

ZYGMUND VECTOR FIELDS, HILBERT TRANSFORM AND FOURIER COEFFICIENTS IN SHEAR COORDINATES

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ABSTRACT. We parametrize the space \mathcal{Z} of Zygmund vector fields on the unit circle in terms of infinitesimal shear functions on the Farey tessellation. Then we express the Hilbert transform and the Fourier coefficients of the Zygmund vector fields in terms of the above parametrization by infinitesimal shear functions. Finally, we compute the Weil-Petersson metric on the Teichmüller space of a punctured surface in terms of shears.

1. INTRODUCTION

The hyperbolic plane \mathbf{H} has the unit circle S^1 as its ideal boundary. Vector fields on the unit circle S^1 are describing infinitesimal deformations of the circle maps. We study Zygmund vector fields which arise as tangent vectors to the space of quasimetric maps of S^1 at the identity map $id : S^1 \rightarrow S^1$ (see [15], [25]). The Teichmüller space $T(\mathbf{H})$ of the hyperbolic plane \mathbf{H} , called the *universal Teichmüller space*, consists of all quasimetric maps of the unit circle S^1 which fix $1, i, -1 \in S^1$. Therefore, the space \mathcal{Z} of all Zygmund vector fields on S^1 which vanish at $1, i, -1 \in S^1$ is the tangent space to the universal Teichmüller space $T(\mathbf{H})$ at the *basepoint* $id \in T(\mathbf{H})$ (see [11]).

The Farey tessellation \mathcal{F} of the hyperbolic plane \mathbf{H} is a locally finite ideal triangulation of \mathbf{H} which is invariant under the group of isometries of \mathbf{H} generated by hyperbolic reflections in the sides of a complementary triangle of \mathcal{F} . The group of orientation preserving isometries of \mathbf{H} that setwise preserve \mathcal{F} is isomorphic to $PSL_2(\mathbf{Z})$ and it acts simply transitively on the oriented edges of \mathcal{F} . A vector field on S^1 naturally induces a real-valued function on the edges of \mathcal{F} , called the *infinitesimal shear function* of V (see Section 4). We parametrize the space \mathcal{Z} of Zygmund vector fields on S^1 that vanish at $1, i, -1 \in S^1$ in terms of the corresponding shear functions. Then we proceed to compute the Hilbert transform of Zygmund vector fields in terms of infinitesimal shear functions. We apply these formulas to compute the Weil-Petersson metric on Teichmüller spaces of finite punctured surfaces and the Fourier coefficients of Zygmund vector fields. Finally, we show that the formula for the Weil-Petersson metric in shear coordinates extend by continuity to the Weil-Petersson metric of the boundary Teichmüller space in its shear coordinates (which is a result of Masur [18]). We give more details below.

A quasimetric map $h : S^1 \rightarrow S^1$ induces a shear function $s : \mathcal{F} \rightarrow \mathbf{R}$ which assigns to each edge $e \in \mathcal{F}$ a real number

$$s(e) = \log cr(a, b, c, d)$$

where the endpoints of e are $b, d \in S^1$, the two adjacent complementary triangles of \mathcal{F} with common side e have third vertices $a, c \in S^1$, and $cr(a, b, c, d) = \frac{(b-c)(d-a)}{(b-a)(d-c)}$. A *fan of geodesics* of \mathcal{F} with *tip* $p \in S^1$ is the set \mathcal{F}_p of all geodesics of \mathcal{F} with one endpoint p . Fix a correspondence of \mathcal{F}_p with the integers \mathbf{Z} such that $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbf{Z}}$ and e_n^p is adjacent to e_{n+1}^p for all $n \in \mathbf{Z}$. We have the following characterization of quasymmetric maps in terms of shear functions.

Theorem 1. [29] *A shear function $s : \mathcal{F} \rightarrow \mathbf{R}$ induces a quasymmetric map of the unit circle if and only if there exists a constant $M \geq 1$ such that for each fan $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbf{Z}}$ and for each $m, k \in \mathbf{Z}$, $k \geq 0$, we have*

$$\frac{1}{M} \leq e^{s(e_m)} \frac{1 + e^{s(e_{m+1})} + \dots + e^{s(e_{m+1}) + \dots + s(e_{m+k})}}{1 + e^{-s(e_{m-1})} + \dots + e^{-s(e_{m-1}) - \dots - s(e_{m-k})}} \leq M.$$

A differentiable path $h_t : S^1 \rightarrow S^1$, $|t| < \epsilon$, of quasymmetric maps with $h_0 = id$ induces a Zygmund vector field V on S^1 by the formula

$$V(z) = \frac{d}{dt} h_t(z)|_{t=0}.$$

If $s_t : \mathcal{F} \rightarrow \mathbf{R}$ is the path of shear functions for h_t , then the derivative

$$\frac{d}{dt} s_t(e)|_{t=0} = \dot{s}(e)$$

for $e \in \mathcal{F}$ induces a *infinitesimal shear function*

$$\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$$

corresponding to the Zygmund vector field V . Our first result given in Theorem 4.1 is a characterization of infinitesimal shear functions which induce Zygmund bounded vector fields on S^1 .

Theorem 2. *Let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be a shear function. Then \dot{s} induces a Zygmund vector field V on S^1 if and only if there exists a constant $C > 0$ such that for all fans $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbf{Z}}$ and for all $m, k \in \mathbf{Z}$, $k \geq 0$, we have*

$$\left| \dot{s}(e_m) + \frac{k}{k+1} [\dot{s}(e_{m+1}) + \dot{s}(e_{m-1})] + \frac{k-1}{k+1} [\dot{s}(e_{m+2}) + \dot{s}(e_{m-2})] + \dots + \frac{1}{k+1} [\dot{s}(e_{m+k}) + \dot{s}(e_{m-k})] \right| \leq C.$$

A vector field on S^1 that vanishes at $1, i, -1$ is uniquely determined by its infinitesimal shear function. Therefore the above theorem gives an explicit (and rather simple) parametrization of the space \mathcal{Z} (of all normalized Zygmund bounded vector fields) which is the tangent space at the basepoint of the universal Teichmüller space $T(\mathbf{H})$. This should be compared to the question of characterizing Zygmund vector fields (or Zygmund maps) in terms of its Fourier coefficients where no explicit condition is known (see [14]). The key idea in the proof of the above theorem is to “decompose” a vector field V into an infinite sum of the vector fields V_p over all fans of \mathcal{F}_p of \mathcal{F} . The infinitesimal shear functions of V_p are zero on all geodesics of \mathcal{F} except on the geodesics of \mathcal{F}_p where they are equal to (the one-half of) the

infinitesimal shear function \dot{s} . We obtain (see Corollary 4.8)

$$V(z) = \sum_{p \in \mathcal{F}^0} V_p(z)$$

where \mathcal{F}^0 is the set of vertices of \mathcal{F} ; the convergence is absolute and uniform; each partial sum of V_p 's and the sum V is Zygmund bounded with a Zygmund constant continuously depending on C . The decomposition of V into vector fields V_p corresponding to the fans of \mathcal{F} is well-suited for the Zygmund vector fields, while writing the sum over all geodesics of \mathcal{F} does not even produce a pointwise convergence on S^1 . This is in an analogy with the Césaro summation of the Fourier series of a continuous function where the sum of the Fourier series does not necessarily converge at a single point of S^1 (see [14]).

An interesting problem in Teichmüller theory is to express the complex structure of Teichmüller spaces in terms of the hyperbolic metric on the underlining surfaces. Kerckhoff proved that the Hilbert transform H of the Zygmund vector fields on S^1 is the almost complex structure on the tangent space \mathcal{Z} at the basepoint of the universal Teichmüller space $T(\mathbf{H})$ (see Nag-Verjovsky [20]). We compute the Hilbert transform in terms of infinitesimal shear functions (which are hyperbolic geometry invariants) which gives a solution to computing of the almost complex structure on the universal Teichmüller space $T(\mathbf{H})$ in terms of the hyperbolic geometry (see Theorem 5.2 and Theorem 5.4).

Theorem 3. *Let V be a Zygmund bounded vector field on S^1 and let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be the corresponding shear function. Then the Hilbert transform of V is given by the series*

$$HV(x) = \sum_{p \in \hat{\mathcal{Q}}} HV_p(x)$$

where HV_p is the Hilbert transform of V_p and V_p is defined as above using the shears in the fan \mathcal{F}_p . The series converges uniformly on S^1 .

Moreover, the Hilbert transform $H(V_p)$ of the vector field V_p corresponding to the fan $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbf{Z}}$ is given by the formula

$$HV_p(z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \dot{s}(e_n^p) \frac{(z - a_n^p)(z - b_n^p)}{a_n^p - b_n^p} \log \left| \frac{z - b_n^p}{z - a_n^p} \right|$$

where a_n^p is the initial point and b_n^p is the terminal point of e_n^p .

Theorem 3 gives the Hilbert transform on Zygmund vector fields on S^1 in terms of infinitesimal shear functions from Theorem 2. Penner [24] gave a related formula for the Hilbert transform on $C^{5/2+\epsilon}$ -smooth vector fields which are contained in the space of Zygmund vector fields. The formula in Theorem 3 is a double sum with the exterior sum over all fans of \mathcal{F} and the interior sum over all geodesic of a single fan. We remark that a single sum over all geodesics in \mathcal{F} does not necessarily converge which forces the double sum. The formula of Theorem 3 can be explicitly given in terms of the hyperbolic geometry and we obtain an expression for the infinitesimal shear function $H(\dot{s})$ of the Zygmund bounded vector field $H(V)$ in terms of the hyperbolic invariants of \mathcal{F} and the infinitesimal shear function \dot{s} (see Corollary 6.1 and Corollary 6.2).

We also compute the Fourier coefficients of Zygmund bounded vector fields in terms of infinitesimal shear functions (see Theorem 7.1).

Theorem 4. *Let $V : S^1 \rightarrow \mathbf{C}$ be a Zygmund vector field and let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be the corresponding infinitesimal shear function. Then the n -th Fourier coefficient $\widehat{V}(n)$ of the vector field V is given by*

$$\widehat{V}(n) = \sum_{p \in \mathcal{F}^0} \widehat{V}_p(n),$$

where the convergence is absolute and

$$\begin{aligned} \widehat{V}_p(n) = \sum_{e_n = (e^{i\phi_0^n}, e^{i\phi_1^n}) \in \mathcal{F}_p} \frac{\dot{s}(e_n)}{4\pi(e^{i\phi_0} - e^{i\phi_1})} & \left[\frac{e^{i(2-n)\phi_1} - e^{i(2-n)\phi_0}}{i(2-n)} - (e^{i\phi_0} + e^{i\phi_1}) \times \right. \\ & \left. \times \frac{e^{i(1-n)\phi_1} - e^{i(1-n)\phi_0}}{i(1-n)} + e^{i(\phi_0 + \phi_1)} \frac{e^{-in\phi_1} - e^{-in\phi_0}}{-in} \right] \end{aligned}$$

Finally, we consider the almost-complex structure on the Teichmüller space $T(S)$ of a finite area hyperbolic surface S with at least one puncture. Let τ be an ideal geodesic triangulation of S and let $\tilde{\tau}$ be the lift of τ to the hyperbolic plane \mathbf{H} . Then a infinitesimal shear function $\dot{s} : \tilde{\tau} \rightarrow \mathbf{R}$ which is invariant under the deck transformations for the universal covering of S and which satisfies the cusp condition (see Section 8) represents a tangent vector V to $T(S)$. We obtain a formula for the infinitesimal shear function $H(\dot{s})$ of the Hilbert transform of V and for the Weil-Petersson metric on $T(S)$ (see Corollary 8.4 and Theorem 9.1).

Theorem 5. *Let $\dot{s} : \tilde{\tau} \rightarrow \mathbf{R}$ be the infinitesimal shear function of a Zygmund bounded vector field V invariant under a co-finite group G , where $\tilde{\tau}$ is an ideal geodesic triangulation of \mathbf{H} invariant under G . Then the infinitesimal shear function*

$$H(\dot{s}) : \tilde{\tau} \rightarrow \mathbf{R}$$

of the vector field $H(V)$ obtained by taking the Hilbert transform of V is given by

$$H(\dot{s})(b, d) = \sum_{n \in \mathbf{N}} \dot{s}(e_n) \Delta_{b,d}(e_n)$$

where $(b, d) \in \tilde{\tau}$ is the common boundary side of the two complementary triangles of $\tilde{\tau}$ with vertices (a, b, d) and (b, c, d) , $\tilde{\tau} = \{e_n\}_{n \in \mathbf{N}}$,

$$\begin{aligned} \Delta_{b,d}(e_n) = \sinh^2 \left(\frac{\delta_{b,c}}{2} \right) \log \coth^2 \left(\frac{\delta_{b,c}}{2} \right) + \cosh^2 \left(\frac{\delta_{a,d}}{2} \right) \log \coth^2 \left(\frac{\delta_{a,d}}{2} \right) - \\ \cosh^2 \left(\frac{\delta_{a,b}}{2} \right) \log \coth^2 \left(\frac{\delta_{a,b}}{2} \right) - \cosh^2 \left(\frac{\delta_{c,d}}{2} \right) \log \coth^2 \left(\frac{\delta_{c,d}}{2} \right) \end{aligned}$$

and $\delta_{b,c}$ is the distance between between the geodesic e_n and the geodesic (b, c) with analogous definition in other cases $\delta_{a,d}, \delta_{a,b}, \delta_{c,d}$. If e_n has at least one endpoint in (a, b, c, d) then $\Delta_{b,d}(e_n)$ is given appropriate interpretation as in Section 6.

Moreover, the Weil-Petersson metric pairing of the two tangent vectors v_1, v_2 to $T(S)$ represented by two infinitesimal shear functions $\dot{s}_1, \dot{s}_2 : \tau \rightarrow \mathbf{R}$ is given by

$$g_{WP}(v_1, v_2) = 2i(\dot{s}_1, H(\dot{s}_2))$$

where $H(\dot{s}_2)$ is the projection of $H(\dot{s}_2)$ onto τ and $i(\cdot, \cdot)$ is Thurston's algebraic intersection number.

We point out that the expression in the above theorem which represents the almost complex structure on the tangent space in terms of infinitesimal shear functions is given by a single sum unlike for arbitrary Zygmund bounded vector fields

where the sum is double. This feature appears because the infinitesimal shear functions are invariant under the action of a cofinite Fuchsian group G . Finally, we use the above expression of the almost complex structure in terms of shear coordinates to conclude that the almost complex structure on $T(S)$ extend continuously to the boundary Teichmüller spaces of $T(S)$ obtained by pinching some closed curves on S (see Section 10). Together with the result of J. Roger [27] which establishes that the Thurston's algebraic intersection number extend by continuity to the boundary of $T(S)$, we obtain the well-known result of Masur [18] that the Weil-Petersson metric extends by continuity to the boundary of $T(S)$. We also point out that G. Riera [26] obtained a formula for the Weil-Petersson metric on Teichmüller spaces of closed surfaces in terms of hyperbolic invariants.

2. ZYGMUND VECTOR FIELDS AND THE TANGENT SPACE TO $T(\mathbf{H})$

Let $\mathbf{H} = \{Im(z) > 0\}$ be the upper half-plane model of the hyperbolic plane with the hyperbolic metric given by $\rho(z) = \frac{|dz|}{Im(z)}$. Then $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is the ideal boundary of \mathbf{H} and $\hat{\mathbf{R}}$ is homeomorphic to the unit circle S^1 . A continuous map

$$V : \mathbf{R} \rightarrow \mathbf{R}$$

such that

$$(1) \quad \lim_{x \rightarrow \infty} \frac{V(x)}{x^2} = \lim_{x \rightarrow -\infty} \frac{V(x)}{x^2} < \infty$$

represents a (*continuous*) *vector field* on $\hat{\mathbf{R}}$. \mathbf{R} is a chart of $\hat{\mathbf{R}}$ and the condition (1) guarantees that V extends by continuity to a well-defined vector field at $\infty \in \hat{\mathbf{R}}$.

A vector field $V : \mathbf{R} \rightarrow \mathbf{R}$ on $\hat{\mathbf{R}}$ is said to be *Zygmund* if there exists $C > 0$ such that both $V(x)$ and $x^2V(1/x)$ satisfy

$$(2) \quad |V(x+t) + V(x-t) - 2V(x)| \leq Ct$$

for $x \in [-2, 2]$ and $t \in [0, 1]$. If V is Zygmund, then V can be normalized by adding a quadratic polynomial in x such that $V(0) = V(1) = 0$ and $V(x) = O(x \log |x|)$ as $|x| \rightarrow \infty$ (see [11]). We are mainly interested in Zygmund vector fields on $\hat{\mathbf{R}}$. A quadratic polynomial in x describes an infinitesimal change in $PSL_2(\mathbf{R})$ which is a trivial change in our context of the universal Teichmüller space $T(\mathbf{H})$. Alternatively, a vector field $V : \mathbf{R} \rightarrow \mathbf{R}$ on $\hat{\mathbf{R}}$ is Zygmund if it satisfies (2) for all $x \in \mathbf{R}$ and $t > 0$. The *Zygmund norm* of a Zygmund vector field $V : \mathbf{R} \rightarrow \mathbf{R}$ is the smallest constant C for which (2) remains in force.

Let $a, b, c, d \in \hat{\mathbf{R}}$ be four distinct points. The *cross-ratio* $cr(a, b, c, d)$ is defined by

$$cr(a, b, c, d) = \frac{(c-b)(d-a)}{(b-a)(d-c)}.$$

The *cross-ratio norm* $\|V\|_{cr}$ of a vector field $V : \mathbf{R} \rightarrow \mathbf{R}$ on $\hat{\mathbf{R}}$ is defined by

$$\|V\|_{cr} = \sup_{cr(a,b,c,d)=1} \left| \frac{V(c) - V(b)}{c-b} + \frac{V(d) - V(a)}{d-a} - \frac{V(b) - V(a)}{b-a} - \frac{V(d) - V(c)}{d-c} \right|.$$

A sequence $\{V_n\}_{n \in \mathbf{N}}$ of Zygmund vector fields converges to a Zygmund vector field V in the topology of the Zygmund norm (the cross-ratio norm) if and only if the Zygmund norm (the cross-ratio norm) of $V - V_n$ goes to zero as $n \rightarrow \infty$. The

convergence in the Zygmund norm is equivalent to the convergence in the cross-ratio norm [12], [11].

Gardiner and Sullivan [15], and Riemann [25] showed that a normalized vector field $V : \mathbf{R} \rightarrow \mathbf{R}$ is Zygmund if and only if there exists a differentiable path of quasymmetric maps

$$h_t : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$$

for $|t| < \epsilon$ and $\epsilon > 0$ such that $h_t : 0, 1, \infty \mapsto 0, 1, \infty$ and

$$\frac{d}{dt}h_t|_{t=0} = V$$

on \mathbf{R} .

We recall that the universal Teichmüller space $T(\mathbf{H})$ consists of all quasymmetric maps $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ such that $h : 0, 1, \infty \mapsto 0, 1, \infty$. Thus the space \mathcal{Z} of normalized Zygmund maps is identified with the tangent space $T_{id}T(\mathbf{H})$ of the universal Teichmüller space $T(\mathbf{H})$ at the identity (*basepoint*) $id \in T(\mathbf{H})$.

3. COMPLEX STRUCTURE ON $T(\mathbf{H})$ AND HILBERT TRANSFORM ON ZYGMUND VECTOR FIELDS

Let $V : \mathbf{R} \rightarrow \mathbf{R}$ be a normalized Zygmund vector field V on $\hat{\mathbf{R}}$, namely V is a tangent vector to $T(\mathbf{H})$ at the basepoint $id \in T(\mathbf{H})$. Let $\mu \in L^\infty(\mathbf{H})$ be a Beltrami differential representing the tangent vector $V \in T_{id}T(\mathbf{H})$. Denote by $[\mu]$ the equivalence class of Beltrami differentials representing the same tangent vector as μ . We have [3], [11] a standard formula

$$V(x) = -\frac{2}{\pi} \operatorname{Re} \int_{\mathbf{H}} R(x, \zeta) \mu(\zeta) d\xi d\eta,$$

where $R(x, \zeta) = \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)}$, $\zeta = \xi + i\eta \in \mathbf{H}$ and $x \in \mathbf{R}$.

The complex structure operator $J : T_{id}T(\mathbf{H}) \rightarrow T_{id}T(\mathbf{H})$ is given by $J([\mu]) = [i\mu]$, where $[i\mu]$ is the tangent vector represented by $\mu \in L^\infty(\mathbf{H})$ and $i \in \mathbf{C}$ is the imaginary unit. The above formula gives that the complex structure on V is

$$J(V)(x) = -\frac{2}{\pi} \operatorname{Re} \int_{\mathbf{H}} R(x, \zeta) i\mu(\zeta) d\xi d\eta$$

or, equivalently

$$J(V)(x) = \frac{2}{\pi} \operatorname{Im} \int_{\mathbf{H}} R(x, \zeta) \mu(\zeta) d\xi d\eta.$$

Let $\tilde{V}(z)$ for $z \in \mathbf{H}$ be the Beurling-Ahlfors extension of V into \mathbf{H} (see [15], [11, pages 316,317]). Then we have that $\partial\tilde{V}(z) = \mu(z)$ a.e. in \mathbf{H} (see [11]). By Stoke's theorem, we get

$$\frac{2i}{\pi} \int_{\mathbf{H}} R(x, \zeta) \mu(\zeta) d\xi d\eta = p.v. \frac{1}{\pi} \int_{\mathbf{R}} R(x, \xi) V(\xi) d\xi - iV(x).$$

By taking the real parts in the above equality, we obtain that the complex structure J on V is

$$J(V)(x) = -\frac{1}{\pi} p.v. \int_{\mathbf{R}} R(x, \xi) V(\xi) d\xi$$

for $x \in \mathbf{R}$ which equals the *Hilbert transform* of V

$$(3) \quad H(V)(x) = -\frac{1}{\pi} p.v. \int_{\mathbf{R}} R(x, \xi) V(\xi) d\xi$$

Kerckhoff was first to observe that the Hilbert transform gives the almost-complex structure operator on $T_{id}T(\mathbf{H})$ and the proof of this statement first appears in [20]. We used the discussion in [10]. The Hilbert transform preserves the space of normalized Zygmund vector fields on $\hat{\mathbf{R}}$ (see [36], [10]).

Let $V_{(0,\infty)} : \mathbf{R} \rightarrow \mathbf{R}$ be the *elementary shear vector field* for the geodesic $(0, \infty)$ defined by

$$(4) \quad V_{(0,\infty)}(x) = \begin{cases} x, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}$$

Then the Hilbert transform of $V_{(0,\infty)}$ is given by

$$(5) \quad HV_{(0,\infty)}(x) = \frac{1}{\pi}x \log |x|$$

for all $x \in \mathbf{R}$.

Some more elementary integration gives

$$(6) \quad H(V_{(a,\infty)})(x) = \frac{1}{\pi}(x-a) \log |x-a| + \frac{(a-1) \log |a-1|}{\pi}x - \frac{a \log a}{\pi}(x-1)$$

for all $x \in \mathbf{R}$, where $a \in \mathbf{R}$, $a > 0$, and the *elementary shear vector field* $V_{(a,\infty)}$ for the geodesic (a, ∞) is defined by

$$V_{(a,\infty)}(x) = \begin{cases} x-a, & \text{for } x > a \\ 0, & \text{for } x \leq a \end{cases}$$

For $a \in \mathbf{R}$, $a < 0$, the *elementary shear vector field* for the geodesic $(-\infty, a)$ is defined by

$$V_{(-\infty,a)}(x) = \begin{cases} -(x-a), & \text{for } x < a \\ 0, & \text{for } x \geq a \end{cases}$$

Then

$$(7) \quad H(V_{(-\infty,a)})(x) = -\frac{1}{\pi}(x-a) \log |x-a| - \frac{(a-1) \log |a-1|}{\pi}x + \frac{a \log |a|}{\pi}(x-1)$$

Let $a, b \in \mathbf{R}$ and $a < b$. For the *elementary shear vector field*

$$V_{(a,b)}(x) = \begin{cases} \frac{(x-a)(x-b)}{a-b}, & \text{for } a < x < b \\ 0, & \text{otherwise} \end{cases}$$

we obtain the Hilbert transform

$$(8) \quad H(V_{(a,b)})(x) = \frac{1}{\pi} \frac{(x-a)(x-b)}{a-b} \log \left| \frac{x-b}{x-a} \right| - x \frac{(1-a)(1-b)}{\pi(a-b)} \log \left| \frac{b-1}{a-1} \right| + (x-1) \frac{ab}{\pi(a-b)} \log \left| \frac{b}{a} \right|$$

for all $x \in \mathbf{R}$.

4. SHEAR PARAMETRIZATION OF ZYGMUND MAPS

Let $V : \mathbf{R} \rightarrow \mathbf{R}$ be a normalized Zygmund vector field. Namely $V(x)$ satisfies (2) and it vanishes at 0, 1 and ∞ , where vanishing at ∞ means that the rate of growth of $|V(x)|$ is $|x| \log |x|$ as $x \rightarrow \pm\infty$ (see [11]). Let $h_t : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$, $|t| < \epsilon$, be a path of quasisymmetric maps such that $h_0 = id$, h_t fixes 0, 1 and ∞ for each t , $|t| < \epsilon$, and

$$\frac{d}{dt}h_t(x)|_{t=0} = V(x)$$

for all $x \in \mathbf{R}$. In addition, we assume that the quasisymmetric constants of h_t are bounded by $1 + Ct$ for some constant C and for each t , $|t| < \epsilon$ (see [15], [11]).

Given two ideal hyperbolic triangles Δ_1 and Δ_2 with common boundary side e and disjoint interiors, the shear of Δ_1 and Δ_2 is the signed hyperbolic distance between the projections of third vertices of Δ_2 and Δ_1 onto e with the orientation of e as a part of the boundary of Δ_1 . A homeomorphism $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ maps the Farey tessellation \mathcal{F} onto a tessellation $h(\mathcal{F})$. Given $e \in \mathcal{F}$, let Δ_1^e and Δ_2^e be the two complementary triangles of $h(\mathcal{F})$ with common boundary side $h(e)$. We define a shear function $s : \mathcal{F} \rightarrow \mathbf{R}$ of h by setting $s(e)$ to be equal to the shear of the triangles Δ_1^e and Δ_2^e .

Let $s_t : \mathcal{F} \rightarrow \mathbf{R}$ be the path of shear functions corresponding to the path h_t of quasisymmetric maps. It follows that

$$\dot{s}(e) := \frac{d}{dt}s_t(e)|_{t=0}$$

is well-defined for each $e \in \mathcal{F}$ by the differentiability of h_t . The function

$$\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$$

associated to V is called the *infinitesimal shear function* of V and it uniquely determines V when V is normalized to vanish at 0, 1 and ∞ .

A *fan of geodesics* in \mathcal{F} with *tip* $p \in \hat{\mathbf{Q}}$ consists of all geodesics of \mathcal{F} with a common endpoint p . We orient each geodesic of the fan with tip p such that p is its initial point. Then for two geodesics e and e' of the fan with tip p we define $e < e'$ if e' is to the left of e and we define $e' < e$, otherwise. This induces a one-to-one correspondence of the fan $\{e_n\}_{n \in \mathbf{Z}}$ with integers \mathbf{Z} such that e_n is adjacent to e_{n+1} and $e_n < e_{n+1}$ for all $n \in \mathbf{Z}$. Under the above assumptions on h_t , there exists $C > 0$ such that

$$(9) \quad \frac{1}{1 + Ct} \leq e^{s_t(m)} \frac{1 + e^{s_t(m+1)} + \dots + e^{s_t(m+1) + \dots + s_t(m+k)}}{1 + e^{-s_t(m-1)} + \dots + e^{-s_t(m-1) - \dots - s_t(m-k)}} \leq 1 + Ct$$

for all $m, k \in \mathbf{Z}$, where $s_t(m) := s_t(e_m)$ (see [29]). We have the expansion

$$s_t(m) = \dot{s}(m)t + o_m(t)$$

where $o_m(t)/t \rightarrow 0$ as $t \rightarrow 0$ and constant in $o_m(t)$ depends on m . From (9) and the above expansion, we obtain

$$(10) \quad \left| \dot{s}(m) + \frac{k-1}{k} [\dot{s}(m+1) + \dot{s}(m-1)] + \frac{k-2}{k} [\dot{s}(m+2) + \dot{s}(m-2)] + \dots + \frac{1}{k} [\dot{s}(m+k) + \dot{s}(m-k)] \right| \leq C$$

for all $m, k \in \mathbf{Z}$, for all fans $\{e_n\}_{n \in \mathbf{Z}}$ of \mathcal{F} and for the fixed constant $C > 0$ from the above. Thus (10) is a necessary condition for $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ to be the infinitesimal shear

function of a Zygmund map V . We show that (10) is also a sufficient condition. Thus the space \mathcal{Z} of all Zygmund vector fields normalized to vanish at $0, 1$ and ∞ is parameterized by the set of functions $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ which satisfy (10) in all fans for some constant $C > 0$ (which is the same for all fans).

Theorem 4.1. *Let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be an arbitrary function. Then \dot{s} is the infinitesimal shear function of a normalized Zygmund vector field $V : \mathbf{R} \rightarrow \mathbf{R}$ if and only if \dot{s} satisfies (10) for all fans in \mathcal{F} with a fixed constant C .*

Proof. We proved above that (10) is a necessary condition for V to be a Zygmund vector field. It remains to prove that (10) is also a sufficient condition for \dot{s} to be induced by a Zygmund vector field $V : \mathbf{R} \rightarrow \mathbf{R}$. We first show that \dot{s} is induced by a continuous vector field $V : \mathbf{R} \rightarrow \mathbf{R}$.

Let Δ_0 be a hyperbolic triangle in the upper half-plane \mathbf{H} with vertices $0, 1$ and ∞ . We orient each edge in \mathcal{F} to the left as seen from Δ_0 . Consider a fan of geodesics $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbf{Z}}$ with tip $p \in \hat{\mathbf{Q}}$. The geodesics of the fan \mathcal{F}_p are divided into two groups with respect to the orientation given by Δ_0 : the geodesics whose initial point is p and the geodesics whose terminal point is p . We enumerate the fan with tip p by $\{e_n^p\}_{n \in \mathbf{Z}}$ such that e_n^p is adjacent to e_{n+1}^p , and that e_0^p has initial point p and e_1^p has terminal point p . This enumeration is unique. In the case $p = \infty$, we have that e_n^∞ is a geodesic with endpoints n and ∞ .

Assume that function

$$\dot{s} : \mathbf{Z} \rightarrow \mathbf{R}$$

satisfies (10). We define

$$(11) \quad V_s^\infty(x) = \begin{cases} 0 & ; x \in [0, 1] \\ \dot{s}(1)(x-1) + \dots + \dot{s}(n)(x-n) & ; x \in (n, n+1] \\ -\dot{s}(0)x - \dots - \dot{s}(-n)(x+n) & ; x \in [-n-1, -n) \end{cases}$$

for $n \in \mathbf{N}$.

Proposition 4.2. *Under the above notation, the vector field $V_s^\infty : \mathbf{R} \rightarrow \mathbf{R}$ is Zygmund with the Zygmund norm $2C + 18\|\dot{s}\|_\infty \leq 20C$, where C is given in (10).*

Proof. It is clear that V_s^∞ is continuous. We consider $x \in \mathbf{R}$ and $t \geq 2$. Let $m \in \mathbf{Z}$ and $k \in \mathbf{N}$ be such that $m \leq x \leq m+1$ and $m+k \leq x+t \leq m+k+1$. Then we have $k-1 \leq t \leq k+1$ and $m-k-1 \leq x-t \leq m-k+2$. A simple estimation shows that

$$|V_s^\infty(x+t) + V_s^\infty(x-t) - 2V_s^\infty(x)| \leq |V_s^\infty(m+k) + V_s^\infty(m-k) - 2V_s^\infty(m)| + 6\|\dot{s}\|_\infty k + 3\|\dot{s}\| \leq C_1 t,$$

where $C_1 = 2C + 18\|\dot{s}\|_\infty$ because $k \geq 1$ implies $2t \geq k$ and condition (10) is equivalent to $|V_s^\infty(m+k) + V_s^\infty(m-k) - 2V_s^\infty(m)| \leq Ck$.

For $0 < t < 2$, we have that V_s^∞ is piecewise linear on each interval $[x-t, x+t]$ with at most three points where it is not differentiable and with the bound $\|\dot{s}\|_\infty$ on the difference between the two adjacent slopes. Thus $|V_s^\infty(x+t) + V_s^\infty(x-t) - 2V_s^\infty(x)| \leq C_1 t$ in this case as well. Thus V_s^∞ is Zygmund bounded with the Zygmund norm C_1 . \square

Continuation of the proof of Theorem 4.1. We introduce an order on $\hat{\mathbf{Q}}$ using the Farey tessellation. Namely, 0 and ∞ in $\hat{\mathbf{Q}}$ are said to be of order 1. The numbers

1 and -1 are said to be of order 2. If the maximum of the orders of a/b and c/d is n and they are adjacent on $\hat{\mathbf{R}}$ (when we consider only numbers in $\hat{\mathbf{Q}}$ of order at most n), then the mediant $(a+c)/(b+d)$ is of the order $n+1$. In this fashion, we obtain the whole $\hat{\mathbf{Q}}$. There are two order 1 and two order 2 numbers in $\hat{\mathbf{Q}}$. For $n \geq 2$, there are 2^{n-1} numbers of order n .

The reference triangle Δ_0 has vertices 0 , 1 and ∞ . Fix $p \in \hat{\mathbf{Q}}$ and let $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbf{Z}}$ be the fan with tip p . Recall that e_n^p 's are oriented to the left as seen from Δ_0 , and that e_0^p has the initial point p and e_1^p has the terminal point p . Let a be the terminal endpoint of e_0^p and let $B \in PSL_2(\mathbf{Z})$ be the unique element such that $B(0) = a$ and $B(\infty) = p$. Let $V_{\dot{s}_p}^\infty$ be given by formula (11) where

$$\dot{s}_p : \mathbf{Z} \rightarrow \mathbf{R}$$

is defined by

$$\dot{s}_p(n) = \frac{1}{2} \dot{s}(e_n^p)$$

for $n \in \mathbf{Z}$. (Note that we use $\frac{1}{2}\dot{s}$ because each geodesic of \mathcal{F} appears in two different fans.) Then $V_{\dot{s}_p}^\infty$ is a Zygmund vector field with the Zygmund constant $10C$ by Proposition 4.2.

Define

$$V_p := B^* V_{\dot{s}_p}^\infty$$

Then V_p is a Zygmund vector field corresponding to the shear function which agrees with $\frac{1}{2}\dot{s}$ at $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbf{Z}}$ and which is zero on $\mathcal{F} \setminus \mathcal{F}_p$. If $p \neq 0, 1$ then V_p is normalized to be zero on $(-\infty, B(0)) \cup (B(1), +\infty)$ if $B(0) < B(1)$, otherwise V_p is zero on $(-\infty, B(1)) \cup (B(0), +\infty)$ if $B(1) < B(0)$. In both cases V_p vanishes at ∞ . It is also true that V_p vanishes at 0 and 1 because geodesics in \mathcal{F}_p do not separate 0 and 1 from Δ_0 . Thus V_p is a normalized Zygmund vector field.

We define a formal series

$$V(x) := \sum_{p \in \hat{\mathbf{Q}}} V_p(x)$$

for $x \in \mathbf{R}$. It is clear that V is well-defined on $\hat{\mathbf{Q}}$ because only finitely many terms $V_p(x)$ in the above series are non-zero. (If we knew that V is a tangent vector to some path of homeomorphisms, then the corresponding infinitesimal shear function would be equal to $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$. However, we still need to show that V is defined on all of \mathbf{R} and that it is Zygmund.)

Proposition 4.3. *The series*

$$(12) \quad V(x) := \sum_{p \in \hat{\mathbf{Q}}} V_p(x)$$

converges absolutely and uniformly on compact subsets of \mathbf{R} .

Proof. Let $x \in \mathbf{R}$. By Proposition 4.2 and the above, the Zygmund norm of each $V_{\dot{s}_p}^\infty$ is bounded by $C_1 = 10C$. This implies that the cross-ratio norm of each $V_{\dot{s}_p}^\infty$ is bounded by $C_2 = C_2(C_1)$ (see [15], [11]). Thus the Zygmund norm of V_b is bounded by C_2 because V_p is normalized to vanish at 0 , 1 and ∞ .

Let r_x be the geodesic ray from the center of Δ_0 to $x \in \mathbf{R}$. Consider the set of fans whose geodesics intersect r_x . If r_x intersects \mathcal{F}_p and the Farey order of p is at least 2, then r_x does not intersect any other fan with the same order. On the other

hand, r_x intersects both \mathcal{F}_0 and \mathcal{F}_∞ if and only if r_x intersects the geodesic with endpoints 0 and ∞ . We order the set of all fans intersecting r_x by the Farey order of their tips. The only ambiguity is if both \mathcal{F}_0 and \mathcal{F}_∞ intersect r_x in which case we set $\mathcal{F}_0 < \mathcal{F}_\infty$. This gives us a linear order on the set of fans intersecting r_x and we can put them in a sequence $\{\mathcal{F}_{p_n}\}_{n \in \mathbf{N}}$ such that $\mathcal{F}_{p_n} < \mathcal{F}_{p_{n+1}}$. Consider the ordered subset $\{e_{i_1}^{p_n}, \dots, e_{i_{k(n)}}^{p_n}\}$ of geodesics of \mathcal{F}_{p_n} that intersect r_x . Then either $i_1 = 0$ and $i_1 > i_2 > \dots > i_k$, or $i_1 = 1$ and $i_1 < i_2 < \dots < i_k$.

Consider the fans \mathcal{F}_{p_n} and $\mathcal{F}_{p_{n+2}}$. The last geodesic $e_{i_{k(n)}}^{p_n}$ of the fan \mathcal{F}_{p_n} which intersects r_x does not share an endpoint with the first geodesic $e_{i_1}^{p_{n+2}}$ of the fan $\mathcal{F}_{p_{n+2}}$ which intersects r_x . Thus the distance between $e_{i_{k(n)}}^{p_n}$ and $e_{i_1}^{p_{n+2}}$ is $\delta = \log(1 + \sqrt{2})^2$.

Assume that $n \geq 3$. Then no geodesic of the fan \mathcal{F}_{p_n} has ∞ as an endpoint. Moreover, the Farey order of p_n is at least 3. Let $e_{i_1}^{p_n} = (a_n, b_n)$ be the first geodesic of the fan \mathcal{F}_{p_n} that intersects r_x . Note that $V_{p_n}(a_n) = V_{p_n}(b_n) = 0$ by the normalization of V_{p_n} . The distance of the center of the triangle Δ_0 to $e_{i_1}^{p_n}$ is at least $\frac{n-2}{2}\delta$ by the above considerations. An elementary hyperbolic geometry implies that $|a_n - b_n|$ is of the order $e^{-\frac{(n-2)\delta}{2}}$ for $n \in \mathbf{N}$, where the constant depends on a compact subset of \mathbf{R} in which x lies. Since V_{p_n} has $\epsilon \log \frac{1}{\epsilon}$ modulus of continuity with the constant depending on its Zygmund norm and since the support of V_{p_n} is the interval in \mathbf{R} with endpoints a_n and b_n , for x in a compact subset of \mathbf{R} , we obtain

$$\sum_{p \in \hat{\mathbf{Q}}} |V_p(x)| = \sum_{n \in \mathbf{N}} |V_{p_n}(x)| \leq C_0 + C_3 \sum_{n \in \mathbf{N}} n e^{-\frac{n\delta}{2}} < \infty$$

where $C_0 = |V_{p_1}(x)| + |V_{p_2}(x)| + |V_{p_3}(x)|$, and $C_3 = C_3(C_2)$ is a constant which depends on the bound C_2 on the Zygmund norm of V_{p_n} 's. \square

Remark 4.4. We showed that the formal series (12) converges to a continuous vector field on \mathbf{R} which is normalized to vanish at 0 and 1 because each V_p vanishes at 0 and 1. To see that $V(x)$ extends to a (continuous) normalized vector field on $\hat{\mathbf{R}}$ we need to show that $V(x) = o(x^2)$ as $x \rightarrow \pm\infty$. In the proof of Proposition 4.3 we showed that $V(x)$ (for x in a compact subset of \mathbf{R}) is bounded in terms of the Zygmund bound on V_p 's. The push-forward of the vector fields V_p under the translation $T : x \mapsto x + a$, $a \in \mathbf{R}$, satisfies $\|V_p\|_\infty = \|T^*(V_p)\|_\infty$ because $T^*(V_p)(x) = V_p(x - a)$. Let $\sum' V_{p_n}(x)$ be the sum (12) without terms V_{p_1} , V_{p_2} and V_{p_3} . Note that each V_{p_n} in the sum $\sum' V_{p_n}(x)$ does not have a positive shear on a geodesic of \mathcal{F} with one endpoint ∞ . If $x \in [k, k+1]$ for $k \in \mathbf{Z}$, define $T_k(x) = x - k$. Then $(T_k)^*(\sum' V_{p_n}(x))$ is equal to $V'_k(x - k)$, where the infinitesimal shear function of the vector field V'_k is the push-forward by T_k of infinitesimal shear function \dot{s} . Since V'_k is bounded on compact subsets of \mathbf{R} by Proposition 4.3, since $x - k \in [0, 1]$ and since $\sum' V_{p_n}(x) = V'_k(x - k)$, it follows that $\sum' V_{p_n}(x)$ is bounded on \mathbf{R} . Since V_{p_1} , V_{p_2} and V_{p_3} are normalized to vanish at ∞ and $\sum' V_{p_n}(x)$ is bounded on \mathbf{R} , it follows that $V(x) = o(x^2)$ as $x \rightarrow \pm\infty$. Thus (12) defines a normalized vector field on $\hat{\mathbf{R}}$.

Remark 4.5. The sum (12) defines a continuous vector field $V : \mathbf{R} \rightarrow \mathbf{R}$ on $\hat{\mathbf{R}}$ which vanishes at 0, 1 and ∞ . This is achieved by normalizing each vector field V_b in the same fashion. To normalize $V(x)$ to vanish at the endpoints a_1, a_2, a_3 of another complementary triangle Δ of \mathcal{F} , we start the construction of each V_p with

respect to the triangle Δ . Equivalently, we need to add a quadratic polynomial to the formula (12) in order to obtain a vector field with the same infinitesimal shear function \dot{s} which vanishes at a_1, a_2, a_3 . The new vector field is of the order $O(x^2)$ as $x \rightarrow \pm\infty$.

Remark 4.6. Let $B \in PSL_2(\mathbf{R})$ and let $\mathcal{F}^* = B(\mathcal{F})$. Then $\dot{s} \circ B^{-1} : \mathcal{F}^* \rightarrow \mathbf{R}$ is a shear function on \mathcal{F}^* . Fix a complementary triangle Δ^* in \mathcal{F}^* . Then there exists $B_1 \in PSL_2(\mathbf{R})$ such that $B_1(\Delta_0) = \Delta^*$ and $B_1(\mathcal{F}) = \mathcal{F}^*$. We define

$$W(x) := \sum_{p \in B_1(\hat{\mathcal{Q}})} W_p(x)$$

where $W_p = (B_1)^*(V_{B_1^{-1}(p)})$. Then the series for $W(x)$ converges absolutely and uniformly on compact subsets of $\mathbf{R} \setminus \{B_1(\infty)\}$ because each term is the push-forward of the series (12) by the same Möbius map $B_1 \in PSL_2(\mathbf{R})$. Vector field $W : \mathbf{R} \rightarrow \mathbf{R}$ is normalized to vanish at the vertices of Δ^* and its infinitesimal shear function is $\dot{s} \circ B_1^{-1} : \mathcal{F}^* \rightarrow \mathbf{R}$.

We continue the proof of Theorem 4.1. Proposition 4.3 implies that $V(x)$ is a continuous vector field on \mathbf{R} because each $V_p(x)$ is a continuous vector field and the convergence is uniform. We showed in Remark 4.4 that $V(x)$ extends to a normalized vector field on $\hat{\mathbf{R}}$. We need to show that $V(x)$ is a Zygmund vector field. We show a slightly stronger statement which will be used later on.

Proposition 4.7. *Let $C > 0$ be fixed. If*

$$\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$$

satisfies (10) with constant C , then there exists $M = M(C) > 0$ such that the vector field

$$V : \mathbf{R} \rightarrow \mathbf{R}$$

induced by \dot{s} is $M(C)$ -Zygmund bounded.

Proof. Assume on the contrary that there exists a sequence $\dot{t}_n : \mathcal{F} \rightarrow \mathbf{R}$ of infinitesimal shear functions satisfying (10) with the fixed constant $C > 0$ such that the cross-ratio norms of the corresponding vector fields $W_n : \mathbf{R} \rightarrow \mathbf{R}$ converge to infinity as $n \rightarrow \infty$. Then there exists a sequence $\{Q_n = (a_n, b_n, c_n, d_n)\}$ of quadruples on $\hat{\mathbf{R}}$ with $cr(Q_n) = \frac{(c_n - a_n)(d_n - b_n)}{(d_n - a_n)(c_n - b_n)} = 2$ such that $|W_n[Q_n]| \rightarrow \infty$ as $n \rightarrow \infty$. (If for a single shear function $\dot{t} : \mathcal{F} \rightarrow \mathbf{R}$ the induced continuous vector field W is not Zygmund, then there exists a sequence Q_n as above such that $|W[Q_n]| \rightarrow \infty$ as $n \rightarrow \infty$. We replace W_n with W and the argument below is the same.) We seek a contradiction.

Let $Q^* := (-e^\delta, -1, 1, e^\delta)$ with $\delta = \log(1 + \sqrt{2})^2$ and note that $cr(Q^*) = 2$. Then there exists a unique Möbius map B_n such that $B_n(Q_n) = Q^*$. Define

$$V_n := (B_n)^* W_n.$$

Then

$$|V_n[Q^*]| \rightarrow \infty$$

as $n \rightarrow \infty$.

Recall that $V_n(x) = \frac{1}{(B_n^{-1})'(x)} W_n(B_n^{-1}(x))$. Let $\mathcal{F}_n = B_n(\mathcal{F})$ and $\dot{s}_n := \dot{t}_n \circ B_n^{-1}$. Then V_n is a vector field corresponding to \dot{s}_n whose support is an ideal triangulation

\mathcal{F}_n and which is normalized to be zero at the vertices of the complementary triangle $\Delta_n := B_n(\Delta_0)$. There are two possibilities to consider:

- (a) \mathcal{F}_n converges to an ideal triangulation \mathcal{F}^* of \mathbf{H} , or
- (b) \mathcal{F}_n does not converge to an ideal triangulation of \mathbf{H} .

In both cases we are allowed to further normalize the sequence V_n to vanish at any three points on $\hat{\mathbf{R}}$ by adding a quadratic polynomial.

Assume we are in the case (a). Then the sequence \dot{s}_n of infinitesimal shear functions has a convergent subsequence in the weak*-topology (if we consider \dot{s}_n as functions on the space of geodesics on \mathbf{H}). Without loss of generality, we assume that \dot{s}_n converges to \dot{s}^* as $n \rightarrow \infty$. The support of \dot{s}^* is \mathcal{F}^* . Moreover, \dot{s}^* satisfies property (10) in each fan of \mathcal{F}^* because the values of \dot{s}^* in each fan of \mathcal{F}^* are limits of the values of \dot{s}_n on the fans of \mathcal{F}_n (and all \dot{s}_n satisfy property (10) with the same constant). Note that all shears of \mathcal{F}^* are 0 because it is the limit of \mathcal{F}_n whose each shear is 0. Thus \mathcal{F}^* is the image of the Farey tessellation under a Möbius map $B^* \in PSL_2(\mathbf{R})$. By Remark 4.6, the infinitesimal shear function \dot{s}^* determines a continuous vector field $V^* : \mathbf{R} \rightarrow \mathbf{R}$. If the sequence V_n pointwise converges to V^* then we get a contradiction with $|V_n[Q^*]| \rightarrow \infty$ because $|V_n[Q^*]| \rightarrow |V^*[Q^*]| < \infty$ as $n \rightarrow \infty$.

To obtain a contradiction in the case (a), it is enough to prove the convergence of V_n to V^* . By Remark 4.6, we have

$$V^*(x) = \sum_{p \in \mathcal{F}_0^*} V_p^*(x)$$

for $x \in \mathbf{R}$, where \mathcal{F}_0^* is the set of vertices of \mathcal{F}^* . The convergence is absolute and uniform on compact subsets of \mathbf{R} . Recall that each V_p^* is normalized to be zero on the vertices of $\Delta_n := B_n(\Delta_0)$. Let Δ^* be the limit of Δ_n . If the sequence Δ_n does not converge, then we normalize V_n to be zero on the vertices Δ'_n such that the sequence Δ'_n converges. We denote the normalization of V_n by V_n again and Δ'_n by Δ_n for simplicity of notation. Then we normalize $V^*(x)$ to be zero at the vertices of the complementary triangle Δ^* of \mathcal{F}^* by normalizing each V_p^* to be zero at these points.

Given a compact subset K of \mathbf{R} and an $\epsilon > 0$, there exists finitely many $p_1, p_2, \dots, p_k \in \mathcal{F}_0^*$ such that

$$(13) \quad |V^*(x) - \sum_{i=1}^k V_{p_i}^*(x)| < \epsilon$$

for all $x \in K$. To see this, consider the sequence of all fans $\mathcal{F}_{p_n}^*$ which intersect all rays r_x from the center of Δ^* to $x \in K$. Given $d > 0$, there exists k such that the support of $V_{p_i}^*$ for $i > k$ has length smaller than d . (This was established for the Farey tessellation \mathcal{F} and the proof is similar for \mathcal{F}^* .) The lefthand side of (13) is less than

$$(14) \quad \sum_{i=k+1}^{\infty} |V_{p_i}^*(x)| \leq C \sum_{i=k+1}^{\infty} i e^{-\frac{id}{2}}$$

where C is given in the last paragraph in the proof of Proposition 4.3. The estimate (13) follows for k large enough.

For each fan $\mathcal{F}_{p_i}^*$, $i = 1, 2, \dots, k$, there exists a sequence of fans $\{(\mathcal{F}_n)_{p_i^n}\}_{n \in \mathbf{N}}$ which converges to $\mathcal{F}_{p_i}^*$ as $n \rightarrow \infty$ such that the functions $\dot{s}_n|_{(\mathcal{F}_n)_{p_i^n}}$ converge in the

weak* topology to $\dot{s}_\infty|_{\mathcal{F}_{p_i}^*}$. We have

$$|V_n(x) - \sum_{i=1}^k (V_n)_{p_i^n}(x)| < \sum_{i=k+1}^{\infty} |(V_n)_{p_i^n}(x)| < C_n \sum_{i=k+1}^{\infty} i e^{-\frac{i\delta}{2}}$$

where C_n is the constant corresponding to V_n similar to the above.

Note that $\{C_n\}_{n \in \mathbf{N}}$ is a bounded sequence. Then

$$(15) \quad |V_n(x) - \sum_{i=1}^k (V_n)_{p_i^n}(x)| < \epsilon$$

for all $x \in K$ and n large enough. Thus k can be chosen simultaneously for V^* and all V_n , $n \in \mathbf{N}$ depending only on $\epsilon > 0$.

By (13) and (15), it is enough to prove that $(V_n)_{p_i^n}(x) \rightarrow V_{p_i}^*(x)$ uniformly for x in the compact subset K of \mathbf{R} as $n \rightarrow \infty$. Note that $(V_n)_{p_i^n}$ and $V_{p_i}^*$ are Zygmund vector fields with uniformly bounded cross-ratio norms and the supports of $(V_n)_{p_i^n}$ on \mathbf{R} converge to the support of $V_{p_i}^*$. Thus the uniform convergence of $(V_n)_{p_i^n}$ to $V_{p_i}^*$ follows from the pointwise convergence on a dense subset of the compact subset K of \mathbf{R} because $\{(V_n)_{p_i^n}\}$ is a normal family. On the other hand, the weak* convergence of $\dot{s}_n|_{(\mathcal{F}_n)_{p_i^n}}$ to $\dot{s}_\infty|_{\mathcal{F}_{p_i}^*}$ implies that $(V_n)_{p_i^n}$ converges pointwise to $V_{p_i}^*$, when $(V_n)_{p_i^n}$ are normalized to vanish at the vertices of the triangle Δ_n in \mathcal{F}_n with $\Delta_n \rightarrow \Delta^*$ as $n \rightarrow \infty$. Thus V_n converges uniformly to V^* on K and case (a) is finished.

It remains to consider the case (b). The sequence \mathcal{F}_n has a convergent subsequence in the Hausdorff topology on closed subset of \mathbf{H} . Without loss of generality we assume that the whole sequence \mathcal{F}_n converges to a subset \mathcal{F}^* of the space of geodesics of \mathbf{H} . The limit \mathcal{F}^* of \mathcal{F}_n is a geodesic lamination of \mathbf{H} because each \mathcal{F}_n is a geodesic lamination. Moreover, \mathcal{F}^* is not empty because there is a bounded neighborhood of the imaginary unit $i \in \mathbf{H}$ which has to intersect an edge of \mathcal{F}_n (since complementary triangles of \mathcal{F}_n cover \mathbf{H} and there is a positive upper bound on the distance of points inside an ideal triangle to its boundary). If \mathcal{F}^* has a non-empty complement, then the connected components of the complement consist of ideal triangles because the connected components of the complement of \mathcal{F}_n are ideal triangles. However, if a connected component of the complement of \mathcal{F}^* is an ideal triangle then \mathcal{F}^* contains all the images of the triangle under the group of isometries of \mathbf{H} generated by inversions in the sides of the triangle. This implies that \mathcal{F}^* is an ideal triangulation which contradicts the case (b). Thus \mathcal{F}^* is necessarily a geodesic foliation of \mathbf{H} .

Let l be a leaf of the foliation \mathcal{F}^* which contains $i \in \mathbf{H}$ and let Δ_n be a complementary triangle of \mathcal{F}_n which contains $i \in \mathbf{H}$. Two boundary sides of Δ_n converge to l as $n \rightarrow \infty$. Let $p_n \in \hat{\mathbf{R}}$ be the common endpoint of the above two boundary sides. Recall that

$$V_n := (B_n)^*(W_n)$$

with

$$|V_n[Q^*]| \rightarrow \infty$$

as $n \rightarrow \infty$. By Proposition 4.3 and Remark 4.6, we have

$$V_n(x) = (V_n)_{p_n}(x) + \sum_{p' \in (B_n(\hat{Q}) \setminus \{p_n\})} (V_n)_{p'}(x).$$

Note that V_n vanishes at vertices of $B_n(\Delta_0)$, where Δ_0 is the ideal triangle with vertices 0, 1 and ∞ . We normalize V_n by adding a quadratic polynomial such that it vanishes at the vertices of Δ_n , where two boundary sides of Δ_n converge to $l \in \mathcal{F}^*$. We denote by V'_n the normalized vector field. Then we have

$$V'_n(x) = (V_n)'_{p_n}(x) + \sum_{p' \in (B_n(\hat{Q}) \setminus \{p_n\})} (V_n)'_{p'}(x)$$

where each $(V_n)'_{p'}$ is defined to vanish at the vertices of the triangle Δ_n . Note that $\dot{s}_n = \dot{t}_n \circ B_n^{-1}$ is the infinitesimal shear function of V'_n and V_n . Since the cross-ratio norm is invariant under the addition of a quadratic polynomial, we have

$$|V'_n[Q^*]| \rightarrow \infty$$

as $n \rightarrow \infty$.

Since the reference triangle Δ_n converges to a single geodesic l , it follows that all geodesics of \mathcal{F}_n except the geodesics in the fan with tip p_n have the supremum of their Euclidean sizes going to zero as $n \rightarrow \infty$. Then, similar to the proof of Proposition 4.3, we have

$$\sum_{p' \in (B_n(\hat{Q}) \setminus \{p_n\})} |(V_n)'_{p'}(x)| \leq C \sum_{i=1}^{\infty} (a(n) + i) e^{\frac{-(a(n)+i)\delta}{2}}$$

with $a(n) \rightarrow \infty$ as $n \rightarrow \infty$. Thus $\sum_{p' \in (B_n(\hat{Q}) \setminus \{p_n\})} |(V_n)'_{p'}(x)|$ is arbitrary small when n is large. By Proposition 4.2, $(V_n)'_{p_n}(x)$ are Zygmund bounded with the cross-ratio norms independent of n . We further normalize $(V_n)'_{p_n}(x)$ by adding a quadratic polynomial $q_n(x)$ such that $(V_n)''_{p_n}(x) := (V_n)'_{p_n}(x) + q_n(x)$ vanishes at 0, 1 and ∞ . It follows that $\{(V_n)''_{p_n}\}_{n \in \mathbf{N}}$ is a normal family. Thus there exists a subsequence of $\{(V_n)''_{p_n}\}_{n \in \mathbf{N}}$ which converges uniformly. Without loss of generality, we assume that $(V_n)''_{p_n}$ converges to a continuous function V_*'' uniformly on compact subsets of \mathbf{R} .

Define $V_n'' := V'_n + q_n$. Then $V_n'' \rightarrow V_*''$ as $n \rightarrow \infty$ uniformly on compact subsets of \mathbf{R} by the above discussion. This implies that $|V_n[Q^*]| = |V'_n[Q^*]| = |V_n''[Q^*]| \rightarrow |V_*''[Q^*]| < \infty$ as $n \rightarrow \infty$. This contradicts $|V_n[Q^*]| \rightarrow \infty$ as $n \rightarrow \infty$. Thus case (b) cannot occur as well. It follows that the sequence $\{V_n(x)\}_{n \in \mathbf{N}}$ has uniformly bounded Zygmund norms. \square

End of the proof of Theorem 4.1. Proposition 4.3 states that \dot{s} is induced by a continuous vector field V . Proposition 4.7 states that V is Zygmund bounded with the Zygmund constant depending on the constant C in (10) which finishes the proof. \square

The proof of the above theorem also establishes the uniform convergence property of the series defining the Zygmund vector field in terms of the infinitesimal shear function $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$. The series giving the vector field corresponding to the infinitesimal shear function $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ is defined by adding over all fans vector fields corresponding to the fans which is a particular order of the summation of the series of elementary shear vector fields for \dot{s} . This is analogous to the convergence of the Césaro sum of the Fourier series of continuous functions on S^1 (see [14]).

Note that there exist continuous functions on the unit disk S^1 whose Fourier series does not converge at a single point of S^1 . Similarly, the series $V(x) =$

$\sum_{e=(a,b) \in \mathcal{F}} \dot{s}(e) \chi_{[a,b]}(x) \frac{(x-a)(x-b)}{a-b}$, where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$, does not necessarily converge on $\mathbf{R} \setminus \hat{\mathbf{Q}}$ because the infinitesimal shear function \dot{s} does not induce a bounded measured lamination in the sense of Thurston (see [30], [28]). Thus we need a summation method similar to the situation for the Fourier series which is given in the following corollary.

Corollary 4.8. *Let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be a shear function which satisfies (10) in each fan of \mathcal{F} . Then the series defining the corresponding vector field*

$$V(x) = \sum_{p \in \hat{\mathbf{Q}}} V_p(x)$$

converges absolutely and uniformly on compact subsets of \mathbf{R} , where $V_p(x)$ is the vector field corresponding to $\dot{s}|_{\mathcal{F}_p}$. The term $V_p(x)$ is a piecewise quadratic polynomial except at the tip p . Moreover, we have

$$|V(x) - \sum_{i=1}^k V_{p_i}(x)| < C \sum_{i=n}^{\infty} i e^{-\frac{(i-2)\delta}{2}}$$

where $\delta = \log(1 + \sqrt{2})^2$, C is a function of the constant in (10) and $\{p_1, \dots, p_k\}$ are all Farey number of order at most n . In addition, the Zygmund norms of $V(x)$ and $\sum_{i=1}^k V_{p_i}(x)$ for $k \in \mathbf{N}$ are bounded by constant $M(C)$, where C is the constant from (10).

5. THE HILBERT TRANSFORM IN SHEARS

Let $V : \mathbf{R} \rightarrow \mathbf{R}$ be a Zygmund vector field on $\hat{\mathbf{R}}$ and let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be the corresponding infinitesimal shear function. We use the formula

$$V(x) = \sum_{p \in \hat{\mathbf{Q}}} V_p(x)$$

to find the Hilbert transform of V in terms of the corresponding infinitesimal shear function \dot{s} , where the order of the summation is given by increasing Farey orders of tips $p \in \hat{\mathbf{Q}}$. Recall that the fan \mathcal{F}_p with tip p is enumerated by $\{e_n\}_{n \in \mathbf{Z}}$ such that the initial point of e_0 is p and the terminal point of e_1 is p , where e_n is oriented to the left as seen from the reference triangle Δ_0 . We define the *tip- p infinitesimal shear function* $\dot{s}_p : \mathcal{F}_p = \{e_n\}_{n \in \mathbf{N}} \rightarrow \mathbf{R}$ by setting $\dot{s}_p(e_n) := \frac{1}{2} \dot{s}(e_n)$. Thus \dot{s}_p satisfies (10) with the constant $\frac{1}{2}C$, where C is the constant of \dot{s} . Therefore

$$V_p(x) = \sum_{n \in \mathbf{Z}} \dot{s}_p(e_n) V_{(a_n, b_n)}(x)$$

is Zygmund bounded with the cross-ratio norm bounded independently of the fan \mathcal{F}_p , where e_n has initial point a_n and terminal point b_n , and $V_{(a_n, b_n)}$ is the elementary shear vector field defined in section 3.

Since V_p is Zygmund bounded, it follows that the Hilbert transform HV_p is well-defined and also Zygmund bounded (see [36], [10], [11]). Moreover, the cross-ratio norm of HV_p is bounded in terms of the cross-ratio norm of V_p .

We first give a lemma which facilitates various convergence arguments in the rest of the paper.

Lemma 5.1. *Let $V_n : \mathbf{R} \rightarrow \mathbf{R}$, for $n \in \mathbf{N}$, and $V : \mathbf{R} \rightarrow \mathbf{R}$ be Zygmund bounded vector fields on $\hat{\mathbf{R}}$ normalized to vanish at 0, 1 and ∞ . Suppose that*

$$V_n(x) \rightarrow V(x)$$

as $n \rightarrow \infty$ uniformly on compact subsets of \mathbf{R} and that the sequence $\{V_n\}_{n \in \mathbf{N}}$ has uniformly bounded cross-ratio norms. Then

$$H(V_n)(x) \rightarrow H(V)(x)$$

as $n \rightarrow \infty$ uniformly on compact subsets of \mathbf{R} .

Proof. Recall that the Hilbert transform of $V(x)$ is given by

$$HV(x) = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left[\int_{-\infty}^{x-\epsilon} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V(\zeta) d\zeta + \int_{x+\epsilon}^{\infty} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V(\zeta) d\zeta \right].$$

Since V, V_n are normalized Zygmund vector fields with uniformly bounded cross-ratio norms, it follows that there exists $M > 0$ such that $|V(x)|, |V_n| \leq M|x| \log|x|$ as $|x| \rightarrow \infty$ (see [11]). Consequently, $\int_R^{\infty} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V(\zeta) d\zeta$, $\int_R^{\infty} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_n(\zeta) d\zeta$, $\int_{-\infty}^{-R} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V(\zeta) d\zeta$ and $\int_{-\infty}^{-R} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_n(\zeta) d\zeta$ are of the order $o(R^{-\alpha})$ for some $0 < \alpha < 1$ (see [11]). Since V_n converges to V uniformly on compact subsets of \mathbf{R} , it follows that (for a fixed $\epsilon > 0$)

$$\int_{-\infty}^{x-\epsilon} \frac{V(\zeta) - V_n(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\zeta + \int_{x+\epsilon}^{\infty} \frac{V(\zeta) - V_n(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\zeta \rightarrow 0$$

as $n \rightarrow \infty$.

To finish the proof, it is enough to show that for any $\delta > 0$ there exists $\epsilon = \epsilon(\delta) > 0$ and $n_0 = n_0(\delta) \in \mathbf{N}$ such that

$$(16) \quad \left| p. v. \int_{x-\epsilon}^{x+\epsilon} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} [V(\zeta) - V_n(\zeta)] d\zeta \right| < \delta$$

for all $n \geq n_0$.

Let \tilde{V} and \tilde{V}_n be the Beurling-Ahlfors extensions to \mathbf{H} of V and V_n respectively (see [15]). The $\bar{\partial}$ -derivatives μ and μ_n of \tilde{V} and \tilde{V}_n are Beltrami differentials on \mathbf{H} corresponding to the vector fields V and V_n (see [15], [11]). The vector fields V and V_n have uniformly (in n) bounded cross-ratio norms which implies the existence of a constant $S > 0$ such that $\|\mu\|_{\infty}, \|\mu_n\|_{\infty} < S$ for all $n \in \mathbf{N}$. Since V_n converges to V uniformly on compact subsets of \mathbf{R} , it follows that \tilde{V}_n converges to \tilde{V} uniformly on compact subsets of $\mathbf{H} \cup \mathbf{R}$ when considered as a subset of \mathbf{C} . Stoke's theorem gives

$$(17) \quad p. v. \int_{x-\epsilon}^{x+\epsilon} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V(\zeta) d\zeta = \iint_{D_{\epsilon}(x)} \frac{x(x-1)\mu(\xi)}{\xi(\xi-1)(\xi-x)} d\zeta d\eta - \int_{C_{\epsilon}(x)} \frac{x(x-1)}{\xi(\xi-1)(\xi-x)} \tilde{V}(\xi) d\xi$$

where $\mu = \bar{\partial}\tilde{V}$, $D_{\epsilon}(x)$ is the upper half-disk with center x and radius ϵ , and $C_{\epsilon}(x)$ is the circular part of the boundary of $D_{\epsilon}(x)$. A similar equation holds for V_n .

By (17), to prove (16) it is enough to prove that

$$(18) \quad \iint_{D_{\epsilon}(x)} \left| \frac{x(x-1)\mu_n(\xi)}{\xi(\xi-1)(\xi-x)} \right| d\zeta d\eta \rightarrow 0$$

as $\epsilon \rightarrow 0$ uniformly in $n \in \mathbf{N}$, and that

$$(19) \quad \int_{C_\epsilon(x)} \frac{x(x-1)}{\xi(\xi-1)(\xi-x)} [\tilde{V}(\xi) - \tilde{V}_n(\xi)] d\xi \rightarrow 0$$

as $n \rightarrow \infty$ for all $\epsilon < 1$. Since $\|\mu_n\|_\infty \leq S$, and the expression $\frac{x(x-1)}{\xi(\xi-1)(\xi-x)}$ has a simple pole at x , and the area of $D_\epsilon(x)$ goes to zero as $\epsilon \rightarrow 0$, we obtain (18).

To obtain (19), we change the variable $\xi = x + \epsilon e^{i\varphi}$ to obtain

$$\begin{aligned} & \int_{C_\epsilon(x)} \frac{[\tilde{V}(\xi) - \tilde{V}_n(\xi)]}{\xi(\xi-1)(\xi-x)} d\xi = \\ &= \int_0^\pi \frac{[\tilde{V}(x + \epsilon e^{i\varphi}) - \tilde{V}_n(x + \epsilon e^{i\varphi})]}{(x + \epsilon e^{i\varphi})(x - 1 + \epsilon e^{i\varphi})\epsilon e^{i\varphi}} \epsilon i e^{i\varphi} d\varphi = \\ &= \int_0^\pi \frac{[\tilde{V}(x + \epsilon e^{i\varphi}) - \tilde{V}_n(x + \epsilon e^{i\varphi})]}{(x + \epsilon e^{i\varphi})(x - 1 + \epsilon e^{i\varphi})} i d\varphi \end{aligned}$$

The last integral converges to zero because $\tilde{V}_n \rightarrow \tilde{V}$ uniformly on compact subsets of $\mathbf{H} \cup \mathbf{R}$. \square

In the following theorem we decompose the Hilbert transform HV into an infinite sum of the Hilbert transforms HV_p for $p \in \hat{\mathbf{Q}}$ analogous to Corollary 4.8. The main tool in the proof is Lemma 5.1.

Theorem 5.2. *Let $V : \mathbf{R} \rightarrow \mathbf{R}$ be a Zygmund vector field on $\hat{\mathbf{R}}$ and let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be the corresponding infinitesimal shear function. Then the Hilbert transform of V is given by the series*

$$HV(x) = \sum_{p \in \hat{\mathbf{Q}}} HV_p(x)$$

where HV_p is the Hilbert transform of V_p and V_p is defined as above using the shears in the fan \mathcal{F}_p . The series converges uniformly on compact subsets of \mathbf{R} .

Remark 5.3. We point out that the uniform convergence of the series is strong property because the Hilbert transform is given by the principal value of an improper integral. Moreover, the infinitesimal shear function has variable sign which further complicates the convergence. On the other hand, it seems that the series does not converge absolutely. One should also note that the “naive” series $HV(x) = \sum_{e=(a,b) \in \mathcal{F}} \dot{s}(e)HV_{(a,b)}(x)$ does not converge, where $HV_{(a,b)}(x)$ is given by (8), (7), (6), or (5).

Proof. Let $V_n(x) = \sum_{i=1}^k V_{p_i}(x)$ such that $\{p_1, p_2, \dots, p_k\}$ is the set of Farey numbers of order at most n . Then V_n converges to V uniformly on compact subsets of \mathbf{R} by Corollary 4.8. Proposition 4.7 states that each V_n is a Zygmund vector field with uniformly bounded Zygmund norms. The theorem follows from Lemma 5.1. \square

We compute the Hilbert transform HV_p of V_p in terms of shears.

Theorem 5.4. *Let $\dot{s}_p : \mathcal{F}_p \rightarrow \mathbf{R}$ be a shear function that satisfies (10) in the fan $\mathcal{F}_p = \{e_n\}_{n \in \mathbf{Z}}$ and let V_p be the induced vector field. Then*

$$HV_p(x) = \sum_{n=-\infty}^{\infty} \dot{s}_p(e_n)H(V_{e_n})(x)$$

where V_{e_n} is the elementary shear vector field for the geodesic e_n oriented to the left as seen from the reference triangle Δ_0 .

Proof. We divide the proof into two cases $p = \infty$ and $p \neq \infty$.

Assume first that $p = \infty$. We have

$$(20) \quad \begin{aligned} V_\infty(\zeta) &= \dot{s}_\infty(1)(\zeta - 1) + \dot{s}_\infty(2)(\zeta - 2) + \cdots + \\ &+ \dot{s}_\infty(n)(\zeta - n) = \left(\sum_{i=1}^n \dot{s}_\infty(i) \right) \zeta - \sum_{i=1}^n i \dot{s}_\infty(i) \end{aligned}$$

for $1 \leq n \leq \zeta \leq n + 1$, we have

$$V_\infty(x) = 0$$

for $0 \leq \zeta \leq 1$, and we have

$$\begin{aligned} V_\infty(\zeta) &= \dot{s}_\infty(0)\zeta + \dot{s}_\infty(-1)(\zeta + 1) + \cdots + \\ &+ \dot{s}_\infty(-n)(\zeta + n) = \left(\sum_{i=0}^n \dot{s}_\infty(-i) \right) \zeta + \sum_{i=0}^n i \dot{s}_\infty(-i) \end{aligned}$$

for $-n - 1 \leq \zeta \leq -n \leq 0$.

Since V_∞ is a Zygmund map, it follows that $H(V_\infty)$ exists and it is also a Zygmund map. Thus

$$-\frac{1}{\pi} \int_{-n}^n \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty(\zeta) d\zeta \rightarrow H(V_\infty)(x)$$

as $n \rightarrow \infty$ for each $x \in \mathbf{R}$. Define

$$(21) \quad V_\infty^n(x) = \begin{cases} V_\infty(x), & \text{for } -n \leq x \leq n \\ \left(\sum_{i=1}^n \dot{s}_\infty(i) \right) x - \sum_{i=1}^n i \dot{s}_\infty(i), & \text{for } x > n \\ \left(\sum_{i=0}^n \dot{s}_\infty(-i) \right) x + \sum_{i=0}^n i \dot{s}_\infty(-i), & \text{for } x < -n \end{cases}$$

It follows that

$$-\frac{1}{\pi} \int_{-n}^n \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty(\zeta) d\zeta = -\frac{1}{\pi} \int_{-n}^n \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty^n(\zeta) d\zeta.$$

Since the Hilbert transform is given by an integral kernel, it follows that

$$H(V_\infty^n)(x) = \sum_{i=-n}^n \dot{s}_\infty(i) H(V_{e_i})(x).$$

To finish the proof of the theorem in this case, it is enough to show that

$$\int_n^\infty \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty^n(\zeta) d\zeta \rightarrow 0$$

and

$$\int_{-\infty}^{-n} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty^n(\zeta) d\zeta \rightarrow 0$$

as $n \rightarrow \infty$.

Using elementary computations, we obtain

$$(22) \quad \begin{aligned} &\int_n^\infty \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty^n(\zeta) d\zeta = \\ &x(x-1) \int_n^\infty \frac{\left(\sum_{i=-n}^n \dot{s}_\infty(i) \right) \zeta - \sum_{i=-n}^n i \dot{s}_\infty(i)}{\zeta(\zeta-1)(\zeta-x)} d\zeta = \\ &\left(\sum_{i=-n}^n \dot{s}_\infty(i) \right) O\left(\frac{1}{n}\right) + \left(\sum_{i=-n}^n i \dot{s}_\infty(i) \right) O\left(\frac{1}{n^2}\right) \end{aligned}$$

If the tip- ∞ infinitesimal shear function which satisfies (10) in addition satisfies

$$(23) \quad \sum_{i=-n}^n \dot{s}_\infty(i) = o(n)$$

then

$$\sum_{i=-n}^n i \dot{s}_\infty(i) = o(n^2)$$

and the quantity (22) converges to zero as $n \rightarrow \infty$. Thus, to finish the proof that $\int_n^\infty \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty^n(\zeta) d\zeta \rightarrow 0$ as $n \rightarrow \infty$ it remains to show (23) which is done in Lemma 5.5 below. The proof of $\int_{-\infty}^{-n} \frac{x(x-1)}{\zeta(\zeta-1)(\zeta-x)} V_\infty^n(\zeta) d\zeta \rightarrow 0$ as $n \rightarrow \infty$ is similar and left to the reader. This finishes the proof for $p = \infty$.

Assume that $p = 0$ and let $\mathcal{F}_0 = \{e_n\}_{n \in \mathbf{Z}}$ be the fan with tip 0. Then $e_1 = (0, \infty)$ and $e_n = (0, \frac{1}{-n+1})$ for $n \in \mathbf{Z} \setminus \{1\}$. Fix $n \in \mathbf{N}$. We have

$$V_0^n(\zeta) = \dot{s}(e_0) \frac{\zeta(\zeta-1)}{1} + \dot{s}(e_{-1}) \frac{\zeta(\zeta-\frac{1}{2})}{\frac{1}{2}} + \cdots + \dot{s}(e_{-n+1}) \frac{\zeta(\zeta-\frac{1}{n})}{\frac{1}{n}}$$

for $0 \leq \zeta \leq \frac{1}{n}$. A short computation gives

$$V_0^n(\zeta) = \left[\sum_{i=1}^n i \dot{s}(-i+1) \right] \zeta^2 - \left[\sum_{i=1}^n \dot{s}(-i+1) \right] \zeta$$

for $0 \leq \zeta \leq \frac{1}{n}$.

By the above and by Lemma (5.5), we get that

$$|V_0^n(\zeta)| \leq o(n^2)\zeta^2 + o(n)\zeta$$

for $0 \leq \zeta \leq \frac{1}{n}$. Then

$$\left| \int_0^{1/n} \frac{V_0^n(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\zeta \right| \leq o(n^2) \frac{1}{n^2} + o(n) \frac{1}{n} \rightarrow 0$$

as $n \rightarrow \infty$. Similarly

$$\left| \int_{-1/n}^0 \frac{V_0^n(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\zeta \right| \rightarrow 0$$

as $n \rightarrow \infty$. This implies $H(V_0^n) \rightarrow H(V_0)$ as $n \rightarrow \infty$.

Assume that $p \neq \infty, 0$. We have that

$$|V_p^n(\zeta)| \leq o(n^2)(\zeta-p)^2 + o(n)|\zeta-p|$$

because the vector field V_p^n is the push-forward by a Möbius map $B : 0 \mapsto p$ of the vector field V_0^n . The argument that

$$\int_p^{p^{n+1}} \frac{V_p^n(\zeta)}{\zeta(\zeta-1)(\zeta-x)} \rightarrow 0$$

as $n \rightarrow \infty$ proceeds similarly as the case $p = 0$ when $x = p$, and it is even easier when $x \neq p$ because the integrand does not have a singularity for n large. Thus $H(V_p^n) \rightarrow H(V_p)$ as $n \rightarrow \infty$.

We obtained that $H(V_p^n) \rightarrow H(V_p)$. Since $H(V_p^n)(x) = \sum_{i=-n}^n \dot{s}_p(e_i) H(V_{e_i})(x)$ the theorem follows. \square

Lemma 5.5. *Let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be a shear function which satisfies (10) in a fan $\{e_n\}_{n \in \mathbb{Z}}$. Then*

$$\sum_{i=k}^{k+n} \dot{s}(e_i) = o(n)$$

for each $k \in \mathbb{Z}$.

Proof. Let $a_0 = \dot{s}(e_k)$ and $a_i = a_{i-1} + \dot{s}(e_{k+i}) + \dot{s}(e_{k-i})$ for $i \geq 1$. Then the condition (10) becomes

$$(24) \quad \left| \frac{1}{n} (a_0 + a_1 + \cdots + a_{n-1}) \right| \leq C.$$

From (10) we also obtain

$$(25) \quad |a_i - a_{i-1}| \leq 2C$$

for all $i \geq 1$. The statement of the lemma translates to $a_n = o(n)$.

Assume on the contrary that there exists a sequence $i_n \rightarrow \infty$ as $n \rightarrow \infty$ and a constant $k > 0$ such that

$$|a_{i_n}| \geq ki_n$$

for all n . Without loss of generality, we assume that

$$a_{i_n} \geq ki_n$$

and seek a contradiction. By (25), we get that

$$(26) \quad \begin{aligned} a_{i_n+1} &\geq a_{i_n} - 2C \geq ki_n - 2C \\ a_{i_n+2} &\geq a_{i_n} - 4C \geq ki_n - 4C \\ &\dots \\ a_{i_n+j} &\geq a_{i_n} - 2jC \geq ki_n - 2jC \end{aligned}$$

for some $j \in \mathbb{Z}$. Adding the inequalities in (26), we obtain

$$a_0 + a_1 + \cdots + a_{i_n} + a_{i_n+1} + \cdots + a_{i_n+j} \geq ki_n j - j(j+1)C - Ci_n.$$

If $j = \lceil \sqrt{i_n} \rceil$ then

$$\frac{1}{i_n + j} (a_0 + \cdots + a_{i_n+j}) \geq \frac{ki_n j}{i_n + j} - \frac{j(j+1)}{i_n + j} C - \frac{i_n C}{i_n + j} \rightarrow \infty$$

as $n \rightarrow \infty$. This is in a contradiction with (24). \square

6. RECOVERING THE SHEARS

We gave a formula for the Hilbert transform of a Zygmund vector field in terms of the corresponding infinitesimal shear function (see theorems 5.2 and 5.4). We describe below how to obtain the infinitesimal shear function \dot{s} corresponding to a vector field $V : \mathbf{R} \rightarrow \mathbf{R}$.

Let $a, b, c, d \in \hat{\mathbf{R}}$ (given in the counter-clockwise order) be four vertices of an ideal hyperbolic quadrilateral which is decomposed into the union of the triangle with vertices a, b, d and the triangle with vertices b, c, d . Then the value of the shear function \dot{s} on the geodesic (b, d) with respect to the above two triangles is given by (see [11])

$$(27) \quad \dot{s}((b, d)) = \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(a)}{d - a} - \frac{V(b) - V(a)}{b - a} - \frac{V(d) - V(c)}{d - c}.$$

Formula (27) is the first variation of the cross-ratio

$$cr(a, b, c, d) = \frac{(c-b)(d-a)}{(b-a)(d-c)}$$

where $a, b, c, d \in \hat{\mathbf{R}}$ are given in the cyclic order on $\hat{\mathbf{R}}$. For this definition of the cross-ratio,

$$\log cr(a, b, c, d)$$

is the shear on the geodesic (b, d) considered as a diagonal of the quadrilateral with vertices a, b, c, d .

Using the equation (27) and theorems 5.2 and 5.4, we immediately obtain

Corollary 6.1. *Let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be the shear function of a Zygmund bounded vector field on $\hat{\mathbf{R}}$. Then the infinitesimal shear function*

$$H(\dot{s}) : \mathcal{F} \rightarrow \mathbf{R}$$

of the vector field $H(V)$ obtained by taking the Hilbert transform of V is given by (28)

$$H(\dot{s})((b, d)) = \sum_{p \in \hat{\mathcal{Q}}} \sum_{n \in \mathbf{Z}} \dot{s}_p(e_n) \left[\frac{H(V_{e_n})(c) - H(V_{e_n})(b)}{c-b} + \frac{H(V_{e_n})(d) - H(V_{e_n})(a)}{d-a} \right. \\ \left. - \frac{H(V_{e_n})(b) - H(V_{e_n})(a)}{b-a} - \frac{H(V_{e_n})(d) - H(V_{e_n})(c)}{d-c} \right]$$

where $(b, d) \in \mathcal{F}$ is the common boundary side of the two complementary triangles of \mathcal{F} with vertices (a, b, d) and (b, c, d) , and e_n is a geodesic of the fan \mathcal{F}_p oriented such that p is the initial point of e_n for $n \geq 1$ and p is the terminal point of e_n for $n \leq 0$.

The above computation of the infinitesimal shear function $H(\dot{s})$ can be given in terms of the hyperbolic geometry of the underlining tessellation \mathcal{F} . Namely, let $(0, \infty)$ be the geodesic e_n and let (b, d) be the geodesic whose shear $H(\dot{s})((b, d))$ we want to compute. Let a and c be the other two endpoints of the ideal quadrilateral in $\mathbf{H} \setminus (\mathcal{F} \setminus \{(b, d)\})$.

Assume first that $a, b, c, d \notin \{0, \infty\}$ and let $V_{0, \infty}$ be given by (4). Then the contribution of $H(V_{0, \infty})$ to $H(\dot{s})((b, d))$ is

$$\frac{b}{c-b} \log \left| \frac{c}{b} \right| + \frac{d}{d-a} \log \left| \frac{d}{a} \right| - \frac{b}{b-a} \log \left| \frac{b}{a} \right| - \frac{d}{d-c} \log \left| \frac{d}{c} \right|.$$

We write the above expression in terms of the cross-ratios as

$$(29) \quad cr(0, \infty, c, b) \log |cr(0, b, \infty, c)| + cr(0, \infty, a, d) \log |cr(0, a, \infty, d)| - \\ cr(0, \infty, a, b) \log |cr(0, a, \infty, b)| - cr(0, \infty, c, d) \log |cr(0, b, \infty, c)|.$$

Let $\delta_{a, d}$ be the hyperbolic distance between $(0, \infty)$ and (a, d) , and similarly for $\delta_{a, b}, \delta_{b, c}, \delta_{c, d}$. Then the expression (29) can be written as

$$(30) \quad \sinh^2 \left(\frac{\delta_{b, c}}{2} \right) \log \coth^2 \left(\frac{\delta_{b, c}}{2} \right) + \cosh^2 \left(\frac{\delta_{a, d}}{2} \right) \log \coth^2 \left(\frac{\delta_{a, d}}{2} \right) - \\ \cosh^2 \left(\frac{\delta_{a, b}}{2} \right) \log \coth^2 \left(\frac{\delta_{a, b}}{2} \right) - \cosh^2 \left(\frac{\delta_{c, d}}{2} \right) \log \coth^2 \left(\frac{\delta_{c, d}}{2} \right)$$

If $a = 0$ and $b, c, d \notin \{0, \infty\}$, then by similar reasoning we get that the contribution is

$$(31) \quad \sinh^2\left(\frac{\delta_{b,c}}{2}\right) \log \coth^2\left(\frac{\delta_{b,c}}{2}\right) + \log \coth^2\left(\frac{\delta_{b,d}}{2}\right) - \cosh^2\left(\frac{\delta_{c,d}}{2}\right) \log \coth^2\left(\frac{\delta_{c,d}}{2}\right).$$

If $d = \infty$ and $a, b, c \notin \{0, \infty\}$, then the contribution is

$$(32) \quad \sinh^2\left(\frac{\delta_{b,c}}{2}\right) \log \coth^2\left(\frac{\delta_{b,c}}{2}\right) - \cosh^2\left(\frac{\delta_{a,b}}{2}\right) \log \coth^2\left(\frac{\delta_{a,b}}{2}\right).$$

If $a = 0$ and $d = \infty$, then the contribution is

$$(33) \quad \cosh^2\left(\frac{\delta_{b,c}}{2}\right) \log \coth^2\left(\frac{\delta_{b,c}}{2}\right).$$

If $b = 0$ and $d = \infty$, then the contribution is

$$(34) \quad \log \tan^2(\theta_{a,c})$$

where $\theta_{a,c}$ is the angle between geodesics $(0, \infty)$ and (a, c) .

Note that all the contributions to $H(\dot{s})((b, d))$ are expressed in terms of the invariants for the positions of the geodesics with endpoints $a, b, c, d, 0, \infty$ and they remain in force when the geodesic $(0, \infty)$ is replaced by an arbitrary geodesic. Thus we obtain

Corollary 6.2. *Let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be the infinitesimal shear function of a Zygmund bounded vector field V on $\hat{\mathbf{R}}$. Then the infinitesimal shear function*

$$H(\dot{s}) : \mathcal{F} \rightarrow \mathbf{R}$$

of the vector field $H(V)$ obtained by taking the Hilbert transform of V is given by

$$(35) \quad H(\dot{s})((b, d)) = \sum_{p \in \hat{\mathbf{Q}}} \sum_{n \in \mathbf{Z}} \dot{s}(e_n) \Delta_{b,d}(e_n)$$

where $(b, d) \in \mathcal{F}$ is the common boundary side of the two complementary triangles of \mathcal{F} with vertices (a, b, d) and (b, c, d) , and e_n is a geodesic of the fan \mathcal{F}_p oriented such that p is the initial point of e_n for $n \geq 1$ and p is the terminal point of e_n for $n \leq 0$, and $\Delta_{b,d}(e_n)$ is one of the expressions (30), (31), (32) (33), (34) depending on the relative positions of (a, b, c, d) with respect to e_n .

7. FOURIER COEFFICIENTS AND INFINITESIMAL SHEAR FUNCTIONS

Let $V : S^1 \rightarrow \mathbf{C}$ be a Zygmund bounded vector field on the unit circle of S^1 . Let \mathcal{F} be the Farey tessellation of the unit disk \mathbf{D} which is obtained by taking the image of the Farey tessellation of \mathbf{H} under the Möbius map $B : \mathbf{H} \rightarrow \mathbf{D}$ which maps $0, 1$ and ∞ onto $1, i$ and -1 . In this section, V is normalized to vanish at $1, i$ and -1 . The infinitesimal shear function $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ corresponding to V satisfies the property (10) since all the considerations are geometric.

We express Fourier coefficients of a Zygmund bounded vector field V on the unit circle S^1 in terms of its infinitesimal shear function defined on the Farey tessellation \mathcal{F} . Parametrize S^1 by associating to each point $z \in S^1$ its angle $\phi = \arg(z)$, where $\phi \in [0, 2\pi]$. Then, for $\phi_0 < \phi_1$, the elementary shear vector field is defined by

$$V_{\phi_0, \phi_1}(z) = \begin{cases} \frac{(z - e^{i\phi_0})(z - e^{i\phi_1})}{e^{i\phi_0} - e^{i\phi_1}}, & \text{for } \phi_0 < \arg(z) < \phi_1 \\ 0, & \text{otherwise} \end{cases}$$

Given a fan \mathcal{F}_p with the tip $p \in S^1$ of the Farey tessellation \mathcal{F} in \mathbf{D} , denote by V_p the vector field corresponding to a infinitesimal shear function which agrees with $\frac{1}{2}\dot{s}$ on \mathcal{F}_p and which is zero on $\mathcal{F} \setminus \mathcal{F}_p$, and V_p is normalized to be zero at 1, i and -1 . Namely,

$$V_p(z) = \sum_{e_n=(e^{i\phi_0^n}, e^{i\phi_1^n}) \in \mathcal{F}_p} \frac{1}{2}\dot{s}(e_n)V_{\phi_0^n, \phi_1^n}(z).$$

Corollary 4.8 implies that

$$V(z) = \sum_{p \in \mathcal{F}^0} V_p(z)$$

where \mathcal{F}^0 is the set of all vertices of \mathcal{F} and the convergence is absolute and uniform on S^1 .

An elementary integration gives the n -th Fourier coefficient $\widehat{V}_{\phi_0, \phi_1}(n)$ of V_{ϕ_0, ϕ_1} as

$$(36) \quad \widehat{V}_{\phi_0, \phi_1}(n) = \frac{1}{2\pi(e^{i\phi_0} - e^{i\phi_1})} \left[\frac{e^{i(2-n)\phi_1} - e^{i(2-n)\phi_0}}{i(2-n)} - (e^{i\phi_0} + e^{i\phi_1}) \times \right. \\ \left. \times \frac{e^{i(1-n)\phi_1} - e^{i(1-n)\phi_0}}{i(1-n)} + e^{i(\phi_0+\phi_1)} \frac{e^{-in\phi_1} - e^{-in\phi_0}}{-in} \right]$$

We have the following theorem.

Theorem 7.1. *Let $V : S^1 \rightarrow \mathbf{C}$ be a Zygmund vector field and let $\dot{s} : \mathcal{F} \rightarrow \mathbf{R}$ be the corresponding infinitesimal shear function. Then the n -th Fourier coefficient $\widehat{V}(n)$ of the vector field V is given by*

$$\widehat{V}(n) = \sum_{p \in \mathcal{F}^0} \widehat{V}_p(n),$$

where the convergence is absolute and

$$\widehat{V}_p(n) = \sum_{e_n=(e^{i\phi_0^n}, e^{i\phi_1^n}) \in \mathcal{F}_p} \frac{1}{2}\dot{s}(e_n)\widehat{V}_{\phi_0^n, \phi_1^n}(n).$$

Proof. Since $\sum_{p \in \mathcal{F}^0} V_p(z)$ is converging uniformly on S^1 to $V(z)$, the first formula of the theorem is immediate. By Lemma 5.5 and the proof of Theorem 5.2, we have that $|V_p(z)|$ in a neighborhood of p is increasing to infinity slower than the distance to p . This implies that the second formula of the theorem holds. \square

8. THE ALMOST COMPLEX STRUCTURE FOR TEICHMÜLLER SPACES OF FINITE SURFACES

Let S be a finite area hyperbolic surface with $s > 0$ punctures. The hyperbolic plane \mathbf{H} is the universal covering of S such that the covering map $\pi : \mathbf{H} \rightarrow S$ is a local isometry. Let G be the group of deck transformations and let τ be a locally finite ideal triangulation of S . Then τ lifts to a locally finite ideal triangulation $\tilde{\tau}$ of \mathbf{H} which is invariant under G .

The Teichmüller space $T(S)$ of a Riemann surface S is the homotopy class of all marked hyperbolic surfaces $f : S \rightarrow X$ up to post-composition by hyperbolic isometries. We denote the Teichmüller class of $f : S \rightarrow X$ by $[X, f]$. The Teichmüller space $T(S)$ maps into the space of functions from the set of geodesics $|\tau|$ of the triangulation τ into the real numbers \mathbf{R} as follows. Let c_1, \dots, c_s be the set

of punctures of S . Given a marked hyperbolic surface $f : S \rightarrow X$, we replace each curve in $|f(\tau)|$ with the geodesic of X homotopic to it relative punctures. We obtain an ideal geodesic triangulation of the marked hyperbolic surface X homotopic to $f(\tau)$. Then we assign to the marked hyperbolic surface X the function from $|\tau|$ into \mathbf{R} which maps each edge of τ to its shear with respect to $f(\tau)$. In this fashion, we obtain an injective map

$$T(S) \rightarrow \mathbf{R}^{|\tau|}$$

which is a real analytic diffeomorphism onto its image (see [30], [22]). The image of $T(S)$ consists of all functions $s : |\tau| \rightarrow \mathbf{R}$ such that for each puncture $c_i, i = 1, \dots, s$, we have

$$\sum_{j=1}^{k(i)} s(e_j^{c_i}) = 0$$

where $e_j^{c_i}, j = 1, \dots, k(i)$, is the set of geodesics from τ having an ideal endpoint the cusp c_i such that a single geodesic appears twice if and only if both of its endpoint are c_i .

A tangent vector to $T(S)$ at point $[X, f] \in T(S)$ is also described by a function in $\mathbf{R}^{|\tau|}$ as follows. Namely, a differentiable (in t) path of shear functions $s_t : |\tau| \rightarrow \mathbf{R}$, $t \in (-\epsilon, \epsilon)$, such that $s_0 : |\tau| \rightarrow \mathbf{R}$ equals the shear function of $[X, f]$ describes a differentiable path in $T(S)$ through the point $[X, f]$. The derivative

$$\frac{d}{dt} s_t|_{t=0} = \dot{s} : |\tau| \rightarrow \mathbf{R}$$

represents a tangent vector $v \in T_{[X, f]}T(S)$. In general, a function $\dot{s} \in \mathbf{R}^{|\tau|}$ represents a tangent vector at $T_{[X, f]}T(S)$ if and only if for each puncture c_i it satisfies

$$\sum_{j=1}^{k(i)} \dot{s}(e_j^{c_i}) = 0$$

where $e_j^{c_i}, j = 1, \dots, k(i)$, is the set of geodesics from τ having an ideal endpoint the cusp c_i with possible repeating if both ends are at the same cusp as before.

Lift the triangulation τ of S to a triangulation $\tilde{\tau}$ of \mathbf{H} . We normalize the covering map such that $\tilde{\tau}$ contains the geodesic $(0, \infty)$ and denote by $a_{-1} < 0$ the endpoint of the geodesic in the fan of $\tilde{\tau}$ with tip ∞ adjacent to $(0, \infty)$ on the left. We choose the complementary triangle Δ_0 with vertices $0, a_{-1}$ and ∞ to be the reference triangle for $\tilde{\tau}$ and orient all geodesics of $\tilde{\tau}$ to the left as seen from Δ_0 . Given a vertex $p \in \hat{\mathbf{R}}$ of $\tilde{\tau}$, let $\{e_n^p\}_{n \in \mathbf{Z}}$ be the set of edges of $\tilde{\tau}$ which have one endpoint p such that e_n^p is adjacent to e_{n+1}^p for all $n \in \mathbf{Z}$, and that e_0^p has the initial point p . The set $\{e_n^p\}_{n \in \mathbf{Z}} = \tilde{\tau}_p$ is said to be a fan of $\tilde{\tau}$ with tip p . Then, a map

$$\tilde{s} : |\tilde{\tau}| \rightarrow \mathbf{R}$$

represent a point in $T(S)$ if and only if \tilde{s} is invariant under G and for each vertex $p \in \tilde{\tau}_0$ we have

$$\sum_{i=1}^{n(p)} \tilde{s}(e_{j+i}^p) = 0$$

where e_{j+1}^p is identified with $e_{j+n(p)+1}^p$ by an element of G and $j \in \mathbf{Z}$ is arbitrary. We choose $n(p)$ to be the smallest positive number such that e_{j+1}^p is identified with

$e_{j+n(p)+1}^p$ by an element of G . A function

$$\dot{s} : |\tilde{\tau}| \rightarrow \mathbf{R}$$

represent a tangent vector to a point in $T(S)$ if and only if \dot{s} is invariant under G and for each vertex $p \in \tilde{\tau}_0$ we have

$$\sum_{i=1}^{n(p)} \dot{s}(e_{j+i}^p) = 0$$

where e_{j+1}^p is identified with $e_{j+n(p)+1}^p$ by an element of G , and again we assume that $n(p)$ is the smallest such number. Note that \dot{s} defines a tangent vector at an arbitrary point of $T(S)$.

Our goal in this section is to compute the Hilbert transform in terms of $\dot{s} : |\tilde{\tau}| \rightarrow \mathbf{R}$. Let Δ_0 be the reference complementary triangle to $\tilde{\tau}$ whose ideal vertices are 0 , a_1 and ∞ . Let $\{e_n^\infty\}_{n \in \mathbf{Z}}$ be the fan of $\tilde{\tau}$ with tip ∞ such that e_0 has initial point ∞ and terminal point 0 , e_1 is to the left of e_0 and e_n is adjacent to e_{n+1} for all $n \in \mathbf{Z}$. Let a_n be the endpoint of e_n different from ∞ . Let V_∞ be the vector field normalized to be zero at 0 , a_1 and ∞ whose shear function agree with \dot{s} on $\tilde{\tau}_\infty$ and equals zero on all other geodesics of $\tilde{\tau}$. The lemma below proves that V_∞ is a Zygmund vector field.

Lemma 8.1. *Let $n \geq 1$ be a fixed integer and let $\dot{s} : \tilde{\tau}_\infty \rightarrow \mathbf{R}$ be a function such that*

$$\dot{s}(e_j^\infty) = \dot{s}(e_{j+n}^\infty)$$

and

$$(37) \quad \sum_{i=j}^{j+n-1} \dot{s}(e_i^\infty) = 0$$

for all $j \in \mathbf{Z}$. Let $\dot{s}_N : \tilde{\tau}_\infty \rightarrow \mathbf{R}$ be defined by $\dot{s}_N(e_i^\infty) = \dot{s}(e_i^\infty)$ if $-Nn \leq i \leq Nn-1$, and $\dot{s}_N(e_i^\infty) = 0$ otherwise. Let V_∞^N be the vector field corresponding to \dot{s}_N and let V_∞ be the vector field corresponding to \dot{s} . Then V_∞ and V_∞^N are Zygmund vector fields whose Zygmund constants are bounded by some $C_3 > 0$ for all $N \in \mathbf{N}$.

Proof. Let $g(x) = x + k$, for $k > 0$, be a generator of the subgroup of G fixing ∞ . Then we have $a_i + k = a_{i+n}$ for all $i \in \mathbf{Z}$ by the invariance of $\tilde{\tau}$ under G . We normalize V_∞ and V_∞^N to be zero on the interval $[a_{-1}, a_0 = 0]$. By (37), the contribution of

$$\dot{s}(e_0^\infty)(x - a_0) + \dot{s}(e_1^\infty)(x - a_1) + \cdots + \dot{s}(e_{n-1}^\infty)(x - a_{n-1})$$

is equal to

$$(38) \quad - \sum_{i=0}^{n-1} \dot{s}(e_i^\infty) a_i.$$

Moreover, for $j > 0$ and $j \equiv 0 \pmod{n}$, we have

$$(39) \quad \sum_{i=0}^{n-1} \dot{s}(e_{j+i}^\infty)(x - a_{j+i}) = \sum_{i=0}^{n-1} \dot{s}(e_i^\infty)(x - \frac{j}{n}k - a_i) = - \sum_{i=0}^{n-1} \dot{s}(e_i^\infty) a_i$$

because $\dot{s}(e_{j+i}^\infty) = \dot{s}(e_i^\infty)$ and $a_{j+i} = a_i + \frac{j}{n}k$ by the invariance of \dot{s} and $\tilde{\tau}$ under G .

If $j < 0$ and $j \equiv 0 \pmod n$, then, in a similar fashion, we obtain

$$\sum_{i=0}^{n-1} \dot{\hat{s}}(e_{j-1-i}^\infty)(x - a_{j-1-i}) = - \sum_{i=0}^{n-1} \dot{\hat{s}}(e_{-i-1}^\infty)a_{-i-1}.$$

Define

$$C = \max_{0 < i_0 < n} \left\{ \sup_{x \in [a_{i_0}, a_{i_0+1}]} \left| \sum_{i=0}^{i_0} \dot{\hat{s}}(e_i^\infty)(x - a_i) \right| + \sup_{x \in [a_{-i_0-1}, a_{-i_0}]} \left| \sum_{i=1}^{i_0} \dot{\hat{s}}(e_{-i}^\infty)(x - a_{-i}) \right| \right\}$$

The above estimates give that

$$(40) \quad |V_\infty(x)| \leq C + \frac{|x|}{k} C_1$$

and

$$(41) \quad |V_\infty^N(x)| \leq C + C_1 N$$

for all $x \in \mathbf{R}$, where C is defined above and

$$C_1 = \max \left\{ \left| \sum_{i=0}^{n-1} \dot{\hat{s}}(e_i^\infty)a_i \right|, \left| \sum_{i=0}^{n-1} \dot{\hat{s}}(e_{-i-1}^\infty)a_{-i-1} \right| \right\}.$$

We show that $V_\infty(x)$ and $V_\infty^N(x)$ are Lipschitz continuous with the constants bounded independently of N . Let $x \in \mathbf{R}$ and $t > 0$. By adding a linear term and by the invariance of $V_\infty(x)$ under the action of $g(x) = x + k$, we assume that $0 \leq x \leq k$.

If $t \leq k$, then an easy argument shows that

$$|V_\infty^N(x+t) - V_\infty^N(x)|, |V_\infty(x+t) - V_\infty(x)| \leq C_2 t$$

and

$$|V_\infty^N(x) - V_\infty^N(x-t)|, |V_\infty(x) - V_\infty(x-t)| \leq C_2 t,$$

where $C_2 = \max\{1, \sum_{i=0}^{n-1} |\dot{\hat{s}}(e_i^\infty)|\}$.

Assume that $t > k$. Then (40) and (41) imply that

$$|V_\infty^N(x+t) - V_\infty^N(x)|, |V_\infty(x+t) - V_\infty(x)| \leq 2 \frac{C_1 + C_2}{k} t$$

and

$$|V_\infty^N(x) - V_\infty^N(x-t)|, |V_\infty(x) - V_\infty(x-t)| \leq 2 \frac{C_1 + C_2}{k} t,$$

where C_1, C_2 are as above. This implies that V_∞ and V_∞^N are Zygmund bounded with bound $C_3 = 4(\frac{C_1 + C_2}{k} + C_2)$. \square

Since V_∞ and V_∞^N have their Zygmund norms bounded independently of N , it follows that their cross-ratio norms are uniformly bounded as well. Let $\tilde{\tau}^0$ be the set of vertices of $\tilde{\tau}$. This implies that V_p and V_p^N are Zygmund bounded with Zygmund bounds independent of N for all $p \in \tilde{\tau}^0$, where V_p^N, V_p are defined analogous to V_∞^N, V_∞ .

Theorem 8.2. *Let S be a finite area punctured hyperbolic Riemann surface equipped with an ideal geodesic triangulation τ . Let $\tilde{\tau} = \{e_n\}_{n \in \mathbf{N}}$ be the lift to the hyperbolic plane \mathbf{H} of τ and let $\{V_{e_n}\}_{n \in \mathbf{N}}$ be the elementary vector fields corresponding to the*

geodesics e_n of $\tilde{\tau}$. Let $V(x)$ be a Zygmund vector field on $\hat{\mathbf{R}}$ representing an infinitesimal deformation of S and let $\dot{\tilde{s}} : \tilde{\tau} \rightarrow \mathbf{R}$ be the corresponding shear function. Then the series

$$(42) \quad \sum_{n \in \mathbf{N}} \dot{\tilde{s}}(e_n) V_{e_n}(x)$$

converges to $V(x)$ uniformly on compact subsets of \mathbf{R} .

Proof. We follow the idea in the proof of Theorem 4.1. Let $\{p_i\}_{i \in \hat{\mathbf{Q}}}$ be the set of vertices of $\tilde{\tau}$. There exists $\delta > 0$, such that for any two fans $\tilde{\tau}_{p_1}$ and $\tilde{\tau}_{p_2}$ which do not have a common geodesic the minimum distance between the geodesics of $\tilde{\tau}_{p_1}$ and $\tilde{\tau}_{p_2}$ is at least δ . This is a direct consequence of the invariance of $\tilde{\tau}$ under a co-finite group G .

Let $I \subset \mathbf{R}$ be a closed, bounded interval in \mathbf{R} . For $x \in I$, let r_x denote a geodesic ray that connects $i \in \mathbf{H}$ to $x \in I \subset \mathbf{R}$. Given $\epsilon > 0$, there exists a finite set of tips $\{p_1, \dots, p_s\} \subset \tilde{\tau}^0$, $s = s(\epsilon)$, such that if a geodesic of the fan with a tip $p \in \tilde{\tau}^0$ intersects r_x for some $x \in I$ then either all geodesics in $\tilde{\tau}_p$ have their Euclidean lengths less than ϵ , or $p \in \{p_1, \dots, p_s\}$ and all fans whose geodesics intersect r_x before the geodesics of $\tilde{\tau}_p$ are also in $\{p_1, \dots, p_s\}$. Let $n(p_i)$ stand for the number of elements in $\tilde{\tau}_{p_i}$ that form a fundamental set for the action by the subgroup of G that stabilizes p_i . There exists $N = N(\epsilon, s) \in \mathbf{N}$ such that all geodesics $e_j^{p_i}$ of $\tilde{\tau}_{p_i}$, $i = 1, \dots, s$, for either $j > Nn(p_i)$ or $j < -Nn(p_i)$ have their Euclidean sizes less than ϵ .

By our construction, the sum

$$\sum_{i=1}^s V_{p_i}^N(x)$$

agrees with $V(x)$ on an ϵ -net for I . Since the Zygmund norms of each V_p^N are uniformly bounded for $p \in \tilde{\tau}^0$ and $N \in \mathbf{N}$ and since the ideal triangulation has positive minimum distance between geodesics of two disjoint fans, it follows that the above sum converges uniformly on I to $V(x)$ as $\epsilon \rightarrow 0$ similar to proof of Theorem 4.1. To finish the proof, it is enough to note that each V_p^N is a finite sum of elementary shear vector fields V_e for $e \in \tilde{\tau}$ with the coefficients $\dot{\tilde{s}}(e)$. \square

We use the above theorem to establish a formula for the Hilbert transform in terms of infinitesimal shear function of Zygmund vector field.

Theorem 8.3. *Let $V : \mathbf{R} \rightarrow \mathbf{R}$ be a Zygmund vector field which is invariant under a co-finite group G and let $\tilde{\tau}$ be an ideal triangulation of \mathbf{H} invariant under G . Let $\dot{\tilde{s}} : \tilde{\tau} \rightarrow \mathbf{R}$ be the infinitesimal shear function of V . Then the Hilbert transform $H(V)$ is given by*

$$H(V)(x) = \sum_{n \in \mathbf{N}} \dot{\tilde{s}}(e_n) H(V_{e_n})(x)$$

where $\{e_n\}_{n \in \mathbf{N}}$ are the geodesics of $\tilde{\tau}$ given in a sequence and $H(V_{e_n})$ is defined by (8).

Proof. We established that the quantity

$$\sum_{i=1}^s V_{p_i}^N(x)$$

in the proof of Theorem 8.2 converges uniformly on compact subsets of \mathbf{R} and that each $V_{p_i}^N$ is Zygmund bounded with uniformly bounded Zygmund norms. Then Proposition 4.7 implies that $\sum_{i=1}^s V_{p_i}^N(x)$ have uniformly bounded Zygmund norms. Then Lemma 5.1 implies that

$$H\left(\sum_{i=1}^s V_{p_i}^N(x)\right) = \sum_{i=1}^s H(V_{p_i}^N)(x) \rightarrow H(V)(x)$$

as $\epsilon \rightarrow 0$. The theorem follows because each $V_{p_i}^N$ is a finite linear combination of simple shear vector fields. \square

We also obtain an expression of the infinitesimal shear function for the Hilbert transform similarly to the case when Zygmund vector field is not invariant.

Corollary 8.4. *Let $\dot{s} : \tilde{\tau} \rightarrow \mathbf{R}$ be the infinitesimal shear function of a Zygmund vector field V invariant under a co-finite group G , where $\tilde{\tau}$ is an ideal triangulation of \mathbf{H} invariant under G . Then the infinitesimal shear function*

$$H(\dot{s}) : \tilde{\tau} \rightarrow \mathbf{R}$$

of the vector field $H(V)$ obtained by taking the Hilbert transform of V is given by

$$(43) \quad H(\dot{s})(b, d) = \sum_{n \in \mathbf{N}} \dot{s}(e_n) \Delta_{b,d}(e_n)$$

where $(b, d) \in \mathcal{F}$ is the common boundary side of the two complementary triangles of \mathcal{F} with vertices (a, b, d) and (b, c, d) , $\tilde{\tau} = \{e_n\}_{n \in \mathbf{N}}$, and $\Delta_{b,d}(e_n)$ is one of the expressions (30), (31), (32), (33), (34) depending on the relative positions of (a, b, c, d) with respect to e_n .

9. THE WEIL-PETERSSON METRIC FOR TEICHMÜLLER SPACES OF FINITE SURFACES IN SHEAR COORDINATES

We described above the Hilbert transform on the tangent vectors to the Teichmüller space $T(S)$ of a finite area hyperbolic surface S with $s > 0$ punctures. The Hilbert transform is the almost complex structure to $T(S)$ (see [20]). The Weil-Petersson metric g_{WP} is Kähler [4], [33], [34]. The symplectic two form of the Weil-Petersson metric in the shear coordinates is simply given as twice the Thurston algebraic intersection form on the weights induced by the tangent vectors on the train track associated to the ideal triangulation τ of S ([8], see also [33], [23]). Therefore, we obtain the following formula in terms of shear coordinates.

Theorem 9.1. *Let S be a finite area hyperbolic surface with $s > 0$ punctures and let τ be an ideal triangulation of S . Given two infinitesimal shear functions*

$$\dot{s}_1, \dot{s}_2 : \tilde{\tau} \rightarrow \mathbf{R}$$

which represent two tangent vectors v_1, v_2 to the Teichmüller space $T(S)$ of S at the basepoint $[S, id]$, the Weil-Petersson inner product is given by

$$g_{WP}(v_1, v_2) = 2i(\dot{s}_1, H(\dot{s}_2))$$

where $H(\dot{s}_2)$ is the infinitesimal shear function of the Hilbert transform $H(v_2)$ and $i(\cdot, \cdot)$ is the Thurston's algebraic intersection number between infinitesimal shear functions defined on τ .

10. CONTINUOUS EXTENSION OF THE HILBERT TRANSFORM AND THE WEIL-PETERSSON METRIC TO THE BOUNDARY OF THE TEICHMÜLLER SPACE

Let S be a finite area hyperbolic surface with $s > 0$ punctures and let $\bar{T}(S)$ be the augmented Teichmüller space of S [1], [5], [35]. Our goal is to show that the Hilbert transform and the Weil-Petersson metric extend by continuity to the completion of $T(S)$. Masur [18] proved that the Weil-Petersson hermitian metric continuously extend to the augmented Teichmüller space $\bar{T}(S)$. More recently, Roger [27] showed that the Weil-Petersson symplectic two form on $T(S)$ extends by continuity to the augmented Teichmüller space $\bar{T}(S)$ using the shear parametrization of $T(S)$ with appropriate degeneration of the ideal triangulation when the point is in $\bar{T}(S) \setminus T(S)$. We show that the Hilbert transform in terms of the shears for $T(S)$ extends by continuity to the augmented Teichmüller space $\bar{T}(S)$.

One description of the manifold structure on the augmented Teichmüller space $\bar{T}(S)$ is given in terms of the plumbing coordinates for a neighborhood of a point on the boundary of $T(S)$ [1], [5]. Another description is given by extending the Fenchel-Nielsen coordinates for $T(S)$ to allow the lengths of some of the closed geodesics of the pants decomposition to be zero [1], [5], [35]. Let $\sigma = \{\gamma_1, \dots, \gamma_k\}$ be a set of mutually non-homotopic, non-intersecting and non-trivial simple closed curves on S . We denote by $\mathcal{S}(\sigma)$ the set of all (marked) noded Riemann surfaces R obtained from S by pinching curves in σ . Then the augmented Teichmüller space consists of $T(S)$ and $\mathcal{S}(\sigma)$ over all σ as above.

We use the plumbing coordinates for our purposes. Let R_t , $0 \leq t < 1$, be a path of marked Riemann surfaces in $T(S)$ such that its limit point R^* as $t \rightarrow 1$ is on the boundary of $T(S)$ in $\mathcal{S}(\sigma)$. Therefore R^* is a noded Riemann surface with nodes corresponding to the curves of σ . The path R_t is described by the plumbing construction in the neighborhood of the nodes in R^* (see [18]). A tangent vector to R^* on the boundary $\bar{T}(S) \setminus T(S)$ is represented by a (smooth) Beltrami differential ν which is supported on $R^* \setminus U$, where U is the union of neighborhoods of nodes of R^* which correspond to the curves in σ . Then the Beltrami differential ν on $R_t \setminus U$ represents tangent vectors at $R_t \in T(S)$ which converge to ν on R^* . We normalize the universal coverings $\pi_t : \mathbf{H} \rightarrow R_t$ and $\pi : \mathbf{H} \rightarrow R^*$ such that they map $i \in \mathbf{H}$ onto a fixed point $p \in R_t \setminus U = R^* \setminus U$. Since the differentiable structures on $R_t \setminus U$ and $R^* \setminus U$ are the same, we can further normalize π_t, π such that they map a fixed tangent direction at $i \in \mathbf{H}$ onto a fixed tangent direction at p .

Let $\tilde{\nu}_t = (\pi_t)_* \nu$ and $\tilde{\nu} = (\pi)_* \nu$ be the pull-backs of ν by the universal coverings. The Zygmund vector fields V_t on $\hat{\mathbf{R}}$ corresponding to $\tilde{\nu}_t$ converge pointwise on $\hat{\mathbf{R}}$ to the Zygmund vector field V corresponding to $\tilde{\nu}$ because $\tilde{\nu}_t$ converges pointwise to $\tilde{\nu}$ on \mathbf{H} . The Zygmund norms of V_t, V are uniformly bounded because $\|\tilde{\nu}_t\|_\infty, \|\tilde{\nu}\|_\infty$ are uniformly bounded.

We describe the convergence of tangent vectors as they approach to the boundary of $T(S)$ in terms of shear parametrization. We first recall the convergence of the path of infinitesimal shear functions corresponding to a path of points in $T(S)$ that converges to a boundary point $R \in \mathcal{S}(\sigma)$ described by J. Roger [27]. Let $\lambda = \{h_1, h_2, \dots, h_n\}$ be an ideal triangulation of the punctured surface S . Consider the arcs $h_i \cap (S \setminus \sigma)$ for $i = 1, 2, \dots, n$. They divide the surface $S \setminus \sigma$ into triangles and bigons [27]. We group the edges of $\lambda \cap (S \setminus \sigma)$ into homotopy classes relative to punctures of S and to σ . In this fashion an ideal triangulation $\mu = \{g_1, g_2, \dots, g_n\}$

of $S \setminus \sigma$ is obtained. Let $k_{i,j}$ be the number of arcs of $h_j \cap (S \setminus \sigma)$ that are homotopic to g_i . Then

Proposition 10.1. [27] *Let $R_t \in T(S)$ for $t \in [0, 1)$ be a path which converges to $R^* \in \mathcal{S}(\sigma) \subset \bar{T}(S)$ as $t \rightarrow 1$. Let $s_t : \lambda \rightarrow \mathbf{R}$ be the infinitesimal shear function for R_t and let $s^* : \mu \rightarrow \mathbf{R}$ be the infinitesimal shear function for R^* . Then*

$$\lim_{t \rightarrow 1} \sum_{j=1}^n k_{i,j} s_t(h_j) = s^*(g_i).$$

In other words, the value $s^*(g_i)$ is the limit of the sum of the values of s_t on the arcs of $\lambda \cap (S \cap \sigma)$ which are homotopic to g_i .

We continue to work with the path $R_t \in T(S)$ for $t \in [0, 1)$ which converges to $R^* \in \mathcal{S}(\sigma)$ as $t \rightarrow 1$. Let u_t and u^* be tangent vectors based at R_t and R^* such that $R_t \rightarrow R^*$ and $u_t \rightarrow u^*$ as $t \rightarrow 1$. Let $s_t, \dot{s}_t : \lambda \rightarrow \mathbf{R}$ and $s^*, \dot{s}^* : \mu \rightarrow \mathbf{R}$ be the infinitesimal shear functions for R_t, u_t, R^* and u^* , respectively. Then the sum of s_t, \dot{s}_t shears for the geodesics in λ at each puncture of S is zero. Similarly, the sum of s^*, \dot{s}^* shears for the geodesics in μ at each puncture of $S \setminus \sigma$ is zero, where $\alpha \in \sigma$ is considered a puncture of $S \setminus \sigma$.

Proposition 10.2. *Assume that $s_t, \dot{s}_t, s^*, \dot{s}^*$ are as above. Then*

$$\lim_{t \rightarrow 1} \sum_{j=1}^n k_{i,j} \dot{s}_t(h_j) = \dot{s}^*(g_i).$$

Proof. Let R_t^k be the component of $R_t \setminus \sigma$ which corresponds to the component of $S \setminus \sigma$ which contains arcs homotopic to g_i . Let $p \in R_t^k \setminus U = R^* \setminus U$ be a fixed point with a fixed tangent direction. Let $\pi_t : \mathbf{H} \rightarrow R_t$ be the universal covering of R_t chosen such that $\pi_t(i) = p$ and a fixed tangent direction at $i \in \mathbf{H}$ is mapped onto the fixed tangent direction at $p \in R_t^k$. Let V_t and V^* be the Zygmund vector fields corresponding to the tangent vectors u_t and u^* . By the above, we have that V_t converges to V^* pointwise on $\hat{\mathbf{R}}$.

The normalization of the universal covering $\pi_t : \mathbf{H} \rightarrow R_t$ implies that the lifts $(\pi_t)^{-1}(\sigma)$ leave every compact subset of \mathbf{H} . It follows that $(\pi_t)^{-1}(\lambda)$ converges to $\pi^{-1}(\mu)$.

Let \tilde{g}_i be a lift of g_i to \mathbf{H} under the covering map π . Recall that g_i corresponds to a homotopy class of arcs in $\lambda \cap (S \setminus \sigma)$. If g_i does not end at σ , then there is a unique $h_i \in \mu$ which corresponds to it. Let $\tilde{h}_i(t)$ be the lift of h_i under π_t such that $\tilde{h}_i(t) \rightarrow \tilde{g}_i$ as $t \rightarrow 1$. If g_i has one or two ends at σ , then there exists finitely many geodesics $h_{i,1}, \dots, h_{i,j} \in \lambda$ written with multiplicity such that $h_{i,l} \cap (S \setminus \sigma)$ for $l = 1, \dots, j$ is homotopic to g_i . Let $\tilde{h}_{i,l}(t)$ be the lift of $h_{i,l}$ under π_t for $l = 1, \dots, j$ such that $\tilde{h}_{i,l}(t)$ converges to \tilde{g}_i as $t \rightarrow 1$.

We can assume that $\tilde{h}_{i,1}(t), \tilde{h}_{i,2}(t), \dots, \tilde{h}_{i,j}(t)$ are given in order such that each $\tilde{h}_{i,l}$ is in between $\tilde{h}_{i,l-1}$ and $\tilde{h}_{i,l+1}$ for $l = 2, \dots, j-1$. Then $\tilde{h}_{i,l}$ and $\tilde{h}_{i,l+1}$ share a common endpoint.

Let $a, b, c, d \in \hat{\mathbf{R}}$ be the endpoints of a quadrilateral with the diagonal (b, d) . Then the shear \dot{s} of the diagonal (b, d) associated to a vector field V on $\hat{\mathbf{R}}$ is given by

$$\dot{s}((b, d)) = \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(a)}{d - a} - \frac{V(d) - V(c)}{d - c} - \frac{V(b) - V(a)}{b - a}.$$

We write $V(c, b)$ instead of $\frac{V(c)-V(b)}{c-b}$ for short and similar for the other expressions. Note that the expression $\dot{s}((b, d))$ is the sum of four terms with positive or negative sign in front. In fact, the sign in front can be determined as follows. Orient the boundary sides of the triangle T with vertices (a, b, d) such that T stays on the left of the boundary sides. If the terminal endpoint of a side t_1 of T coincides with the initial point of a side t_2 of T then we say that t_1 comes before t_2 , otherwise t_1 comes after t_2 . The quadruple (a, b, c, d) divides $\hat{\mathbf{R}}$ into four intervals. Each expression above corresponds to one of the four intervals. The sign in front of the expression is positive if the geodesic whose endpoints agree with the endpoints of the interval comes after the diagonal (b, d) , and it is negative otherwise.

Each geodesic of the sequence $\tilde{h}_{i,1}(t), \tilde{h}_{i,2}(t) \dots, \tilde{h}_{i,j}(t)$ intersects one lift $\tilde{\alpha}_t$ of some $\alpha \in \sigma$ under the covering map π_t . Thus $\tilde{\alpha}_t$ separates one endpoint of each geodesic of the sequence from $i \in \mathbf{H}$ when t is close enough to 1. The set of other endpoints is either separated from $i \in \mathbf{H}$ by another lift $\tilde{\alpha}'$ of a geodesic $\alpha' \in \sigma$, or all endpoints are equal. In both cases, the sequence of geodesics $\tilde{h}_{i,1}(t), \tilde{h}_{i,2}(t) \dots, \tilde{h}_{i,j}(t)$ converges to \tilde{g}_i as $t \rightarrow 1$. Each two consecutive geodesics $\tilde{h}_{i,l}(t), \tilde{h}_{i,l+1}(t)$ form a hyperbolic wedge and we make a hyperbolic triangle by adding a geodesic to this wedge whose endpoints are the endpoints of the two geodesics which are not common. The added geodesic is separated from $i \in \mathbf{H}$ by either $\tilde{\alpha}$ or $\tilde{\alpha}'$. These added geodesics are small as $t \rightarrow 1$ in the Euclidean metric on $\mathbf{H} \cup \mathbf{R}$. We add a triangle T_1 to the side of $\tilde{h}_{i,1}$ opposite $\tilde{h}_{i,2}$ and a triangle T_2 to the side of $\tilde{h}_{i,j}$ opposite $\tilde{h}_{i,j-1}$. The added triangles T_1 and T_2 have all their sides in $\tilde{\lambda}$. We obtained a sequence of adjacent triangles such that the union of each two adjacent triangles forms a quadrilateral whose diagonal is in the sequence $\tilde{h}_{i,1}(t), \tilde{h}_{i,2}(t) \dots, \tilde{h}_{i,j}(t)$. The added geodesics are small as $t \rightarrow 1$ and each of them appears as a boundary side of exactly two quadrilaterals which share a triangle.

We consider the sum $\sum_{l=1}^j \dot{s}_t(h_{i,l}(t))$, where $\dot{s}_t(\tilde{h}_{i,l}) = V_t(d_l, a_l) + V_t(c_l, b_l) - V_t(d_l, c_l) - V_t(b_l, a_l)$ as before and (a_l, b_l, c_l, d_l) are the vertices of the quadrilateral whose diagonal is $\tilde{h}_{i,l}(t)$. Each small added geodesic (a', b') is contained in exactly two adjacent quadrilaterals. The sign of $V(a', b')$ is determined with respect to the diagonal of the quadrilateral. Since two quadrilaterals have different diagonals it follows that the two terms for $V_t(a', b')$ have different signs in the sum and they cancel out.

We continue the analysis of the terms in the above sum. Note that $\tilde{h}_{i,1}(t)$ and $\tilde{h}_{i,j}(t)$ appear as boundary sides of exactly one quadrilateral and that $\tilde{h}_{i,l}(t)$ for $l = 2, \dots, j-1$ appear as boundary sides of exactly two quadrilaterals. Let $\tilde{h}_{i,l}(t) = (b_l(t), d_l(t)), \tilde{h}_{i,l+1}(t) = (b_{l+1}(t), d_{l+1}(t))$ be two adjacent geodesics in the above sequence which are boundary sides of the above triangle with one small side, where either $b_l(t) = b_{l+1}(t)$ and $d_l(t) \neq d_{l+1}(t)$, or $b_l(t) \neq b_{l+1}(t)$ and $d_l(t) = d_{l+1}(t)$. The triangle appears as one half of two quadrilaterals which implies that $V_t(b_l(t), d_l(t))$ and $V_t(b_{l+1}(t), d_{l+1}(t))$ have different signs in the above sum. It is clear that $|V_t(b_l(t), d_l(t)) - V_t(b_{l+1}(t), d_{l+1}(t))| \rightarrow 0$ as $t \rightarrow 1$ because $b_l(t)$ and $b_{l+1}(t)$, and $d_l(t)$ and $d_{l+1}(t)$ are close for t close to 1. It follows that the total sum of terms $V_t(b_l(t), d_l(t))$ for the sequence $\{\tilde{h}_{i,l}(t)\}_{l=1}^j$ converges to zero as $t \rightarrow 1$.

Let $a(t)$ be the third vertex of T_1 and let $c(t)$ be the third vertex of T_2 . Then $a(t)$ and $c(t)$ converge as $t \rightarrow 1$ to the vertices of the quadrilateral which is the

union of two adjacent triangles in $\mathbf{H} \setminus \pi^{-1}(\mu)$ and which has \tilde{g}_i as its diagonal. The above considerations imply that

$$\begin{aligned} \lim_{t \rightarrow 0} \sum_{l=1}^j \dot{s}_t(h_{i,l}) &= \lim_{t \rightarrow 0} \left[V_t(c(t), b_{i,j}(t)) + V_t(d_{i,1}(t), a(t)) - \right. \\ &\quad \left. V_t(b_{i,1}(t), a(t)) - V_t(d_{i,j}(t), c(t)) \right] = V(c, b) + \\ &\quad V(d, a) - V(b, a) - V(d, c) = \dot{s}^*(g_i) \end{aligned}$$

where $g_i = (b, d)$. The proposition follows because $\sum_{l=1}^j \dot{s}_t(h_{i,l}) = \sum_{j=1}^n k_{i,j} \dot{s}_t(h_j)$ by the definition of $k_{i,j}$. \square

Let u_t for $t \in [0, 1)$ be a path of tangent vectors to $T(S)$ that limits to a tangent vector u^* at a point $R \in \mathcal{S}(\sigma)$ as $t \rightarrow 1$. Let V_t, V^* be the Zygmund vector fields corresponding to the covering maps $\pi_t, \pi : \mathbf{H} \rightarrow R_t, R^*$ normalized as above. This implies that V_t converges pointwise to V^* on \mathbf{R} as $t \rightarrow 1$ and the supremum of the Zygmund norms of V_t is finite. Then $H(V_t)(x) \rightarrow H(V)(x)$ as $t \rightarrow 1$ uniformly on compact subsets of \mathbf{R} by Lemma 5.1.

Let \tilde{s}_t, \tilde{s}^* be the infinitesimal shear functions corresponding to the tangent vectors u_t, u^* . Then by Proposition 10.2 we have that

$$\lim_{t \rightarrow 0} \sum_{j=1}^l k_{i,j} H(\dot{s}_t)(h_{i,j}) = \dot{s}^*(g_i).$$

This combines with the results of Roger [27] to imply a topological version of a theorem of Masur:

Theorem 10.3. [18] *The Weil-Petersson metric on $T(S)$ extends by continuity to the closure $\bar{T}(S)$.*

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