hyperbolic Riemann surfaces, e.g. the hyperbolic plane $\mathbb{H}$, an infinite genus surface, a surface with an interval of ideal boundary points. All these surfaces have infinite hyperbolic area.

This paper is mainly concerned with the dependence of the earthquake measures on homeomorphisms of S^1.

A measured lamination λ of the hyperbolic plane $\mathbb{H}$ is said to be *bounded* if

$$\sup_I \lambda(I) < \infty$$

where the supremum is over all geodesic arcs I of unit length that transversely intersect the support of λ. Then a homeomorphism is quasisymmetric if and only if $h = E^\lambda|_{S^1}$ for a bounded earthquake measure λ (see [8], [13] and [15]).

We denote by $ML_b(\mathbb{H})$ the space of all bounded measured laminations. The above statement gives a well-defined *earthquake measure map*

$$\mathcal{EM} : T(\mathbb{H}) \to \mathcal{ML}_b(\mathbb{H})$$

by $\mathcal{EM}([h]) = \lambda$, where the quasisymmetric map h is continuous extension to S^1 of the earthquake E^λ with the earthquake measure λ. The earthquake measure map is a bijection by the above. Our main result establishes a natural topology on $\mathcal{ML}_b(\mathbb{H})$, called *uniform weak* topology* for which $\mathcal{EM}$ is a homeomorphism.

If $\mathcal{ML}_b(\mathbb{H})$ is given the weak* topology, then $\mathcal{EM}^{-1} : \mathcal{ML}_b(\mathbb{H}) \to T(\mathbb{H})$ is discontinuous. The problem is that the Teichmüller topology on $T(\mathbb{H})$ is uniform on the quadruples of points in S^1 of a fixed cross-ratio, while the weak* topology measures with respect to finitely many quadruples at one time. The remedy is to pull-back, by hyperbolic isometries, measures from all quadruples of a certain size to a fixed (standard) quadruple of the same size and to require convergence of all pull-backs simultaneously. The requirement for the quadruples to be of the same size is essential because a uniform convergence on quadrilaterals of all sizes makes the topology on $\mathcal{ML}_b(\mathbb{H})$ too large. In this case the map $\mathcal{EM} : T(\mathbb{H}) \to \mathcal{ML}_b(\mathbb{H})$ would be discontinuous. This leads to our definition of a uniform weak* topology in Section 5.

Our main result makes a connection between the uniform weak* topology and earthquake maps in the hyperbolic plane $\mathbb{H}$. Namely, we show that the uniform weak* topology on $\mathcal{ML}_b(\mathbb{H})$ is capturing the subtleties of the Teichmüller topology on $T(\mathbb{H})$ and the earthquake maps in the hyperbolic plane $\mathbb{H}$.

Theorem 1 (Earthquake measure map is a homeomorphism)**.** *The earthquake measure map*

$$\mathcal{EM} : T(\mathbb{H}) \to \mathcal{ML}_b(\mathbb{H})$$

is a homeomorphism for the Teichmüller topology of $T(\mathbb{H})$ *and the uniform weak* topology on* $\mathcal{ML}_b(\mathbb{H})$.

The above theorem also holds for any geometrically infinite Riemann surfaces by simply noting that a quasisymmetric map which is invariant under a Fuchsian group is induced by an earthquake whose earthquake measure is invariant under the same Fuchsian group. In the case of a closed hyperbolic surface S, Kerckhoff [11] showed that the earthquake measure map is a homeomorphism for the weak* topology on $ML(S)$. Using the techniques in the paper, it is easy to prove that $\mathcal{EM} : M\ddot{o}b(\mathbb{H})/Homeo(S^1) \to ML(\mathbb{H})$ is a homeomorphism for the topology of pointwise convergence on the space of homeomorphisms $Homeo(S^1)$ of S^1 and the weak* topology on the (not necessarily bounded) measured laminations $ML(\mathbb{H})$ of $\mathbb{H}$, where $M\ddot{o}b(\mathbb{H})$ are Möbius maps that preserve $\mathbb{H}$. We note that the weak* topology on $\mathcal{ML}_b(\mathbb{H})$ is strictly weaker than the uniform weak* topology.

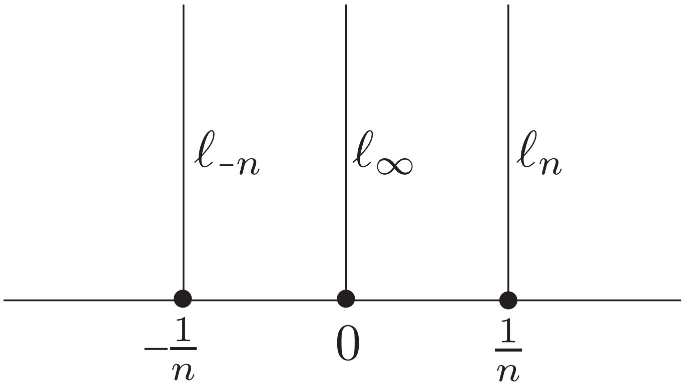

FIGURE 1. $\lambda_n \nrightarrow \lambda$ in the uniform weak* topology.

To illustrate the difference between the weak* topology and the uniform weak* topology on $\mathcal{ML}_b(\mathbb{H})$ we consider the following example. Identify the hyperbolic plane $\mathbb{H}$ with the upper half-plane and its boundary $\partial\mathbb{H}$ with $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Let l be the vertical line connecting 0 and ∞, and let l_n be the vertical line connecting $\frac{1}{n}$ and ∞. Both l and l_n are geodesics in $\mathbb{H}$. Let δ_l and δ_{l_n} denote the measured laminations in $\mathbb{H}$ with supports l and l_n and weights 1 (Figure 1). Then $\frac{\delta_{l_n}+\delta_{l_{-n}}}{2}$ converges in the weak* topology to δ_l as $n \to \infty$, but it does *not* converge in the uniform weak* topology (cf. §6.1). Let E_n be an earthquake whose earthquake measure is $\frac{\delta_{l_n}+\delta_{l_{-n}}}{2}$ and let E be an earthquake whose earthquake measure is δ_l. Then $E_n|_{S^1}$ pointwise converges to $E|_{S^1}$ as $n \to \infty$, but it does *not* converge in the uniform weak* topology (cf. §6.2).

An earthquake is said to be *finite* if its earthquake measure is supported on finitely many geodesics of $\mathbb{H}$. Thurston [21] proved that the graph of any earthquake $E : \mathbb{H} \to \mathbb{H}$ is approximated by the graphs of finite earthquakes. Gardiner-Hu-Lakic [8] proved that each monotone map from an n-tuple of points in S^1 into S^1 can be realized by a finite earthquake whose support geodesics have ideal endpoints in the n-tuple (finite earthquake theorem). We say that an earthquake is *discrete* if the support geodesic lamination $\mathcal{L}$ of its earthquake measure is discrete; namely any compact subset of $\mathbb{H}$ intersect only finitely many geodesics of $\mathcal{L}$. Next to finite earthquakes, discrete earthquakes are the simplest possible earthquakes and, by definition, finite earthquakes are discrete. We prove that each earthquake E can be approximated by a sequence of discrete earthquakes E_n in the sense that $E|_{S^1} \to E_n|_{S^1}$ in the Teichmüller topology as $n \to \infty$. Theorem below is a direct consequence of Theorem 5 (cf. §7.2) and Theorem 1.

Theorem 2 (Countable Earthquake Theorem)**.** *Let $ML_b^{disc}(\mathbb{H})$ be the set of all bounded measured laminations whose supports are discrete geodesic laminations. Then the set*

$$\{[E^\lambda|_{S^1}] : \lambda \in ML_b^{disc}\}$$

is a dense subset of $T(\mathbb{H})$ in the Teichmüller topology.

Zygmund maps on the unit circle S^1 represent infinitesimal deformations of the space of quasisymmetric maps at the identity map of S^1. In other words, a map V

is Zygmund if and only if there exists a differentiable path $t \mapsto h_t$, for $t \in (-\epsilon, \epsilon)$, of quasisymmetric maps such that

$$V = \frac{d}{dt} h_t|_{t=0}$$

and $h_0 = id$ (for example, see [7]). Let $\mathcal{Z}(S^1)$ be the vector space of all Zygmund maps on S^1 modulo the closed subspace of quadratic polynomials equipped with the cross-ratio norm, see §9.2. (Note that quadratic polynomials are infinitesimal deformations of the paths of Möbius maps.)

Given $\lambda \in \mathcal{ML}_b(\mathbb{H})$, the earthquake path $t \mapsto E^{t\lambda}|_{S^1}$ is differentiable and

$$\dot{E}^{\lambda}|_{S^1} := \frac{d}{dt}(E^{t\lambda}|_{S^1})|_{t=0}$$

is called the *infinitesimal earthquake*. Gardiner [6] proved that each Zygmund map arises as an infinitesimal earthquake.

The *infinitesimal earthquake measure map*

$$\dot{\mathcal{EM}} : \mathcal{ML}_b(\mathbb{H}) \to \mathcal{Z}(S^1)$$

defined by

$$\dot{\mathcal{EM}} : \lambda \mapsto \dot{E}^{\lambda}|_{S^1}$$

is a bijection. We prove that the uniform weak* topology on $\mathcal{ML}_b(\mathbb{H})$ makes $\dot{\mathcal{EM}}$ into a homeomorphisms analogous to the case of quasisymmetric maps.

Theorem 3 (Uniform weak* and Zygmund)**.** *Let $\mathcal{ML}_b(\mathbb{H})$ be given the uniform weak* topology and $\mathcal{Z}(S^1)$ be given the cross-ratio norm topology. Then, the infinitesimal earthquake measure map*

$$\dot{\mathcal{EM}} : \mathcal{ML}_b(\mathbb{H}) \to \mathcal{Z}(S^1)$$

is a homeomorphism.

An infinitesimal version of the countable earthquake theorem immediately follows from Theorem 5 in §7 and Theorem 3.

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2. Measured laminations in $\mathbb{H}$

2.1. **Space of geodesics.** From this point on, $\mathbb{H}$ is the unit disk model of the hyperbolic plane. The unit circle S^1 is identified with the set of ideal boundary points $\partial\mathbb{H}$ of the hyperbolic plane. Fix $z_0 \in \mathbb{H}$. Define the distance between $z_1, z_2 \in S^1$ to be smaller angle between the geodesic rays connecting z_0 with z_1 and z_2, respectively. This gives an angle metric on S^1 which depends on z_0. By varying $z_0 \in \mathbb{H}$ we obtain a biLipschitz class of metrics on S^1.

A complete oriented geodesic g in $\mathbb{H}$ is uniquely determined by an ordered pair of its distinct ideal endpoints on S^1, the initial and the terminal point of g. Conversely, given an ordered pair of points on S^1, there is a unique oriented hyperbolic geodesic with its initial endpoint being the first point and its terminal endpoint being the second point of the pair. Thus the space $\tilde{\mathcal{G}}$ of all oriented geodesics on $\mathbb{H}$ is naturally identified with $S^1 \times S^1 \setminus diag$. Let $\mathcal{G}$ be the set of all unoriented complete hyperbolic geodesic on $\mathbb{H}$. The set $\mathcal{G}$ is identified with $(S^1 \times S^1 \setminus diag)/\sim$, where the equivalence is defined by $(a, b) \sim (b, a)$ and $diag$ is the diagonal set of the product. We denote by $\lceil a, b \rceil$ the equivalence class of $(a, b) \in S^1 \times S^1 \setminus diag$. An angle metric d_{z_0} on S^1

with respect to $z_0 \in \mathbb{H}$ induces a metric $\bar{d}_{z_0}$ on $\mathcal{G}$ as follows. Let $\lceil a,b\rceil, \lceil c,d\rceil \in \mathcal{G}$. Define $\bar{d}_{z_0}(\lceil a,b\rceil, \lceil c,d\rceil) = \min\{\max\{d_{z_0}(a,c), d_{z_0}(b,d)\}, \max\{d_{z_0}(a,d), d_{z_0}(b,c)\}\}$. The set of geodesics $\mathcal{G}$ has a biLipschitz class of metrics obtained by varying $z_0 \in \mathbb{H}$.

A quasiconformal map $f : \mathbb{H} \to \mathbb{H}$ continuously extends to a quasisymmetric map $h : S^1 \to S^1$. Mori's theorem [1] implies that h is a Hölder continuous homeomorphism of S^1 whose Hölder constant depends only on the maximal dilatation of f. Thus a quasisymmetric mapping of S^1 also induces a Hölder continuous homeomorphism of $\mathcal{G}$ for the angle metric $\bar{d}_{z_0}$. Since each quasisymmetric map induces a biholomorphic isometry of the universal Teichmüller space, it is natural to work with the class of Hölder equivalent metrics to the metric $\bar{d}_{z_0}$. For our purposes it will be enough to work with the homeomorphism class of $\bar{d}_{z_0}$.

2.2. **Measured laminations.** A *geodesic lamination* $\mathcal{L}$ is a closed subset of $\mathbb{H}$ together with a foliation of this subset by disjoint complete geodesics. We recall that the information of the foliation of the closed subset is necessary for the definition of a geodesic lamination in $\mathbb{H}$. For example, the hyperbolic plane can be foliated by complete hyperbolic geodesics in infinitely many different ways and each different foliation determines a different geodesic lamination. Equivalently, a geodesic lamination $\mathcal{L}$ is a closed subset of $\mathcal{G}$ such that no two geodesics in $\mathcal{L}$ intersect in $\mathbb{H}$ (they can have common ideal endpoints).

Each complete geodesic in $\mathcal{L}$ is called a *leaf* of $\mathcal{L}$. A *stratum* of $\mathcal{L}$ is either a geodesic of $\mathcal{L}$ or a component of the complement of $\mathcal{L}$ in $\mathbb{H}$.

A *measured lamination* λ is a positive, locally finite, Borel measure on the space of geodesics $\mathcal{G}$ whose support $|\lambda|$ is a geodesic lamination. Each measured lamination λ induces a *transverse measure* to its support $|\lambda|$, namely an assignment of a positive, Borel measure to each closed finite hyperbolic arc I in $\mathbb{H}$ whose support is $I \cap |\lambda|$ and which is invariant under homotopies which preserve the strata of $|\lambda|$. More precisely, the λ-mass of an arc I, denoted by $\lambda(I)$, is the λ-measure of the set of geodesics in $\mathcal{G}$ which intersect I. Conversely, a transverse measure to a geodesic lamination $\mathcal{L}$ determines a unique measured lamination λ whose support is $\mathcal{L} = |\lambda|$. For this correspondence we refer the reader to §1 of [3]. A measured lamination λ is *bounded* if the *Thurston's norm*

$$\|\lambda\|_{Th} = \sup_I \lambda(I)$$

is finite, where I runs over all geodesic arcs in $\mathbb{H}$ with unit length. Let $\mathcal{ML}_b(\mathbb{H})$ be the set of bounded measured laminations on $\mathbb{H}$. When the support of a measured lamination λ consists of one geodesic, we say that λ is an *elementary measured lamination.*

Möbius transformations act isometrically on the set of bounded measured laminations by the *pull-backs* as follows. Let $\gamma \in \text{Möb}(\mathbb{H})$ and λ a measured lamination. We define $\gamma^*\lambda$ as the measured lamination with support $\gamma^{-1}(|\lambda|)$ and the transverse measure $\lambda \circ \gamma$, where $(\lambda \circ \gamma)(I) = \lambda(\gamma(I))$ for all geodesic arcs I. Clearly,

$$\|\gamma^*\lambda\|_{Th} = \|\lambda\|_{Th}$$

holds for any measured lamination λ, and hence $\text{Möb}(\mathbb{H})$ acts by isometry on $\mathcal{ML}_b(\mathbb{H})$.

2.3. **Boxes and the Liouville measure.** The *cross ratio* of a quadruple (a,b,c,d) is given by $cr(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$. A *box of geodesics* Q in $\mathcal{G}$ is the quotient under the equivalence $\sim$ of the product $[a,b]\times[c,d]$ of two disjoint closed arcs in S^1, where $[a,b]$ (resp. $[c,d]$) is the arc in S^1 from a (resp. c) to b (resp. d) for the orientation of S^1. We will write somewhat incorrectly $Q=[a,b]\times[c,d]$ instead of a more correct $Q=([a,b]\times[c,d])/\sim$. The *Liouville measure* L is a non-trivial, Möbius group invariant Borel measure on $\mathcal{G}$ defined by

$$L(Q)=|\log|cr(a,b,c,d)||=\left|\log\left|\frac{(a-c)(b-d)}{(a-d)(b-c)}\right|\right|$$

for all boxes $Q=[a,b]\times[c,d]$. The infinitesimal form of the Liouville measure on $\mathcal{G}=(S^1\times S^1\setminus diag)\sim$ is given by (see [2])

$$dL=\frac{d\alpha d\beta}{|e^{i\alpha}-e^{i\beta}|^2}.$$

For instance, when we consider the upper half-plane model of the hyperbolic plane instead of $\mathbb{H}$ and let $Q=[-1,1]\times[e^D,-e^D]$, the Liouville measure of Q is

$$L(Q)=-2\log\tanh\frac{D}{2}. \tag{2.1}$$

Thus, for a general box $Q=[a,b]\times[c,d]$, the Liouville measure $L(Q)$ is inversely related to the hyperbolic distance between the geodesics $\lceil a,b\rceil$ and $\lceil c,d\rceil$. Furthermore, a box $Q=[a,b]\times[c,d]$ satisfies $L(Q)=\log 2$ if and only if the distance D between $\lceil a,b\rceil$ and $\lceil c,d\rceil$ satisfies $e^D=\omega_0$ $(=(1+\sqrt{2})^2)$ if and only if the distance between $\lceil a,b\rceil$ and $\lceil c,d\rceil$ equals the distance between $\lceil a,d\rceil$ and $\lceil b,c\rceil$. A short computation shows that the box $Q=[-1,1]\times[3+2\sqrt{2},-(3+\sqrt{2})]\subset(\hat{\mathbb{R}}\times\hat{\mathbb{R}}\setminus diag)\sim$ has the Liouville measure $\log 2$.

We again consider the unit disk model $\mathbb{H}$ of the hyperbolic plane and define the *standard box*

$$Q^*=[-i,1]\times[i,-1].$$

Let $\ell_{Q^*}=\lceil e^{-\pi/4},e^{3\pi/4}\rceil\in Q^*$. Let Q be a box with $L(Q)=\log 2$ and γ_Q a Möbius transformation of $\mathbb{H}$ with $\gamma_Q(Q^*)=Q$. The geodesic $\ell_Q:=\gamma_Q(\ell_{Q^*})$ is called the *center* of the box Q.

2.4. **Bounded measured laminations as distributions.**

2.4.1. *Weak* convergence.* We say that a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of Borel measures on $\mathcal{G}$ *converges in the weak* topology* to a Borel measure λ if

$$\lim_{n\to\infty}\int_{\mathcal{G}}f\,d\lambda_n=\int_{\mathcal{G}}f\,d\lambda$$

for all continuous functions f on $\mathcal{G}$ with compact support.

2.4.2. *Measures of squares.* The following lemma is well-known. However we give a proof for readers convenience.

Lemma 2.1 (Comparison with Thurston norm)**.** *There is a universal constant C_0 such that for any measured lamination λ, we have*

$$\frac{1}{C_0}\|\lambda\|_{Th}\le\sup_Q\lambda(Q)\le\|\lambda\|_{Th},$$

where the supremum is taken over all boxes Q with Liouville measure $L(Q)=\log 2$.

Proof. Let I be a geodesic arc in $\mathbb{H}$ of unit length which intersects transversely a leaf ℓ of λ. Since the support $|\lambda|$ consists of disjoint geodesics, there is a universal constant L_0 with the following property: Let J be a geodesic arc in $\mathbb{H}$ of length L_0 which is orthogonal to ℓ at the midpoint of J and let the midpoint of J be equal to $I\cap\ell$. Then, any leaf of $|\lambda|$ with non-trivial intersection with I also intersects J.

One can check that any leaf of $|\lambda|$ ($\subset\mathcal{G}$) which intersects J is contained in a box Q' with center ℓ satisfying $L(Q')=2\log\cosh(L_0/2)$. To see this, we identify $\mathbb{H}$ with the upper half-plane and normalize J and ℓ such that $J=[1,e^{L_0}]i$ and $\ell=\{|z|=e^{L_0/2}\}\cap\mathbb{H}$. Any complete geodesic which is disjoint from ℓ and which intersects J is in the box $Q'=[e^{3L_0/2},-e^{L_0/2}]\times[e^{L_0/2},e^{3L_0/2}]$. This means that $\lambda(I)\leq\lambda(J)\leq\lambda(Q')$ and hence we conclude

$$\|\lambda\|_{Th}\leq C_0\sup_Q\lambda(Q)$$

with universal constant $C_0>0$, where the supremum runs over all boxes Q with $L(Q)=\log 2$.

To show the converse, let $Q=[a,b]\times[c,d]$ be a box in $\mathcal{G}$. The measure $\lambda(Q)$ is obtained as follows. Suppose for simplicity that a, b, c and d are lying on S^1 in this order. Let $\ell_1=\lceil a,d\rceil$ and $\ell_2=\lceil b,c\rceil$ and I the geodesic segment which intersects orthogonally to ℓ_1 and ℓ_2 at endpoints. Then, any complete geodesic in Q intersects I. Since the length of I is $\log 2<1$, there is a geodesic arc I' of unit length which contains I and hence we obtain

$$\lambda(Q)\leq\lambda(I')\leq\|\lambda\|_{Th},$$

for all boxes Q with $L(Q)=\log 2$ which implies the desired inequality. □

3. Earthquakes and earthquake measures

3.1. **Earthquakes.** Let $\mathcal{L}$ be a geodesic lamination in $\mathbb{H}$. An *earthquake* E with support $\mathcal{L}$ is a surjective map $E:\mathbb{H}\to\mathbb{H}$ such that E is a hyperbolic isometry when restricted to any stratum of $\mathcal{L}$ and, for any two strata A and B, the *comparison isometry*

$$\mathrm{cmp(A,B)=(E\,|_A)^{-1}\circ E\,|_B}$$

is a hyperbolic translation whose axis weakly separates A and B, and which translates B to the left as seen from A. An earthquake E of $\mathbb{H}$ continuously extends to a homeomorphism of the boundary S^1 (see [21]). We denote by $E\,|_{S^1}$ the extension.

Given an earthquake E with support $\mathcal{L}$, there is an associated positive transverse measure λ to $\mathcal{L}$ as follows. Let I be a closed geodesic arc transversely intersecting $\mathcal{L}$ with arbitrary orientation. For given n, choose a closed geodesic arc I_n which contains I in its interior such that $I_{n+1}\subsetneq I_n$ and $\cap_n I_n=I$. Furthermore, choose strata $\mathcal{A}_n=\{A_0,A_1,\cdots,A_{k(n)},A_{k(n)+1}\}$ of the support of E such that A_0 contains the left end point of I_n, A_1 contains the left endpoint of I, $A_{k(n)}$ contains the right endpoint of I, $A_{k(n)+1}$ contains the right endpoint of I_n, A_i's intersect I in the given order and the maximum of the distances between the consecutive intersections of $\mathcal{A}_n$ with I_n goes to zero as $n\to\infty$. The summation of the translation lengths of the comparison isometries $\mathrm{cmp(A_i,A_{i+1})=(E\,|_{A_i})^{-1}\circ E\,|_{A_{i+1}}}$ for $i=0,1,\cdots,k(n)+1$ is the approximate measure of I. If $n\to\infty$ and $\mathcal{A}_n$ are chosen such that $(\cup_{i=1}^{k(n)}A_i)\cap I$ is dense in I for all n, the limit of approximate measure is a well-defined positive finite Borel measure ([21] and [8]). (Note that if $E:\mathbb{H}\to\mathbb{H}$ is continuous at the endpoints of I then we can replace I_n with I for each n in the above construction.)

This transverse measure defines a measured lamination λ with support $\mathcal{L}$. We call the measured lamination λ the *earthquake measure* for E. We denote by E^λ an earthquake map with earthquake measure λ. An earthquake map is (essentially) uniquely determined by its earthquake measure. The ambiguity is up to post-composition of the earthquake map by a Möbius map and on each leaf where the earthquake has a discontinuity there is a range of possibilities (but the extension to S^1 gives the same map regardless of the choices in this range.) The set of strata where an earthquake map has a discontinuity consists of an at most countable family of leaves of $\mathcal{L}$.

In [21], Thurston showed that for any orientation preserving homeomorphism h on $\partial\mathbb{H} = S^1$, there is a unique earthquake map E^λ such that $h = E^\lambda|_{S^1}$. Thurston's theorem induces an injective map from the space of right cosets of Möb($\mathbb{H}$) in the group of orientation preserving homeomorphisms into the space of measured laminations in $\mathbb{H}$ by the formula Möb($\mathbb{H}$) $\circ\, h \mapsto \lambda$ where $h = E^\lambda|_{S^1}$.

For an orientation preserving homeomorphism $h : S^1 \to S^1$ and an earthquake map E^λ such that $E^\lambda|_{S^1} = h$, we have that $h \circ \gamma = E^{\gamma^*(\lambda)}|_{S^1}$ for any $\gamma \in$ Möb($\mathbb{H}$).

3.2. **Convergence of earthquakes.** Notice from the definition that for any $\gamma \in$ Möb($\mathbb{H}$), the earthquake measure of $\gamma \circ E$ coincide with that of E. Hence, E^λ is determined *up to* postcomposition of Möbius transformations. Because of this ambiguity, we should give a remark on the symbol E^λ. Namely, when E^λ is treated as a map, this E^λ is always chosen suitably for the content. For instance, we have used the equation "$h = E^\lambda$" with a homeomorphism h on S^1. This equation means that we can choose an earthquake map with earthquake measure λ which coincides with h on S^1. When we say that "$E^{\lambda_n} \to E^\lambda$ as $n \to \infty$", a sequence consisting of choices of the earthquake maps for λ_n ($n \in \mathbb{N}$) converges to one of those for λ.

4. The universal Teichmüller space and the earthquake measure map

4.1. **Quasisymmetic maps.** An orientation preserving homeomorphism $h : S^1 \to S^1$ is said to be *quasisymmetric* if there is a constant $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{|h(J_1)|}{|h(J_2)|} \leq M \tag{4.1}$$

for all adjacent intervals $J_1, J_2 \subset S^1$ with $|J_1| = |J_2|$, where $|J_i|$ is the arc length with respect to the angle measure on $S^1 = \partial\mathbb{H}$. Let $\mathcal{QS}$ be the set of all quasisymmetic maps on S^1 and let Möb($\mathbb{H}$) be the group of Möbius transformations that preserve $\mathbb{H}$. The *universal Teichmüller space* $T(\mathbb{H})$ is the quotient space

$$T(\mathbb{H}) = \text{Möb}(\mathbb{H})\backslash\mathcal{QS}$$

where Möb($\mathbb{H}$) acts on $\mathcal{QS}$ via post-compositions. For any $h \in \mathcal{QS}$, we denote by $[h]$ its class in $T(\mathbb{H})$. The universal Teichmüller space $T(\mathbb{H})$ admits a natural (metric) topology induced by considering maximal dilatations of all quasiconformal extensions to $\mathbb{H}$ of quasisymmetric maps of S^1. Namely, two quasisymetric maps h_1 and h_2 are close if there exists a quasiconformal extension of $h_2 \circ h_1^{-1}$ whose maximal dilatation is near one. This topology on $T(\mathbb{H})$ is the same one inherited from quasisymetric constants. See [4] or [7].

4.2. **The earthquake measure map.** In this subsection, we define the earthquake measure map. We first recall the following theorem, which is proved by Gardiner-Hu-Lakic [8] and in [15].

Theorem 4 (Gardiner-Hu-Lakic, Šarić)**.** *Let h be an orientation preserving homeomorphism of $\partial\mathbb{H} = S^1$ and let E^λ be the earthquake of $\mathbb{H}$ whose continuous extension to S^1 equals h. Then the following are equivalent.*

(1) *The earthquake measure λ of the earthquake $E^\lambda|_{S^1} = h$ is bounded.*
(2) *h is quasisymmetric.*

The *earthquake measure map*

$$\mathcal{EM} : T(\mathbb{H}) \to \mathcal{ML}_b(\mathbb{H})$$

is defined by $\mathcal{EM}([h]) = \lambda$ where $h = E^\lambda|_{S^1}$. As noted in §3.2, every earthquake is determined by its earthquake measure up to post-composition by Möbius maps. Hence, together with the uniqueness of the earthquake measures for homeomorphisms [21], Theorem 4 tells us that the earthquake measure map $\mathcal{EM}$ is well-defined and bijective.

In [8] and [9], it is proved that for a quasisymmetric map h, the Thurston norm of the earthquake measure of h is comparable with the quasisymmetric constant of h. We will give a brief proof of a weaker result than the comparison statement which we need here (cf. Lemma 7.1).

5. Uniform weak* topology

We define a topology on $\mathcal{ML}_b(\mathbb{H})$ which is natural for the correspondence between quasisymmetric maps of S^1 and the earthquake measures. This topology is the main object of study in this paper.

A sequence $\lambda_m \in \mathcal{ML}_b(\mathbb{H})$ *converges to* $\lambda \in \mathcal{ML}_b(\mathbb{H})$ *in the uniform weak* topology* if for any continuous function f on $\mathcal{G}$ with $\mathrm{supp}(f) \subset Q^*$,

$$\sup_Q \int_{Q^*} f d((\gamma_Q)^*(\lambda_m) - (\gamma_Q)^*(\lambda)) \to 0$$

as $m \to \infty$, where the supremum is over all boxes Q with the Liouville measure $L(Q) = \log 2$, $\gamma_Q \in \text{Möb}(\mathbb{H})$ is such that $\gamma_Q(Q^*) = Q$ and $Q^* = [-i, 1] \times [i, -1]$.

The definition of the uniform weak* topology has two important features. Namely, it is uniform on an infinite family of boxes of geodesics and the family is restricted to boxes of a fixed size. These two conditions together make the uniform weak* topology useful for our purposes.

6. Examples

In this section, we consider the example from the Introduction of a sequence in the space of bounded measured laminations which *does not converge* in the uniform weak* topology and yet it *does converge* in the weak* topology.

6.1. **Uniform weak* topology vs weak* topology.** For simplicity, we use the upper half-plane model for the hyperbolic plane in place of $\mathbb{H}$. Let $\ell_n = \lceil 1/n, \infty \rceil$ $(n \in \mathbb{Z} \setminus \{0\})$ and $\ell_\infty = \lceil 0, \infty \rceil$ be two geodesics $\mathbb{H}$. Namely, ℓ_n is the vertical line which connects n and ∞, and ℓ_∞ is the vertical line which connects 0 and ∞.

Example 1. Let λ_n be the measured lamination whose support is ℓ_n with $\lambda_n(\ell_n) = 1$. Let λ_∞ be the measured lamination whose support is ℓ_∞ such that

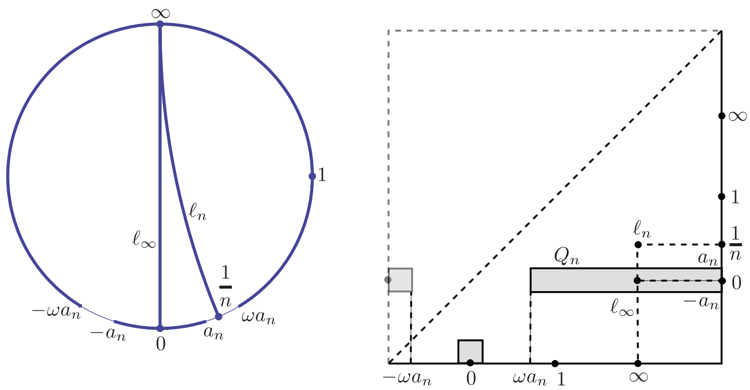

FIGURE 2. ℓ_∞, ℓ_n, and the box Q_n with center ℓ_∞ and $L(Q_n) = \log 2$ such that $\ell_n \notin Q_n$. The right picture represents how Q_n distributes in the space $\mathcal{G}$.

$\lambda_\infty(\ell_\infty) = 1$. We claim that λ_n does *not* converge to λ_∞ in the uniform weak* topology as $n \to \infty$, while it does converge in the weak* topology on measures on $\mathcal{G}$.

Indeed, for $n \geq 1$ and $\omega_0 = (1 + 2\sqrt{2})^2$, we define a box $Q_n = [-a_n, a_n] \times [\omega_0 a_n, -\omega_0 a_n]$ with $1/(\omega_0 n) < a_n < 1/n$, where $[\omega_0 a_n, -\omega_0 a_n]$ is the interval in $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ which contains ∞ and connects $\omega_0 a_n$ and $-\omega_0 a_n$ (cf. Figure 2).

Then, one can check that $L(Q_n) = \log 2$, $\lambda_\infty(Q_n) = 1$ and $\lambda_n(Q_n) = 0$ since $\ell_n \notin Q_n$. We take a positive continuous function φ on $\mathcal{G}$ with support in the *standard box for the upper half-plane* $Q^u = [-1, 1] \times [3 + 2\sqrt{2}, -(3 + 2\sqrt{2})]$ such that $\|\varphi\|_\infty \leq 1$ and the value at the center $\ell_{Q^u} = \ell_\infty$ of φ is positive. From the symmetries of Q_n and Q^u, one can see that $\gamma_{Q_n}(\ell_\infty) = \ell_{Q_n}$ for all n. Then

(6.1)
$$\left| \int_{Q^u} \varphi d(\gamma_{Q_n})^*(\lambda_n - \lambda_\infty) \right| = \left| \int_{Q_n} \varphi \circ \gamma_{Q_n}^{-1} d(\lambda_n - \lambda_\infty) \right| = \varphi \circ \gamma_{Q_n}^{-1}(\ell_\infty) = \varphi(\ell_\infty)$$

for all n which implies that λ_n does not converge to λ_∞ in the uniform weak* topology. The weak* convergence of λ_n to λ_∞ is immediate. By the same reason, we can see that the "midpoint approximation" $\frac{1}{2}(\lambda_n + \lambda_{-n})$ does not converge to λ_∞ in the uniform weak* topology either.

The above example motivates the following necessary condition for a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ to converge (in the uniform weak* topology) to a measured lamination λ_∞ whose support is a single leaf.

Proposition 6.1. *Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of bounded measured laminations which converges in the uniform weak* topology to a measured lamination λ_∞ whose support is a single geodesic. Then, for all sufficiently large n, each endpoint of $|\lambda_\infty|$ is contained in the closure the set of endpoints of leaves of λ_n.*

Proof. Let $|\lambda_\infty| = [0, \infty]$. Suppose on the contrary that there is a $\delta_n > 0$ such that any leaf of λ_n does not have endpoints in an open interval $(-\delta_n, \delta_n)$. We take a sufficiently small $a_n > 0$ such that $\omega_0 a_n < \delta_n$, where $\omega_0 = (1 + \sqrt{2})^2$ as before. Define Q_n by

$$Q_n = [-a_n, a_n] \times [\omega_0 a_n, -\omega_0 a_n]$$

Then, the center of Q_n is ℓ_∞, $L(Q_n) = \log 2$ and $Q_n \cap |\lambda_n| = \emptyset$. Thus, by the same calculation as (6.1), we get

$$\left| \int_{Q^u} \varphi d(\gamma_{Q_n})^*(\lambda_n - \lambda_\infty) \right| \geq \varphi(\ell_\infty) > 0$$

for some continuous function φ independent of n. This means that $\{\lambda_n\}_{n=1}^\infty$ cannot converge to λ_∞ in the uniform weak* topology. □

However, a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ which converges in the weak* topology to a (single geodesic support) measured lamination λ_∞ and which satisfies the property in Proposition 6.1 does not necessarily converge to λ_∞ in the uniform weak* topology which is illustrated by an example similar to the above.

6.2. **Elementary earthquakes.** We shall check the behavior of earthquakes whose supports are single geodesics given in the above section to clarify the connection between the uniform weak* topology and the weak* topology on the measured laminations and the Teichmüller topology on the extensions to S^1 of their corresponding earthquake maps.

Let $\ell_n = \lceil 1/n, \infty \rceil$ for $n \in \mathbb{N} \cup \{\infty\}$. Then the earthquake map E^{λ_n} for elementary measures λ_n with single geodesic support ℓ_n and mass 1 (normalized to fix three points $\{-1, 0, \infty\}$) is

$$E^{\lambda_n}(z) = \begin{cases} e(z - 1/n) + 1/n & (\mathrm{Re}(z) > 1/n) \\ z & (\mathrm{Re}(z) \leq 1/n) \end{cases}$$

for $z \in \mathbb{H}$, where we set $1/\infty = 0$. Clearly $h_n := E^{\lambda_n}|_{\partial\mathbb{H}}$ converges to $h_\infty = E^{\lambda_\infty}|_{\partial\mathbb{H}}$ pointwise. However, h_n does not converge to h_∞ in the Teichmüller topology. Indeed, for $n \in \mathbb{N}$ and boxes $Q_n = [\infty, -e/n] \times [0, e/n]$, we get $L(Q_n) = \log 2$ and

$$L(h_n \circ h_\infty^{-1}(Q_n)) = \log(e + 1) - 1.$$

This means that the maximal dilatation of any quasiconformal extension of $h_n \circ h_\infty^{-1}$ is uniformly greater than 1. Thus, the sequence $\{h_n\}_{n=1}^\infty$ does not converge to h_∞ in $T(\mathbb{H})$, which also follows from Theorem 1 and Example 1 above.

7. The earthquake measure map is a homeomorphism

In this section, we prove Theorem 1. We need the following lemma which is a special case of a result in [9].

Lemma 7.1. *For any $C_1 > 0$, there is $C_2 > 0$ depending only of C_1 such that for any bounded measured lamination λ with $\|\lambda\|_{Th} \leq C_1$, the quasisymmetric constant of $E^\lambda|_{S^1}$ is at most C_2.*

Proof. This follows from the results in [13]. The earthquake path $t \mapsto E^{t\lambda}|_{S^1}$ is a real analytic path in the universal Teichmüller space $T(\mathbb{H})$ which extends to a holomorphic motion $\tau \mapsto E^{\tau\lambda}|_{S^1}$ of S^1 in $\hat{\mathbb{C}}$. Moreover, the holomorphic motion is well-defined for τ in a neighborhood of the real line $\mathbb{R}$ whose shape depends only on $\|\lambda\|_{Th}$ (see [13]). Then the essential supremum norm of the Beltrami coefficient of the extension of the holomorphic motion of S^1 to a holomorphic motion of $\hat{\mathbb{C}}$ for $\tau = 1$ depends only on the shape of the domain in which τ is defined. As we noted above, this in turn only depends on $\|\lambda\|_{Th}$. Thus the quasisymmetric constant of $E^\lambda|_{S^1}$ depends only on $\|\lambda\|_{Th}$ which proves the lemma. □

7.1. **Proof of Theorem 1.** We first show that the earthquake measure map $\mathcal{EM}$ is continuous. Let $[h] \in T(\mathbb{H})$ and $\{[h_m]\}_{m=1}^{\infty} \subset T(\mathbb{H})$ with $[h_m] \to [h]$ as $m \to \infty$. Let $\lambda_m = \mathcal{EM}([h_m])$ and $\lambda = \mathcal{EM}([h])$. Then, it follows from Lemma 4.1 of [16] that for any continuous function f on $\mathcal{G}$ with $\mathrm{supp}(f) \subset Q^*$,

$$\sup_Q \int_{Q^*} fd((\gamma_Q)^*(\lambda_m) - (\gamma_Q)^*(\lambda)) \to 0$$

as $m \to \infty$, where Q runs over all boxes whose Liouville measures are $\log 2$ and γ_Q is a Möbius map which sends the standard box $Q^* = [-i, 1] \times [i, -1]$ onto the box Q. This means that $\mathcal{EM}$ is continuous for the uniform weak* topology on $\mathcal{ML}_b(\mathbb{H})$ and the Teichmüller topology on $T(\mathbf{H})$.

Next, we show that the inverse $\mathcal{EM}^{-1}$ is continuous. Suppose $\lambda_n = \mathcal{EM}([h_m]) \to \lambda = \mathcal{EM}([h])$ in the uniform weak* topology. Assume on the contrary that $\mathcal{EM}^{-1}$ is not continuous. Namely, there are $\epsilon_0 > 0$ and a sequence $\{Q_m\}_{m=1}^{\infty}$ of boxes with the Liouville measure $L(Q_m) = \log 2$ such that

$$|L(h_m(Q_m)) - L(h(Q_m))| \geq \epsilon_0 \tag{7.1}$$

for all m, where h and h_m are normalized to fix 1, i and -1. Take Möbius transformations β_m and β_m^* such that $g_m = \beta_m \circ h_m \circ \gamma_{Q_m}$ and $g_m^* = \beta_m^* \circ h \circ \gamma_{Q_m}$ fix 1, i and -1. By (7.1), we have

$$|L(g_m(Q^*)) - L(g_m^*(Q^*))| \geq \epsilon_0 \tag{7.2}$$

for all m. Since $\lambda_n \to \lambda$ in the uniform weak* topology, it follows that $\|\lambda_n\|_{Th}$ is uniformly bounded (using Lemma 2.1). Lemma 7.1 implies that the constants of quasisymmetry of g_m and g_m^* are uniformly bounded. The compactness of normalized quasisymmetric mappings with uniformly bounded quasisymmetric constants imply that g_m and g_m^* have two subsequences which are index by the same set that converge to normalized quasisymmetric mappings g and g^*, respectively. For simplicity of notation, we rename the subsequences to be g_m and g_m^*. By (7.2), g does not coincide with g^*.

We claim

Claim. The limits, in the weak* topology, of any pair of converging subsequences $\{(\gamma_{Q_{m_j}})^*\lambda_{m_j}\}_{j=1}^{\infty}$ and $\{(\gamma_{Q_{m_j}})^*\lambda\}_{j=1}^{\infty}$ of $\{(\gamma_{Q_m})^*\lambda_m\}_{m=1}^{\infty}$ and $\{(\gamma_{Q_m})^*\lambda\}_{m=1}^{\infty}$ are the same bounded measured lamination λ'.

Proof of the Claim. From the compactness of probability measures under the weak* topology, one sees that two sequences $\{(\gamma_{Q_m})^*\lambda_m\}_{m=1}^{\infty}$ and $\{(\gamma_{Q_m})^*\lambda\}_{m=1}^{\infty}$ contain a pair $\{(\gamma_{Q_{m_j}})^*\lambda_{m_j}\}_{j=1}^{\infty}$ and $\{(\gamma_{Q_{m_j}})^*\lambda\}_{j=1}^{\infty}$ of converging subsequences in the weak* topology. Since λ_m converges to λ in the uniform weak* topology, it follows tha $\{(\gamma_{Q_m})^*\lambda_m - (\gamma_{Q_m})^*\lambda\}_{m=1}^{\infty}$ converges to zero measure in the weak* sense. Hence the weak* limits of the pair of converging subsequences $\{(\gamma_{Q_{m_j}})^*\lambda_{m_j}\}_{j=1}^{\infty}$ and $\{(\gamma_{Q_{m_j}})^*\lambda\}_{j=1}^{\infty}$ are same. □

We continue the proof of Theorem 1. By Lemma 3.2 of [15], we can choose representatives of earthquakes $E^{(\gamma_{Q_m})^*\lambda_m}$ and $E^{(\gamma_{Q_m})^*\lambda}$ such that the two sequences $\{E^{(\gamma_{Q_m})^*\lambda_m}|_{S^1}\}_{m=1}^{\infty}$ and $\{E^{(\gamma_{Q_m})^*\lambda}|_{S^1}\}_{m=1}^{\infty}$ converge to the same (representative of) earthquake map $E^{\lambda'}|_{S^1}$ pointwise on S^1 (cf. §3.2). Then we take Möbius transformations $\hat{\beta}_m$ and $\hat{\beta}_m^*$ such that $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda_n}$ and $\hat{\beta}_m^* \circ E^{(\gamma_{Q_m})^*\lambda}$ fix 1, i and -1. Since the limits of two sequences $\{E^{(\gamma_{Q_m})^*\lambda_n}\}_{m=1}^{\infty}$ and $\{E^{(\gamma_{Q_m})^*\lambda_n}\}_{m=1}^{\infty}$

are same, $\hat{\beta}_m$ and $\hat{\beta}^*_m$ converge the same Möbius transformation. Hence, the limits of $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda_n}$ and $\hat{\beta}^*_m \circ E^{(\gamma_{Q_m})^*\lambda}$ also agree.

On the other hand, from the definition of earthquakes we have that

$$\begin{aligned}\mathcal{EM}([\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda_m}|_{S^1}]) &= \mathcal{EM}([E^{(\gamma_{Q_m})^*\lambda_m}|_{S^1}]) = (\gamma_{Q_m})^*\lambda_m \\ &= \mathcal{EM}([h_m \circ \gamma_{Q_m}]) = \mathcal{EM}([g_m])\end{aligned}$$

and

$$\begin{aligned}\mathcal{EM}([\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda}|_{S^1}]) &= \mathcal{EM}([E^{(\gamma_{Q_m})^*\lambda}|_{S^1}]) = (\gamma_{Q_m})^*\lambda \\ &= \mathcal{EM}([h \circ \gamma_{Q_m}]) = \mathcal{EM}([g^*_m]).\end{aligned}$$

Since the earthquake measure map is bijective and all maps $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda_m}$, $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda}$, g_m, and g^*_m fix 1, i and -1, we conclude $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda_m}|_{S^1} = g_m$ and $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^*\lambda}|_{S^1} = g^*_m$. However, this contradicts that the limits g and g^* of $\{g_m\}_{m=1}^\infty$ and $\{g^*_m\}_{m=1}^\infty$ are distinct. The contradiction proves Theorem 1.

8. Approximations by discrete laminations

The purpose of this section is to propose a candidate for a class of *simple* measured laminations in order to better understand the universal Teichmüller space using earthquake maps. Indeed, we will show that *discrete measured laminations* are dense in $\mathcal{ML}_b(\mathbb{H})$ with respect to the uniform weak* topology. Discrete measured laminations are close relative of finite measured laminations and earthquakes supported on discrete measured laminations are easier to visualize.

8.1. **Discrete laminations.** A geodesic lamination $\mathcal{L}$ is said to be *discrete* if any compact set $K \subset \mathbb{H}$ intersects only finitely many leaves of $\mathcal{L}$. Equivalently, $\mathcal{L}$ is a discrete geodesic lamination if it is discrete subset of $\mathcal{G}$. A measured lamination λ is, by definition, *discrete* if its support $|\lambda|$ is a discrete subset of $\mathcal{G}$. To show the density of discrete measured laminations in $\mathcal{ML}_b(\mathbb{H})$, we give some notations needed in the proof of the density theorem.

Extreme geodesics and peaks. We recall that a box of geodesics is the product set $I \times J \in \mathcal{G}$ where I and J are disjoint closed intervals of $\partial\mathbb{H} = S^1$. In this proof, we generalize the notion of boxes such that either I or J is allowed to be a point, open or half-open interval. For a generalized box $Q = I \times J$, we define the *extreme geodesics* $\{\ell^1_Q, \ell^2_Q\}$ for Q as follows. Suppose that both I and J are non-degenerate intervals. Let $\mathrm{Int}(I) = (a, b)$ and $\mathrm{Int}(J) = (c, d)$. Then, we set $\ell^1_Q = \lceil b, c \rceil$ and $\ell^2_Q = \lceil a, d \rceil$. When exactly one of the intervals is degenerate, say when $I = \{a\}$ and $\mathrm{Int}(J) = (c, d)$, we set $\ell^1_Q = \lceil a, c \rceil$ and $\ell^2_Q = \lceil a, d \rceil$. When I and J are both degenerate, ℓ^1_Q and ℓ^2_Q are defined to be the geodesic connecting I and J. See Figure 3.

Let $Q = I \times J$ be a generalized box in $\mathcal{G}$ and $\mathcal{L}$ a geodesic lamination. Let $\bar{Q} = \bar{I} \times \bar{J}$ be the closure of Q, where $\bar{I}, \bar{J}$ are closures of I, J. A leaf g of $\mathcal{L}$ is said to be *peak with respect to* Q if $g \in \bar{Q}$ and one of the two components of $\mathbb{H} \setminus g$ does not contain leaves of $\mathcal{L} \cap Q$. By definition, when $\mathcal{L} \cap \bar{Q}$ contains at least two leaves, there is exactly two peak geodesics of $\mathcal{L}$ with respect to Q. In addition, if an extreme geodesic of Q is a leaf of $\mathcal{L}$, it is also a peak geodesic of $\mathcal{L}$ with respect to Q.

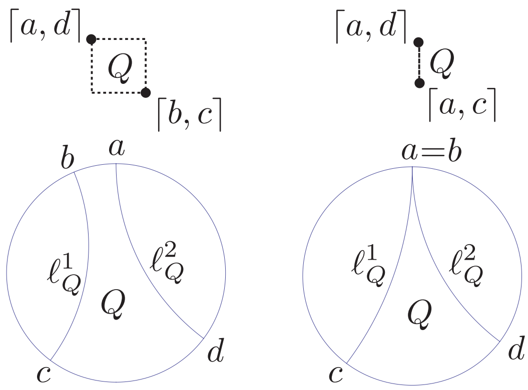

FIGURE 3. Generalized boxes in $\mathcal{G}$ and their extreme geodesics.

8.2. **Density of discrete laminations.** We are ready to prove the density of discrete laminations.

Theorem 5 (Discrete laminations are dense)**.** *The set of discrete bounded measured laminations is dense in $\mathcal{ML}_b(\mathbb{H})$ in the uniform weak* topology.*

Proof. Fix $\lambda \in \mathcal{ML}_b(\mathbb{H})$. Let λ^0 and λ^1 be the discrete and continuous parts of the measure λ on $\mathcal{G}$, respectively. By definition, λ^0 is the (possibly countably infinite) sum of Dirac measures (atoms). Note that the support of λ^0 is not necessarily a discrete geodesic lamination which implies that λ^0 is a not necessarily a discrete bounded measured lamination according to our definition. We identify Dirac measures appearing as terms of λ^0 with their supports (each of them is a positive number assigned to a point in $\mathcal{G}$).

We now fix n and partition $\mathcal{G}$ into a locally finite, countable family of boxes $\{B_s'\}_{s=1}^{\infty}$ with mutually disjoint interiors such that their Liouville measures satisfy $L(B_s') \leq \log 2$. We enumerate the terms of λ^0:

$$\lambda^0 = \sum_{s=1}^{\infty} \sum_{m} \mu_m^s$$

such that $\mathrm{supp}(\mu_m^s) \subset B_s'$. If an atom belongs to the boundary side of a box, then it is shared by at least two boxes and at most four boxes. We fix one of the possible boxes to which the atom belongs and write it in the above sum only once. It is possible that $\{\mu_m^s\}_m$ consists of infinitely many Dirac measures, for any s. For each s, we take $m_{s,n}$ such that

$$\sum_{s=1}^{\infty} \sum_{m \geq m_{s,n}} \mu_m^k(B_s') < 1/n. \tag{8.1}$$

Notice from the definition that

$$\lambda_n^0 := \sum_{s=1}^{\infty} \sum_{m \leq m_{s,n}} \mu_m^s$$

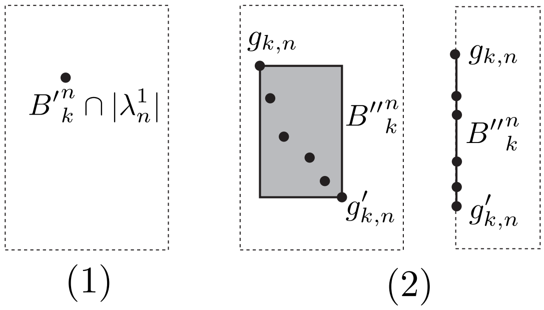

FIGURE 4. Boxes bounded by broken lines represents B'^{n}_{k}.

is a discrete sub-measured lamination of λ. We define a measured lamination λ^1_n by

$$\lambda^1_n := \lambda - \lambda^0_n = \lambda^1 + \sum_{s=1}^{\infty} \sum_{m > m_{s,n}} \mu^k_m$$

We claim the following.

Claim 1. For any n, there is a locally finite collection $\{B^n_k\}_{k=1}^{\infty}$ of countably many, mutually disjoint generalized boxes with the following properties.

(1) $\{B^n_k\}_{k=1}^{\infty}$ covers $|\lambda^1_n|$.
(2) $\lambda^1_n(B^n_k) < 1/n$ and $L(B^n_k) \leq \log 2$ for all k, and
(3) extreme geodesics of B^n_k are leaves of $|\lambda^1_n|$.

Proof of Claim 1. By the definition of λ^1_n, we can divide each B'_s into a finite collection of non-degenerate closed boxes such that its λ^1_n-measure is less than $1/n$ and interiors of distinct boxes are disjoint. We define a sub-collection $\{B'^{n}_{k}\}_{k=1}^{\infty}$ to consist of all the above boxes (running all s) which intersect the support $|\lambda^1_n|$ of λ^1_n.

We now fix one box B'^{n}_{k} and modify it appropriately to get the collection of generalized boxes as in the claim.

Case 1.1 : $B'^{n}_{k} \cap |\lambda^1_n|$ consists of one point. When $B'^{n}_{k} \cap |\lambda^1_n|$ is not an atom, then it has to belong to a boundary side B'^{n}_{k}. We drop B'^{n}_{k} from the family of boxes. Suppose $B'^{n}_{k} \cap |\lambda^1_n|$ is an atom $\lambda'_{k,n}$ of λ, we again drop B'^{n}_{k} from the collection of boxes and add $\lambda'_{k,n}$ to λ^0_n. Since $\{B'^{n}_{k}\}_{k=1}^{\infty}$ is locally finite, even if we continue this procedure infinitely (but countably) many times, λ^0_n is still a locally finite sublamination of λ (cf. (1) in Figure 4).

Case 1.2 : $B'^{n}_{k} \cap |\lambda^1_n|$ contains at least two points. Let $g_{k,n}$ and $g'_{k,n}$ be peak geodesics of $|\lambda^1_n|$ with respect to B'^{n}_{k}. We replace the box B'^{n}_{k} by a box $B''^{n}_{k} \subset B'^{n}_{k}$ whose extreme geodesics are $g_{k,n}$ and $g'_{k,n}$ (cf. (2) in Figure 4). If it happens that $g_{k,n}$ and $g'_{k,n}$ share the same endpoint, then B''^{n}_{k} is a generalized box in our sense (cf. the right figure of (2) in Figure 4).

From the definition, the family $\{B''^{n}_{k}\}_{k=1}^{\infty}$ of the resulting boxes is locally finite and satisfies the properties (1), (2) and (3) in the claim.

It is possible that some of the obtained closed boxes intersect along their boundaries. In this case, we divide the closed box into an open box which is the interior and into boundary sides which are generalized boxes. Each of the boundary sides

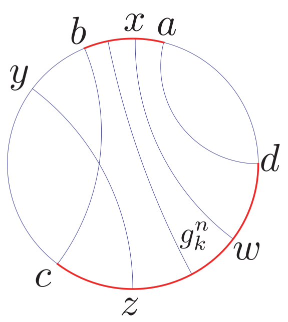

FIGURE 5. (1) in Claim 2 of the proof of Theorem 5.

is divided further into finitely many generalized boxes such that the new family of generalized boxes is pairwise mutually disjoint. Thus, after renumbering with respect to k if necessary, we finally obtain the family of generalized boxes $\{B_k^n\}_{k=1}^{\infty}$ as we claimed. □

Let us continue the proof of the density theorem. Fix $n \in \mathbb{N}$. Let $\{B_k^n\}_{k=1}^{\infty}$ be the family of boxes from Claim 1. We fix $g_k^n \in B_k^n \cap |\lambda|$ arbitrary, and define

$$\lambda_n^2 := \sum_{k=1}^{\infty} \lambda_n^1(B_k^n) \cdot \delta_{g_k^n} \quad \text{and}$$
$$\lambda_n := \lambda_n^0 + \lambda_n^2,$$

where $\delta_{g_k^n}$ is the dirac measure on $\mathcal{G}$ with support g_k^n. Since $\{B_k^n\}_{k=1}^{\infty}$ is locally finite, so is λ_n. Furthermore, λ_n is a measured geodesic lamination, because leaves of λ_n are leaves of λ.

We will prove that, as n tends to ∞, λ_n converges to λ in the uniform weak* topology, which implies that discrete bounded measured laminations are dense in $\mathcal{ML}_b(\mathbb{H})$. We need the following claim to show the convergence.

Claim 2. The following holds.

(1) For any box Q in $\mathcal{G}$, there are at most two boxes from the family $\{B_k^n\}_{k=1}^{\infty}$ such that $B_k^n \cap Q \neq \emptyset$ but $B_k^n \not\subset Q$.
(2) The sequence $\{\lambda_n\}_{n=1}^{\infty}$ has uniformly bounded Thurston norms. In particular, $\lambda_n \in \mathcal{ML}_b(\mathbb{H})$.

Proof of Claim 2. (1) Let B_k^n be a box satisfying $g_k^n \in Q$ but $B_k^n \not\subset Q$. Let $Q = [a, b] \times [c, d]$ and $B_k^n = [x, y] \times [z, w]$. Without loss of generality, we may assume that b is in the interior of $[x, y]$. Then, there is no box $B_{k'}^n = I' \times J'$ such that $B_{k'}^n \cap Q \neq \emptyset$ and $I' \cap [c, z] \neq \emptyset$ or $J' \cap [c, z] \neq \emptyset$. This follows because the extreme geodesics of $B_{k'}^n$ are contained in a component of $\mathbb{H} \setminus \lceil y, z \rceil$ whose closure contains c, and hence, no geodesic in $B_{k'}^n$ can connect $[a, b]$ and $[c, d]$. (Figure 5). If there is another box $B_{k_1}^n = [x_1, y_1] \times [z_1, w_1]$ such that $g_{k_1}^n \in Q$ and $B_{k_1}^n \not\subset Q$, then either $a \in [x_1, y_1]$ or $d \in [x_1, y_1]$ or $a \in [z_1, w_1]$ or $d \in [z_1, w_1]$. The above reasoning implies that there could be no more boxes with the above property. Thus, there are at most two boxes with the property that $B_k^n \cap Q \neq \emptyset$ but $B_k^n \not\subset Q$.

(2) Let Q be a box with Liouville measure $L(Q) = \log 2$. From the definition of λ_n, we get

$$\lambda_n(Q) \leq \lambda_n^0(Q) + \sum_{B_k^n \cap Q \neq \emptyset} \lambda_n^1(B_k^n) \leq \lambda(Q) + (\lambda(Q) + (1/n) \times 2),$$

because λ_n^0 is a sub-measured lamination of λ and the number of boxes B_k^n that intersect Q and are not contained in Q is at most 2. By Lemma 2.1, we deduce that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ has uniformly bounded Thurston norms. □

Let us continue with the proof that λ_n converges to λ in the uniform weak* topology. Let Q be a box with Liouville measure $L(Q) = \log 2$ and let f be a continuous function on $\mathcal{G}$ whose support is in the standard box Q^*. Let $\epsilon > 0$. We take $\delta > 0$ such that $|f(\ell) - f(\ell')| < \epsilon$ when $d(\ell, \ell') \leq \delta$, where d is the fixed metric on $\mathcal{G}$ induced by the angle metric on S^1 with respect to $0 \in \mathbb{H}$ (cf. §2.1).

Take B_k^n with $Q \cap B_k^n \neq \emptyset$. Recall that $\gamma_Q : Q^* \mapsto Q$. Let $\gamma_Q^{-1}(B_k^n) = I \times J$. Suppose that $I \cap [-i, 1]$ and $J \cap [i, -1]$ are non-empty. We set

$$\hat{\lambda}_{Q,n} := (\gamma_Q)^*(\lambda_n) - (\gamma_Q)^*(\lambda) = (\gamma_Q)^*(\lambda_n^2) - (\gamma_Q)^*(\lambda_n^1)$$

for simplicity. We consider the following three cases for B_k^n.

Case 1. $B_k^n \subset Q$ and the lengths of I and J are less than δ.

In this case, we have

$$\left| \int_{\gamma_Q^{-1}(B_k^n)} f \, d\hat{\lambda}_{Q,n} \right| = \left| f(\gamma_Q^{-1}(g_k^n))\lambda_n^1(B_k^n) - \int_{\gamma_Q^{-1}(B_k^n)} f \, d((\gamma_Q)^*(\lambda_n^1)) \right| \leq \epsilon \lambda_n^1(B_k^n).$$

Therefore, the summation over all boxes B_k^n in this case gives

$$\sum_{\{B_k^n\text{'s in Case 1}\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f \, d\hat{\lambda}_n \right| \leq \epsilon \lambda_n^1(Q) \leq \epsilon \lambda(Q). \tag{8.2}$$

Case 2. $B_k^n \subset Q$ and, if $\gamma_Q^{-1}(B_k^n) = I \times J$ then either I or J has length at least δ.

Notice that the number of such B_k^n's in this case is $O(1/\delta)$ because the extreme geodesics of each B_k^n are the leaves of λ which implies that no two B_k^n's can have a side in common. In fact, sides of two B_k^n's can have at most one point in common. Hence, we have

$$\sum_{\{B_k^n\text{'s in Case 2}\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f \, d\hat{\lambda}_{Q,n} \right| \leq O\left(\|f\|_\infty/(n\delta)\right) \tag{8.3}$$

Case 3. $B_k^n \not\subset Q$.

Notice that

$$\left| \int_{\gamma_Q^{-1}(B_k^n)} f \, d\hat{\lambda}_n \right| \leq (\lambda_n^2(B_k^n) + \lambda_n^1(B_k^n))\|f\|_\infty \leq 2\|f\|_\infty/n.$$

By (1) of Claim 2, there are at most two such boxes. Hence, we have

$$\sum_{\{B_k^n\text{'s in Case 3}\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f \, d\hat{\lambda}_{Q,n} \right| \leq 4\|f\|_\infty/n. \tag{8.4}$$

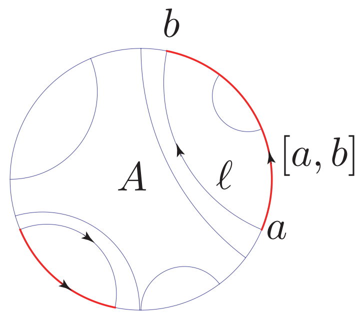

FIGURE 6. Orientations of leaves and associated intervals.

We can now complete the proof of the convergence. Indeed, we take n sufficiently large such that $n\delta > 1/\epsilon$. Then, from the three cases above, we conclude

$$\begin{aligned}\sup_Q \left| \int_{Q^*} f\, d\hat{\lambda}_{Q,n} \right| &\leq \sup_Q \left\{ \sum_{\{B_k^n \cap Q \neq \emptyset\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f\, d\hat{\lambda}_{Q,n} \right| \right\} \\ &\leq \sup_Q \left\{ \epsilon\lambda(Q) + O\left(\|f\|_\infty/(n\delta)\right) + 4\|f\|_\infty/n \right\} \\ &= \epsilon \left(\sup_Q \lambda(Q) \right) + O(\epsilon) = O(\epsilon),\end{aligned}$$

where the supremum is taken over all Q with $L(Q) = \log 2$. Since $\hat{\lambda}_{Q,n} = (\gamma_Q)^*(\lambda_n) - (\gamma_Q)^*(\lambda)$, we have that λ_n converges to λ in the uniform weak* topology. □

Theorem 5 and Theorem 1 immediately imply Theorem 2.

9. INFINITESIMAL EARTHQUAKES AND VECTOR FIELDS

In this section, we consider the vector fields on $\partial\mathbb{H} = S^1$ which arise by differentiating the paths of earthquakes. The aim is to prove the equivalence between the uniform weak* topology on earthquake measures and the Zygmund topology on the vector fields (cf. Theorem 3) which is an analogy to Theorem 1.

9.1. **Vector fields.** Let λ be a bounded measured lamination. From now on, we fix a stratum A of λ such that A is either a gap or a geodesic which is not an atom of λ. Every leaf ℓ of λ is oriented as a part of the boundary of the component of $\mathbb{H} \setminus \ell$ containing A. Let a be the initial point and b the terminal point of ℓ for the given orientation. Let $[a, b]$ be an oriented interval connecting endpoints of ℓ (cf. Figure 6). Then, we set

$$\dot{E}_\ell^\lambda(z) = \begin{cases} 0 & \text{for } z \text{ outside of } [a,b] \\ \frac{(z-a)(z-b)}{a-b} & \text{for } z \in [a,b]. \end{cases}$$

When $\ell \in \mathcal{G}$ is not a leaf of λ, we put $\dot{E}_\ell^\lambda(z) = 0$ for all $z \in \partial\mathbb{H} = S^1$. For any point $z \in \partial\mathbb{H} = S^1$, $\dot{E}_\ell^\lambda(z)$ is a function of $\ell \in \mathcal{G}$.

We consider the integral

$$\dot{E}^\lambda(z) := \int_{\mathcal{G}} \dot{E}_\ell^\lambda(z) d\lambda(\ell) \tag{9.1}$$

for a measured lamination λ. For a finite lamination $\lambda = \sum_{i=1}^{m} \lambda_i \ell_i$, by definition, it holds

$$\dot{E}^{\lambda}(z) = \sum_{i=1}^{m} \lambda_i \dot{E}^{\lambda}_{\ell_i}(z).$$

One can show that the integral $\dot{E}^{\lambda}$ in (9.1) is well-defined for all $\lambda \in \mathcal{ML}_b(\mathbb{H})$ by an approximation argument (see [6]). We give a more direct proof of the convergence of the integral in the Appendix (cf. §10).

9.1.1. *Infinitesimal earthquakes.* For $\lambda \in \mathcal{ML}_b(\mathbb{H})$ and $t > 0$, we normalize $E^{t\lambda}$ to be the identity on the stratum A which we have fixed before. Gardiner-Hu-Lakic [8] proved that the integral (9.1) gives the tangent vector fields to the paths of earthquake deformations:

$$\dot{E}^{\lambda}(z) = \frac{d}{dt} E^{t\lambda}(z)\Big|_{t=0} \tag{9.2}$$

for $z \in \partial\mathbb{H} = S^1$ (cf. [8]). Let $\mathcal{Z}(\partial\mathbb{H})$ be the Banach space of Zygmund functions on $\partial\mathbb{H}$ modulo the subspace of quadratic polynomials (cf. §**??**). Gardiner [6] also proved the *infinitesimal earthquake theorem*, which states that the map

$$\mathcal{ML}_b(\mathbb{H}) \ni \lambda \mapsto \dot{E}^{\lambda} \in \mathcal{Z}(\partial\mathbb{H}) \tag{9.3}$$

is bijective (Theorem 5.1 of [6]).

9.1.2. *Convergence of vector fields.* The following proposition is well-known.

Proposition 9.1. *Let $\lambda \in \mathcal{ML}_b(\mathbb{H})$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{ML}_b(\mathbb{H})$ with uniformly bounded Thurston norms. If $\{\lambda_n\}_{n=1}^{\infty}$ converges to λ in the weak* topology, then $\dot{E}^{\lambda_n}$ pointwise converges to $\dot{E}^{\lambda}$ on $\partial\mathbb{H}$.*

We shall give a proof of Proposition 9.1 in the Appendix (§10.3) for completeness. After that, we will give a simple proof of the formula (9.2) using holomorphic motions and Proposition 9.1 in §10.4.

9.2. **Uniform weak* and Zygmund topologies.** Let V be a continuous function on $\partial\mathbb{H} = S^1$ satisfying $V(z)/(iz) \in \mathbb{R}$ for $z \in \partial\mathbb{H} = S^1$. We say that V is in *the Zygmund class* if there is an $M > 0$ such that

$$|V(e^{i(x+t)}) + V(e^{i(x-t)}) - 2V(e^{ix})| \le M|t| \tag{9.4}$$

for all $0 \le x < 2\pi$ and $0 < t < \pi$. The infimum of the constant M in (9.4) is called the *Zygmund norm* of V and we denote it by $\|V\|_{Zyg}$. Recall that $\|V\|_{Zyg} = 0$ if and only if V is a quadratic polynomial. The quotient of the class of continuous functions satisfying $V(z)/(iz) \in \mathbb{R}$ for $z \in \partial\mathbb{H}$ and inequality (9.4) by the subspace consisting of the quadratic polynomials becomes a Banach space $\mathcal{Z}(\partial\mathbb{H})$ with the norm $\|\cdot\|_{Zyg}$. We call $\mathcal{Z}(\partial\mathbb{H})$ the *Zygmund space.*

We define the *cross-ratio norm* on $\mathcal{Z}(\partial\mathbb{H})$ as follows. Let $Q = [a,b] \times [c,d]$ be a box of geodesics such that 4-points a, b, c, d lie on $\partial\mathbb{H}$ in the counter-clockwise. For $V \in \mathcal{Z}(\partial\mathbb{H})$, we set

$$V[Q] = \frac{V(a) - V(c)}{a - c} + \frac{V(b) - V(d)}{b - d} - \frac{V(a) - V(d)}{a - d} - \frac{V(b) - V(c)}{b - c}.$$

Then, the *cross-ratio norm* $\|V\|_{cr}$ of V is defined by

$$\|V\|_{cr} = \sup_{Q} |V(Q)|$$

where Q runs over all boxes with Liouville measure $L(Q) = \log 2$. The Zygmund norm is equivalent to the cross-ratio norm on $\mathcal{Z}(\partial\mathbb{H})$ (see [7]).

9.3. **Proof of Theorem 3.** By Gardiner's infinitesimal earthquake theorem the map (9.3) is bijective. Hence it suffices to show that the map and its inverse are both continuous.

We first check that the map (9.3) is continuous. Let $\lambda_n \to \lambda$ as $n \to \infty$ in the uniform weak* topology. Then $\|\lambda_n\|_{Th}$ is uniformly bounded. It follows that the sequence $V_n := \dot{E}^{\lambda_n}|_{S^1}$ has uniformly bounded cross-ratio norms. Indeed, the cross-ratio norm gives the infinitesimal change in the cross-ratios under the earthquake path $t \mapsto E^{t\lambda_n}|_{\partial\mathbb{H}}$. Assume on the contrary that $\|V_n\|_{cr} \to \infty$ as $n \to \infty$. Then there exists a sequence Q_n of boxes in $\mathcal{G}$ with the Liouville measure $L(Q_n) = \log 2$ such that $|V_n[Q_n]| \to \infty$ as $n \to \infty$. Let $\gamma_{Q_n} : Q^* \mapsto Q_n$ be a Möbius map from the standard box Q^* and let $\lambda_n' := (\gamma_n)^*(\lambda_n)$. Then there exists a subsequence of λ_n', denoted by λ_n' for simplicity, which converges in the weak* topology to a bounded measured lamination λ'. Then, by Proposition 9.1, there exists an appropriate normalization of the earthquake vector fields such that $\dot{E}^{\lambda_n'}|_{S^1} \to \dot{E}^{\lambda'}|_{S^1}$ pointwise as $n \to \infty$. Since $|V[Q_n]| = |\dot{E}^{\lambda_n'}|_{S^1}[Q^*]| \to \infty$ as $n \to \infty$, this gives a contradiction. Thus the vector fields V_n have uniformly bounded cross-ratio norms.

A family of normalized Zygmund bounded maps (normalized to be zero at three fixed points of S^1) whose cross-ratio norms are uniformly bounded is a normal family (see [7]). If necessary, we normalize $\dot{E}^{\lambda_n}|_{S^1}$ by adding a quadratic polynomial, such that $\dot{E}^{\lambda_n}|_{S^1}$ is a normal family. Assume on the contrary that $\dot{E}^{\lambda_n}|_{S^1} \not\to \dot{E}^{\lambda}|_{S^1}$ in the cross-ratio norm topology. Then there are $C > 0$ and a sequence of quadruples Q_n in S^1 with $L(Q_n) = \log 2$ such that $|\dot{E}^{\lambda_n}[Q_n] - \dot{E}^{\lambda}[Q_n]| \geq C$. Let γ_{Q_n} be the Möbius map such that $\gamma_{Q_n} : Q^* \mapsto Q_n$, where $Q^* = [-i, 1] \times [i, -1]$ is the standard box. Then $|\gamma_{Q_n}^*(\dot{E}^{\lambda_n})[Q^*] - \gamma_{Q_n}^*(\dot{E}^{\lambda})[Q^*]| \geq C$ for all n. Since $\|\gamma_{Q_n}^*(\lambda_n)\|_{Th} = \|\lambda_n\|_{Th}$ and $\|\gamma_{Q_n}^*(\lambda)\|_{Th} = \|\lambda\|_{Th}$, it follows that the Thurston norms of $\gamma_{Q_n}^*(\lambda_n)$ and $\gamma_{Q_n}^*(\lambda)$ are uniformly bounded. Therefore, we can extract convergent subsequences of $\gamma_n^*(\lambda_n)$ and $\gamma_n^*(\lambda)$ in the weak* topology, which we denote by the same letters for simplicity. The assumption on the convergence $\lambda_n \to \lambda$ in the uniform weak* topology implies that the limit of $\gamma_n^*(\lambda_n)$ equals to the limit of $\gamma_n^*(\lambda)$. On the other hand, the two sequences of vector fields $\gamma_n^*(\dot{E}^{\lambda_n})$ and $\gamma_n^*(\dot{E}^{\lambda})$ converge pointwise to different limits (even different up to addition of a quadratic polynomial) because they differ on the standard box Q^*. This implies that a single measured lamination represents two different earthquake vector fields which is impossible. Thus the map $\lambda \mapsto \dot{E}^{\lambda}|_{S^1}$ is continuous.

It remains to show that the inverse map is continuous. From this point until the end of the proof we replace $\mathbb{H}$ with the upper half-plane model. The ideal boundary of the upper half-pane is $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Assume that $\dot{E}^{\lambda_n}|_{\hat{\mathbb{R}}} \to \dot{E}^{\lambda}|_{\hat{\mathbb{R}}}$ as $n \to \infty$ in the cross-ratio norm. We claim that there exists $C > 0$ such that $\|\lambda_n\|_{Th} < C$ for all n. Suppose on the contrary that $\|\lambda_n\|_{Th} \to \infty$ as $n \to \infty$. Then there exists a sequence I_n of closed geodesic arcs in the upper half-plane whose length is $1/n$ such that the λ_n-mass of the geodesics intersecting I_n goes to infinity as $n \to \infty$. Let l_n and r_n be the leftmost and the rightmost geodesic of $|\lambda_n|$ which intersect I_n. It is possible that $l_n = r_n$. Let γ_n be the Möbius map such that the endpoints of $\gamma_n(l_n)$ are fixed points $b, d \in \mathbb{R}$, say $b < d$, and such that the endpoints of $\gamma_n(r_n)$ converge to b and d, respectively. Let $a, c \in \mathbb{R}$ with $a < b$ and $b < c < d$ be such that box $Q = [a,b] \times [c,d]$ has the Liouville measure $L(Q) = \log 2$. We normalize $\dot{E}^{(\gamma_n^{-1})^*(\lambda_n)}|_{\hat{\mathbb{R}}} = (\gamma_n^{-1})^*(\dot{E}^{\lambda_n}|_{\hat{\mathbb{R}}})$ by orienting all the leaves of $|\gamma_n(\lambda_n)|$ to the left with respect to the geodesic $\lceil b, d\rceil$.

The cross-ratio norm is invariant under the push-forward by Möbius maps. This implies that $\|\dot{E}^{(\gamma_n^{-1})^*(\lambda_n)}|_{\hat{\mathbb{R}}}\|_{cr} = \|\dot{E}^{\lambda_n}|_{\hat{\mathbb{R}}}\|_{cr}$ is bounded. Let $V_n = \dot{E}^{(\gamma_n^{-1})^*(\lambda_n)}|_{\hat{\mathbb{R}}}$ for short. The normalization that we imposed on V_n gives that

$$V_n[Q] = V_n(a)[\frac{1}{a-c} - \frac{1}{a-b}] + V_n(c)[\frac{-1}{a-c} + \frac{-1}{c-d}].$$

Both terms are non-negative. Moreover, $V_n(c) \geq \lambda_n(I_n) \to \infty$ as $n \to \infty$, where $\lambda_n(I_n)$ is the λ_n-mass of geodesics intersecting I_n. Thus $V_n[Q] \to \infty$ as $n \to \infty$ which is a contradiction. Thus $\|\lambda_n\|_{Th}$ is uniformly bounded.

Assume on the contrary that $\lambda_n \nrightarrow \lambda$ as $n \to \infty$ in the uniform weak* topology. Then, after possibly taking a subsequence and renaming it, there exists a sequence Q_n of quadruples on $\hat{\mathbb{R}}$ such that $L(Q_n) = \log 2$ and

$$|\dot{E}^{\lambda_n}|_{S^1}[Q_n] - \dot{E}^{\lambda}|_{S^1}[Q_n]| \geq c > 0. \tag{9.5}$$

Let γ_n be Möbius map which maps $Q = (-a,-1,1,a)$ onto Q_n, where $a > 1$ is chosen such that $L(Q) = \log 2$. Let $\mu_n = (\gamma_n)^*(\lambda_n)$ and $\xi_n = (\gamma_n)^*(\lambda)$. Since $\|\mu_n\|_{Th}$ and $\|\xi_n\|_{Th}$ are uniformly bounded, there exist two subsequences of μ_n and ξ_n with common indexing which converge in the weak* topology. We can assume that μ_n and ξ_n converge in the weak* topology to μ and ξ, respectively. By (9.5) we get that $|\dot{E}^{\mu}|_{\hat{\mathbb{R}}}[Q] - \dot{E}^{\xi}|_{\hat{\mathbb{R}}}[Q]| \geq c > 0$ which implies that $\mu \neq \xi$. On the other hand, since $\dot{E}^{\lambda_n}|_{\hat{\mathbb{R}}} \to \dot{E}^{\lambda}|_{\hat{\mathbb{R}}}$ in the uniform weak* topology, it follows that if the push-forwards of $\dot{E}^{\lambda_n}|_{\hat{\mathbb{R}}}$ and $\dot{E}^{\lambda}|_{\hat{\mathbb{R}}}$ by a sequence of Möbius maps pointwise converge then the limits have to be equal. This is a contradiction with $\mu \neq \xi$ by the uniqueness of the earthquake measures. Thus $\lambda_n \to \lambda$ as $n \to \infty$ in the uniform weak* topology which is what we needed.

10. Appendix : The integral $\dot{E}^{\lambda}$

In this section, we consider the integral presentation of the earthquake vector field. We prove (see §10.2) that the integral in (9.1) is well-defined.

10.1. **Strata and restricted measures.**

Recall that a stratum of a (measured) geodesic lamination λ is either a leaf of λ or the closure of a component of $\mathbb{H} \setminus \lambda$. By a *generalized stratum*, we mean either a stratum of λ or a point of $\partial\mathbb{H}$.

Let λ be a measured lamination. Let A and B be two generalized strata of λ. We denote by $\lambda_{A,B}$ a measured lamination whose support consists of leaves of λ separating A and B in $\mathbb{H}$, and a leaf in ∂A (resp. ∂B) facing B (resp. A), if A (resp. B) is a gap. The measure is defined to be the restriction of λ on the above set of geodesics. Thus, $\lambda_{A,B}$ is a measured geodesic lamination.

Alternatively, take a geodesic I connecting A and B where $A \cap I$ and $B \cap I$ are points. When either A or B, say B, is a point of $\partial\mathbb{H}$, we set I to be a geodesic ray from a point of A terminating at B such that $A \cap I$ consists of a point. When both A and B are points of $\partial\mathbb{H}$, then I is the bi-infinite geodesic connecting them. Let $|\lambda|_I$ be leaves of λ intersecting I. Notice that the set $|\lambda|_I$ is independent of the choice of the geodesic I. Since I is closed, $|\lambda|_I$ is a geodesic lamination, that is, it is a closed subset of $\mathcal{G}$. Hence the restriction of λ to $|\lambda|_I$ defines a Borel measure on $\mathcal{G}$ and hence it is recognized as a measured lamination $\lambda_{A,B}$ on $\mathbb{H}$. When we specify the geodesic I, we denote $\lambda_{A,B}$ by λ_I.

In this notation, if B is a point of $\partial\mathbb{H}$ and $B \in \partial A$, we recognize $\lambda_I = \lambda_{A,B}$ as the zero measure. This notation will appear in Proposition 10.1.

10.2. **The integral is well-defined.** In this section, we prove that the integral

$$\int_{\mathcal{G}} \dot{E}^{\lambda}_{\ell}(z)d\lambda(\ell) \tag{10.1}$$

is well-defined for all $z \in \partial\mathbb{H}$, when $\lambda \in \mathcal{ML}_b(\mathbb{H})$.

Remark 10.1. *Recall that when we fix $z \in \partial\mathbb{H}$,*

$$\mathcal{G} \ni \ell \mapsto \dot{E}^{\lambda}_{\ell}(z)$$

is a function with the domain $\mathcal{G}$. Notice from the definition that for $z \in \partial\mathbb{H}$, $\dot{E}^{\lambda}_{\ell}(z)$ is independent of the measure λ, depends only on the support $|\lambda|$ of λ. Hence we can define $\dot{E}^{\lambda}_{\ell}(z)$ for any geodesic lamination λ.

10.2.1. *Support of the integral.* Let A be the fixed stratum which we used to define $\dot{E}^{\lambda}_{\ell}(z)$ in §9. Let ℓ_A be the leaf of λ contained in the closure of A which is closest to z. Let z_0 be a point of ℓ_A.

Let I be the geodesic connecting z_0 and z. If $z \in \partial\mathbb{H} \cap \overline{A}$, $\dot{E}^{\lambda}_{\ell}(z)$ is identically 0 on $\mathcal{G}$. Hence the integral (10.1) converges in this case. Hence we may assume that z is not in $\overline{A}$. This means that $I \cap A = \{z_0\}$ and I is not contained in any leaf of λ.

We define a measured lamination λ_I as before. As above, we denote by $|\lambda|_I$ the support of λ_I. Namely, $|\lambda|_I = |\lambda_I| = |\lambda_{A,z}|$.

The following lemma is immediate from the definition of $\dot{E}^{\lambda}_{\ell}(z)$.

Lemma 10.1. *Suppose λ is a geodesic lamination. Then, for $z \in \partial\mathbb{H}$, the support of the function $\mathcal{G} \ni \ell \mapsto \dot{E}^{\lambda}_{\ell}(z)$ is equal to $|\lambda|_I = |\lambda_{A,z}|$.*

10.2.2. *Function $\tilde{e}_z$ on $\mathcal{G}$.* For $z \in \partial\mathbb{H}$, we define a function $\tilde{e}_z$ on $\mathcal{G}$ as follows. Let $\ell = \lceil a, b \rceil$. We set

$$\tilde{e}_z(\ell) := \begin{cases} \frac{(z-a)(z-b)}{a-b} & a \neq z \text{ and } b \neq z \\ 0 & \text{otherwise,} \end{cases} \tag{10.2}$$

where in the first row of the right-hand side of (10.2), a and b are chosen such that the ordered triple (a, z, b) lies on $\partial\mathbb{H}$ counterclockwise. For instance, in Figure 7, we have $\tilde{e}_z(\ell) = \frac{(z-a)(z-b)}{a-b}$ and $\tilde{e}_{z'}(\ell) = \frac{(z'-b)(z'-a)}{b-a}$. Notice that $\tilde{e}_z$ is well-defined and continuous on $\mathcal{G}$. Since $\tilde{e}_z(\ell) = \dot{E}^{\lambda}_{\ell}(z)$ on the support $|\lambda|_I$ of λ_I, by Lemma 10.1, we conclude the following.

Lemma 10.2. *Let λ be a measured lamination. Then, the function $\mathcal{G} \ni \ell \mapsto \dot{E}^{\lambda}_{\ell}(z)$ is measurable with respect to λ. Furthermore, for any $z \in \partial\mathbb{H}$, if the geodesic ray I above is not contained in any leaf of λ, it holds*

$$\int_{\mathcal{G}} \dot{E}^{\lambda}_{\ell}(z)d\lambda(\ell) = \int_{\mathcal{G}} \tilde{e}_z(\ell)d\lambda_I(\ell) = \int_{\mathcal{G}} \tilde{e}_z(\ell)d\lambda_{A,z}(\ell), \tag{10.3}$$

if either the middle term or the right-hand side of (10.3) are defined.

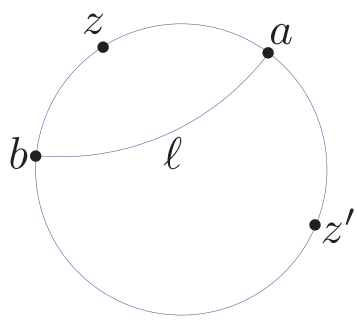

FIGURE 7. Geodesics ℓ and ℓ'.

In particular, the integral (10.1) is represented as the integration of a continuous function defined independently of λ, but depending only on z. Thus, to check the convergence of the integral (10.1), we may prove the integrability of $\tilde{e}_z$ with respect to $\lambda_{A,z}$.

We describe the properties of the function $\tilde{e}_z$. One can easily see that

$$\tilde{e}_{T(z)}(T(\ell))T'(z)^{-1} = \tilde{e}_z(\ell)$$

for all $\ell \in \mathcal{G}$, $z \in \partial\mathbb{H}$ and $T \in \text{Möb}(\mathbb{H})$. Let J be the radial geodesic ray emanating from 0 to $z \in \partial\mathbb{H}$. Let w_d $(d \geq 0)$ be the length parametrization of J with $w_0 = 0$. The function $\tilde{e}_z$ has the following property.

Lemma 10.3. *Let $z \in \partial\mathbb{H}$. For $D_0 > 0$,*

$$|\tilde{e}_z(\ell)| \leq (8\cosh(D_0))e^{-d}$$

when ℓ intersects the D_0-neighborhood of w_d.

Proof. Notice that the set $K_0 \subset \mathcal{G}$ of all geodesics intersecting the hyperbolic disk of center 0 and radius D_0 is compact. By a hyperbolic trigonometry formula, we have

$$|\tilde{e}_z(\ell)| = |(z-a)(z-b)|/|a-b| \leq 4/|a-b| \leq 8\cosh(D_0)$$

for all $\ell = \lceil a, b \rceil \in K_0$ and $z \in \partial\mathbb{H}$.

Let ℓ be a geodesic which intersects the D_0-neighborhood of w_d. Let T be a Möbius transformation acting on $\mathbb{H}$ with $T(w_d) = 0$ and fixing z. Since w_d is on J, $w_d = |w_d|z$. Since $T(\ell) \in K_0$, we obtain

$$\begin{aligned} |\tilde{e}_z(\ell)| &= |\tilde{e}_{T(z)}(T(\ell))||T'(z)|^{-1} \leq (8\cosh(D_0))|1-\overline{w_d}z|^2/(1-|w_d|^2) \\ &= (8\cosh(D_0))\frac{1-|w_d|}{1+|w_d|} = (8\cosh(D_0))e^{-d}, \end{aligned}$$

which implies what we wanted. □

10.2.3. *Proof that the integral is well-defined.* Recall that A is the stratum which we fixed in the beginning and $z_0 \in A$ is the initial point of I. Let z_d $(d \geq 0)$ be the length parametrization of I. We set $I_d = \{z_k \mid k \geq d\}$. We can define a measured lamination λ_{I_d} as above. Notice that if the support $|\lambda_I|$ of λ_I is compact then λ_{I_d} becomes the zero measure for d large enough.

The integral (10.1) for bounded measured laminations converges because of the following estimate.

Proposition 10.1 (Rate of decay)**.** *Let $\lambda \in \mathcal{ML}_b(\mathbb{H})$ and $z \in \partial\mathbb{H}$. Let ℓ_A be the leaf of λ in A facing z. Let $z_0 \in \ell_A$ and I be the geodesic ray emanating from z_0 and terminating at z as above. Then, there is a constant C_2 depending only on the hyperbolic distance between 0 and z_0 such that*

$$\int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_d}(\ell) \leq C_2 \|\lambda\|_{Th} \cdot e^{-d} \tag{10.4}$$

for $d \geq 0$.

Proof. When z is in the closure of A, the interval I is contained in A. Hence λ_I is the zero measure, and (10.4) holds for all $d \geq 0$. In this case $\dot{E}^{\lambda}_{\ell}(z)$ is identically zero on $\mathcal{G}$. Therefore, the integral in (10.1) converges and equals to zero (and the equation (10.3) also holds). Hence we may assume that $z \in \partial\mathbb{H} \setminus \overline{A}$. This assumption means that I transversely intersects some leaves of λ in $\mathbb{H}$. However, note that z may be an endpoint of some leaf of λ.

Divide I_d into a sequence $\{I_{n,d}\}_{n=0}^{\infty}$ of consecutive subintervals of I_d of unit length such that $I_{d,0}$ contains z_d. Then $I_{n,d} \cap I_{n+1,d} = \{z_{d+n}\}$. We define a measured sublamination $\lambda_{I_{n,d}}$ of λ_I as above. When there is no leaf of λ intersecting $I_{n,d}$, we define $\lambda_{I_{n,d}}$ to be the zero measure as we noted before. Let ℓ_n be a leaf of the support of $\lambda_{I_{n,d}}$ and $\{z_{d(n)}\} = \ell_n \cap I_{n,d}$. Note that $d(n)$ is the distance between $z_0 \in I$ to $\ell_n \cap I_{n,d} \in I$ and $d + n \leq d(n) \leq d + n + 1$.

As in Lemma 10.3, we denote by J the radial geodesic ray emanating from 0 to z, and w_d ($d \geq 0$) the length parametrization of J with $w_0 = 0$. Then, by the triangle inequality, we have $d_{\mathbb{H}}(0, z_{d(n)}) \geq n + d - D_0$, where $D_0 = d_{\mathbb{H}}(z_0, w_0)$. Since J shares the endpoint z with I, $d_{\mathbb{H}}(w_{d(n)}, z_{d(n)}) \leq d_{\mathbb{H}}(z_0, w_0) = D_0$, which means that any leaf of $\lambda_{I_{n,d}}$ intersects the $D_0 + 1$-neighborhood of $w_{d(n)}$. By Lemma 10.3, we have

$$|\tilde{e}_z(\ell)| \leq (8\cosh(D_0+1))e^{-d_{\mathbb{H}}(0,z_{d(n)})} \leq (8\cosh(D_0+1))e^{-(d+n-D_0)} = C_1 e^{-(d+n)},$$

where $C_1 = 8e^{D_0}\cosh(D_0+1)$.

Therefore, we get

$$\begin{aligned}\int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_{n,d}}(\ell) &\leq C_1 e^{-(d+n)} \lambda_{I_{n,d}}(\mathcal{G}) = C_1 e^{-(d+n)} \lambda_{I_{n,d}}(I_{n,d}) \\ &\leq C_1 \|\lambda\|_{Th}\, e^{-d} \cdot e^{-n},\end{aligned}$$

since each $I_{n,d}$ has unit length and the support of $\lambda_{I_{n,d}}$ is contained in $I_{n,d}$. Thus, we conclude

$$\int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_d}(\ell) \leq \sum_{n=0}^{\infty} \int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_{n,d}}(\ell) \leq C_2 \|\lambda\|_{Th} e^{-d},$$

where $C_2 = (1 - e^{-1})C_1$. □

10.3. **Weak* convergence and pointwise convergence.** In this section, we prove the continuity of the integral (9.1) on $\mathcal{ML}_b(\mathbb{H})$ with respect to the weak* topology.

Proposition 10.2 (Pointwise convergence)**.** *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of bounded measured laminations which converges in the weak* topology to a measured lamination $\lambda \in \mathcal{ML}_b(\mathbb{H})$. If the Thurston norms of the sequence $\{\lambda_n\}_{n=1}^{\infty}$ of measured* *laminations are uniformly bounded, then there is a choice of normalizations for* $\dot{E}_\ell^\lambda$ *and* $\dot{E}_\ell^{\lambda_n}$ *such that*

$$\lim_{n\to\infty}\int_{\mathcal{G}}\dot{E}_\ell^{\lambda_n}(z)d\lambda_n(\ell)=\int_{\mathcal{G}}\dot{E}_\ell^{\lambda}(z)d\lambda(\ell)$$

for all $z\in\partial\mathbb{H}=S^1$.

Proof. The proof follows the same outline as the proof of [15, Lemma 3.2]. We first fix the normalizations of $\dot{E}_\ell^\lambda$ and $\dot{E}_\ell^{\lambda_n}$. Let A be a fixed stratum of λ which is either a gap of λ or a leaf of λ whose λ-measure is zero (i.e. A is not an atom of λ). Let $z_0\in A$ be a point in the interior of A if it is a gap, or any point of A if it is a leaf of λ. Let A_n be the stratum of λ_n which contains z_0. We orient each $\ell\in|\lambda|$ to the left as seen from A. If A is a geodesic, then we orient A arbitrary. This gives a well-defined function $\dot{E}_\ell^\lambda$ for $\ell\in|\lambda|$ which in turn implies

$$\int_{\mathcal{G}}\dot{E}_\ell^{\lambda}(z)d\lambda(\ell)=\int_{\mathcal{G}}\tilde{e}(\ell)d\lambda_{A,z}(\ell).$$

We define $\dot{E}_\ell^{\lambda_n}$ by giving the left orientation to each ℓ with respect to the stratum A_n in the same fashion.

Let I be a geodesic ray from z_0 to z and let $z_d\in I$ be such that the distance between z_0 and z_d is $d\geq 0$. We fix $d>0$ such that z_d is contained in a stratum A_d of $|\lambda|$ which is either a gap or a leaf which is not an atom of λ.

Given $i\in\mathbb{N}$, let $I_i=(z_l^i,z_r^i)$ be an open geodesic arc whose endpoints are on the distance $1/i$ from z_0 and z_d, and which contains z_0,z_d. The set of geodesics of $\mathbb{H}$ which intersect I_i is open in $\mathcal{G}$ and contains all geodesics of $|\lambda|$ which intersect the closed geodesic arc with endpoints z_0 and z_d. Since the lengths of (z_l^i,z_0) and (z_d,z_r^i) are going to zero as $i\to\infty$, it follows that the λ-measure of the set of geodesics intersecting (z_l^i,z_0) and (z_d,z_r^i) is going to zero as $i\to\infty$ by the choice of z_0 and z_d (namely, A and A_{z_d} are either gaps or non-atomic leaves). Let $\varphi_i:\mathcal{G}\to\mathbb{R}$ be a non-negative continuous function whose support consists of geodesics intersecting $I_i=(z_l^i,z_r^i)$ and which is identically equal to 1 on the set of geodesics intersecting $[z_0,z_d]$. Then the function $\ell\mapsto\varphi_i(\ell)\tilde{e}_\ell(z)$ is a continuous function on $\mathcal{G}$ with compact support. It follows that

$$\int_{\mathcal{G}}\varphi_i(\ell)\tilde{e}_\ell(z)d\lambda_n(\ell)\to\int_{\mathcal{G}}\varphi_i(\ell)\tilde{e}_\ell(z)d\lambda(\ell)$$

as $n\to\infty$ by the weak* convergence $\lambda_n\to\lambda$.

Note that

$$\int_{\mathcal{G}}\varphi_i(\ell)\tilde{e}_\ell(z)d\lambda_n(\ell)\leq\int_{\mathcal{G}}|\tilde{e}_\ell(z)|d[(\lambda_n)_{(z_l^i,z_0)}+(\lambda_n)_{(z_d,z_r^i)}](\ell)+\int_{\mathcal{G}}\tilde{e}_\ell(z)d(\lambda_n)_{(z_0,z_d)}(\ell)$$

and

$$\int_{\mathcal{G}}\varphi_i(\ell)\tilde{e}_\ell(z)d\lambda(\ell)\leq\int_{\mathcal{G}}|\tilde{e}_\ell(z)|d[\lambda_{(z_l^i,z_0)}+\lambda_{(z_d,z_r^i)}](\ell)+\int_{\mathcal{G}}\tilde{e}_\ell(z)d\lambda_{(z_0,z_d)}(\ell).$$

The choice of z_0 and z_d is such that the total masses of $\lambda_{(z_l^i,z_0)}$ and $\lambda_{(z_d,z_r^i)}$ on $\mathcal{G}$ converge to zero as $i\to\infty$. Since λ_n converges to λ in the weak* sense, it follows that given $\epsilon>0$ there exist $i_0,n_0\in\mathbb{N}$ such that the total masses of $\lambda_{(z_l^i,z_0)}$, $\lambda_{(z_d,z_r^i)}$,

$(\lambda_n)_{(z_l^i,z_0)}$ and $(\lambda_n)_{(z_d,z_r^i)}$ on $\mathcal{G}$ are less than ϵ for $i \geq i_0$ and $n \geq n_0$. The above three inequalities imply that

$$\int_{\mathcal{G}} \tilde{e}_\ell(z) d(\lambda_n)_{(z_0,z_d)}(\ell) \to \int_{\mathcal{G}} \tilde{e}_\ell(z) d\lambda_{(z_0,z_d)}(\ell)$$

as $n \to \infty$.

Since $|\int_{\mathcal{G}} \tilde{e}_\ell(z) d(\lambda_n)_{(z_0,z_d)}(\ell) - \int_{\mathcal{G}} \tilde{e}_\ell(z) d\lambda_n(\ell)| \leq Ce^{-d}$ and $|\int_{\mathcal{G}} \tilde{e}_\ell(z) d\lambda_{(z_0,z_d)}(\ell) - \int_{\mathcal{G}} \tilde{e}_\ell(z) d\lambda(\ell)| \leq Ce^{-d}$, the conclusion follows. □

10.4. **Differentiation of earthquake paths.** In this section, we reprove the formula (9.2).

10.4.1. *Holomorphic motions and complex earthquakes.* Let S be a subset of $\hat{\mathbb{C}}$ and let D be a domain in $\hat{\mathbb{C}}$. A *holomorphic motion of S over D with base point $t_0 \in D$* is, by definition, a map $h : S \times D \to \hat{\mathbb{C}}$ satisfying the following three properties:

1. $h(x, t_0) = x$ for all $x \in S$.
2. For all $t \in D$, $h_t(\cdot) := h(\cdot, t)$ is injective on S.
3. For all $s \in S$, $h(s, \cdot) : D \to \hat{\mathbb{C}}$ is holomorphic.

By Slodkowski's theorem ([20]), if D is conformally equivalent to the unit disk, any holomorphic motion h of S over D with base point $t_0 \in D$ extends to a holomorphic motion $\tilde{h}$ of $\hat{\mathbb{C}}$ over D and for each $t \in D$, $\tilde{h}_t$ is K_t-quasiconformal mapping where $K_t = \exp(d_D(t_0, t))$ and d_D is the Poincaré distance on D normalized such that it has curvature -1.

The following theorem is proved in [13].

Theorem 6 (Theorem 2 in [13]). *Let $\lambda \in \mathcal{ML}_b(\mathbb{H})$. The earthquake map $(z,t) \mapsto E^{t\lambda}(z)$ for $t > 0$ and $z \in \partial\mathbb{H}$ extends to a holomorphic motion $(z,\tau) \mapsto E^{\tau\lambda}(z)$ of $\partial\mathbb{H}$ over a neighborhood S_λ of $\mathbb{R}$ in $\mathbb{C}$ with base point $\tau = 0$.*

The domain S_λ in the theorem above is concretely defined by

$$S_\lambda = \{\tau = t + is \mid |s| < \epsilon_0/[C_0 \exp(\|t\lambda\|_{Th})\|\lambda\|_{Th}]\}, \tag{10.5}$$

where ϵ_0 and C_0 are independent of λ.

Proof of Proposition 9.1. We first show the convergence in the case when $\{\lambda_n\}_{n=1}^\infty$ is a finite approximation of λ. From the proof of Theorem 2 in [13], we know that there is a neighborhood V_0 of $\partial\mathbb{H}$ such that the complement of V_0 contains at least 3 points and $E^{\tau\lambda_n}(z) \in V_0$ for all $\tau \in S_\lambda$, $z \in \partial\mathbb{H}$ and $n \in \mathbb{N}$, where we assume in the definition that the restriction of $E^{t\lambda_n}$ is the identity on a stratum of λ_n containing A. This implies that $\{E^{\tau\lambda_n}(z)\}_{\tau\in S_\lambda}$ is normal family and converges to $E^{\tau\lambda}(z)$ on any compact set of S_λ. From the Weierstrass' theorem, we have

$$\left.\frac{d}{d\tau}E^{\tau\lambda}(z)\right|_{\tau=0} = \lim_{n\to 0} \left.\frac{d}{d\tau}E^{\tau\lambda_n}(z)\right|_{\tau=0}.$$

On the other hand, by Theorem 10.2, the integral in (9.1) varies continuously on $\mathcal{ML}_b(\mathbb{H})$. Hence, we get the formula (9.2). □

References

[1] L. V. Ahlfors, *Lectures on quasiconformal Mappings*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1966.
[2] F. Bonahon, Geodesic currents and Teichmüller space, Invent. Math. 92 (1988), no. 1, 139–162.
[3] F. Bonahon, Geodesic laminations with transverse Hölder distributions, Ann. Sci. Ecole Norm. Sup. (4) 30 (1997), no. 2, 205–240.
[4] A. Douady and C.Earle, Conformally natural extension of homeomorphisms of the circle, Acta Math. 157 (1986), no. 1-2, 23–48.
[5] D. Epstein and A.Marden, Convex hulls in hyperbolic Space, a theorem of Sullivan, and measured pleated surfaces, *Analytical and geometric aspects of hyperbolic space*, 113–253, London Math. Soc. Lecture Note Ser., 111, Cambridge Univ. Press, Cambridge (1987).
[6] F.Gardiner, Infinitesimal bending and twisting in one-dimensional dynamics, Trans. Amer. Math. Soc. 347 (1995), no. 3, 915–937.
[7] F.Gardiner and N.Lakic, Quasiconformal Teichmüller theory, Mathematical Surveys and Monographs, **76**. American Mathematical Society, Providence, RI (2000).
[8] F. Gardiner, J. Hu, and N. Lakic, Earthquake curves, complex manifolds and hyperbolic geometry (Guanajuato, 2001), 141–195, Contemp. Math., 311, Amer. Math. Soc., Providence, RI, 2002.
[9] J. Hu, Earthquake measure and cross-ratio distortion, In the tradition of Ahlfors and Bers, III, 285–308, Contemp. Math., 355, Amer. Math. Soc., Providence, RI, 2004.
[10] J. Hu, Norms on earthquake measures and Zygmund functions, Proc. Amer. Math. Soc. 133 (2005), no. 1, 193–202
[11] S. Kerckhoff, Earthquakes are analytic, Comment. Math. Helv. 60 (1985), no. 1, 17–30.
[12] J.P. Otal, About the embedding of Teichmüller space in the space of geodesic Hölder distributions, *Handbook of Teichmüller theory* Vol. I, 223–248, IRMA Lect. Math. Theor. Phys., **11**, Eur. Math. Soc., Zürich (2007).
[13] D. Šarić, Real and complex earthquakes, Trans. Amer. Math. Soc. 358 (2006), no. 1, 233–249.
[14] D. Šarić, Infinitesimal Liouville distributions for Teichmüller space, Proc. London Math. Soc. (3) 88 (2004), no. 2, 436–454.
[15] D. Šarić, Bounded earthquakes, Proc. Amer. Math. Soc. 136 (2008), no. 3, 889–897
[16] D. Šarić, Some remarks on bounded earthquakes, Proc. Amer. Math. Soc. 138 (2010), no. 3, 871–879.
[17] D. Šarić, Geodesic currents and Teichmüller space, Topology 44 (2005), no. 1, 99–130.
[18] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J (1970).
[19] L. Schwartz, Radon measures on arbitrary topological spaces and cylindrical measures, Tata Institute of Fundamental Research, Oxford University Press (1973).
[20] Z. Slodkowski, Holomorphic motions and polynomial hulls, Proc. Amer. Math. Soc. 111 (1991), no. 2, 347–355.
[21] W. Thurston, Earthquakes in two-dimensional hyperbolic geometry, *Low-dimensional topology and Kleinian groups*, LMS. Lecture Note Ser., 112, Cambridge Univ. Press, Cambridge (1986), 91–112.
[22] A. Zygmund, Smooth functions, Duke Math. J. **12**, (1945). 47–76.

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