FENCHEL-NIELSEN COORDINATES ON UPPER BOUNDED PANTS DECOMPOSITIONS

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ABSTRACT. Let X_0 be an infinite type hyperbolic surface (whose boundary components, if any, are closed geodesics or punctures) which has an upper bounded pants decomposition. The length spectrum Teichmüller space $T_{ls}(X_0)$ consists of all surfaces X homeomorphic to X_0 such that the ratios of the corresponding simple closed geodesics are uniformly bounded from below and from above. Alessandrini, Liu, Papadopoulos and Su [1] described the Fenchel-Nielsen coordinates for $T_{ls}(X_0)$ and using these coordinates they proved that $T_{ls}(X_0)$ is path connected. We use the Fenchel-Nielsen coordinates for $T_{ls}(X_0)$ to induce a locally biLipschitz homeomorphism between l^∞ and $T_{ls}(X_0)$ (which extends analogous results by Fletcher [9] and by Allessandrini, Liu, Papadopoulos, Su and Sun [2] for the unreduced and the reduced $T_{qc}(X_0)$). Consequently, $T_{ls}(X_0)$ is contractible. We also characterize the closure in the length spectrum metric of the quasiconformal Teichmüller space $T_{qc}(X_0)$ in $T_{ls}(X_0)$.

1. INTRODUCTION

Let X_0 be a complete hyperbolic surface of infinite type whose boundary components, if any, are closed geodesics or punctures. Assume that there exists a pants decomposition $\mathcal{P} = \{\alpha_n\}$ of X_0 by simple closed geodesics such that their lengths are bounded from the above by a fixed constant $M_0 > 0$. We say that such a pants decomposition is *upper bounded*. By [2], any other pants decomposition of X_0 by simple closed curves can be straightened to a pants decomposition by simple closed geodesics.

The *quasiconformal* Teichmüller space $T_{qc}(X_0)$ consists of all quasiconformal maps $f : X_0 \to X$ up to post composition by isometries and up to bounded homotopies which setwise fix boundary components of X. Note that bounded homotopies do not fix boundary geodesics pointwise since the distance between any two points (on a boundary geodesic) is finite. Thus $T_{qc}(X_0)$ is the reduced quasiconformal Teichmüller space of the surface X_0.

The *length spectrum* Teichmüller space $T_{ls}(X_0)$ consists of all homeomorphisms $h : X_0 \to X$ up to post compositions by isometries and up to bounded homotopies that setwise preserve the boundary components of X such that

$$L(X_0, X) := \sup_{\beta} \max\{\frac{l_\beta(X)}{l_\beta(X_0)}, \frac{l_\beta(X_0)}{l_\beta(X)}\} < \infty,$$

where the supremum is over all simple closed curve β on X_0, and where $l_\beta(X), l_\beta(X_0)$ are the lengths of the geodesic representatives of β on X, X_0, respectively. Note

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that $T_{qc}(X_0) \subset T_{ls}(X_0)$ because for each simple closed geodesic β on X_0 and a K-quasiconformal map $f : X_0 \to X$ we have (cf. Wolpert [13])

$$\frac{1}{K} l_\beta(X_0) \leq l_\beta(X) \leq K l_\beta(X_0).$$

The *length spectrum metric* d_{ls} is defined by

$$d_{ls}(X,Y) = \frac{1}{2}\log L(X,Y)$$

for $X, Y \in T_{ls}(X_0)$. Shiga [12] was the first to study the length spectrum metric on quasiconformal Teichmüller spaces of surfaces of infinite type and he proved that d_{ls} in general is not complete on $T_{qc}(X_0)$. This implies that $T_{qc}(X_0)$ could be a proper subset of $T_{ls}(X_0)$.

For a fixed upper bounded pants decomposition $\mathcal{P} = \{\alpha_n\}$ on X_0, the assignment of Fenchel-Nielsen coordinates $\{(l_{\alpha_n}(X), t_{\alpha_n}(X))\}$ to each $X \in T_{qc}(X_0)$ or $X \in T_{ls}(X_0)$ completely determines the (marked) surface X. Alessandrini, Liu, Papadopoulus, Su and Sun [2] proved that the Fenchel-Nielsen coordinates for $T_{qc}(X_0)$ satisfy $\sup_n |\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}| < \infty$ and $\sup_n |t_{\alpha_n}(X) - t_{\alpha_n}(X_0)| < \infty$, and that the map from $T_{qc}(X_0)$ to the Fenchel-Nielsen coordinates is a locally biLipschitz homeomorphism onto l^∞. Alessandrini, Liu, Papadopuolus and Su [1] proved that the Fenchel-Nielsen coordinates for $X \in T_{ls}(X_0)$ satisfy

$$\sup_{\alpha_n \in \mathcal{P}} |\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}| < \infty$$

and

$$\sup_{\alpha_n \in \mathcal{P}} \frac{|t_{\alpha_n}(X) - t_{\alpha_n}(X_0)|}{\max\{1, |\log l_{\alpha_n}(X_0)|\}} < \infty.$$

Moreover, they proved that if X_0 (equipped with an upper bounded pants decomposition) contains a sequence of simple closed geodesics whose lengths go to 0 then $T_{qc}(X_0) \subsetneq T_{ls}(X_0)$ and that $T_{qc}(X_0)$ is nowhere dense in $T_{ls}(X_0)$ (cf. [1]).

Let $F : T_{ls}(X_0) \to l^\infty$, called the *normalized Fenchel-Nielsen map*, be defined by

$$F(X) = \Big\{ (\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}, \frac{t_{\alpha_n}(X) - t_{\alpha_n}(X_0)}{\max\{1, |\log l_{\alpha_n}(X_0)|\}}) \Big\}_n$$

where if α_n is a boundary geodesic of X_0 then we take only the length component of the Fenchel-Nielsen coordinates.

By [2], the Fenchel-Nielsen coordinates give a locally biLipschitz homeomorphism between the (reduced) quasiconformal Teichmüller space $T_{qc}(X_0)$ and l^∞ which implies that $T_{qc}(X_0)$ is contractible. Fletcher [9] used complex analytic methods to prove that the unreduced quasiconformal Teichmüller space is locally biLipschitz to l^∞. Our main result is (cf. §2, Theorem 2.1 and Corollary 2.2):

Theorem 1. *The normalized Fenchel-Nielsen map*

$$F : T_{ls}(X_0) \to l^\infty$$

is a locally biLipschitz homeomorphism.

In particular, the length spectrum Teichmüller space $T_{ls}(X_0)$ *is contractible.*

Thus [9], [2] and Theorem 1 imply that the unreduced quasiconformal Teichmüller space, the reduced quasiconformal Teichmüller space and the length spectrum Teichmüller space are locally biLipschitz to l^∞ and thus to each other.

A problem of characterizing the closure of $T_{qc}(X_0)$ inside $T_{ls}(X_0)$ for the length spectrum metric d_{ls} is raised in [1]. We use the Fenchel-Nielsen coordinates to characterize the closure $\overline{T_{qc}(X_0)}$ of $T_{qc}(X_0)$.

Theorem 2. *Let X_0 be an infinite type hyperbolic surface with an upper bounded pants decomposition $\mathcal{P} = \{\alpha_n\}$. Then $X \in \overline{T_{qc}(X_0)}$ if and only if*

$$\sup_{\alpha_n \in \mathcal{P}} |\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}| < \infty$$

and

$$|t_{\alpha_n}(X) - t_{\alpha_n}(X_0)| = o(|\log l_{\alpha_n}(X_0)|)$$

as $|\log l_{\alpha_n}(X_0)| \to \infty$.

2. The Fenchel-Nielsen coordinates

We prove that the normalized Fenchel-Nilesen map is a localy biLipschitz homeomorphism onto l^∞.

Theorem 2.1. *Let X_0 be an infinite type complete hyperbolic surface equipped with an upper bounded geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}_{n\in\mathbb{N}}$. The normalized Fenchel-Nielsen map*

$$F(X) = \Big\{\Big(\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}, \frac{t_{\alpha_n}(X) - t_{\alpha_n}(X_0)}{\max\{1, |\log l_{\alpha_n}(X_0)|\}}\Big)\Big\}_{n\in\mathbb{N}} \tag{1}$$

for $X \in T_{ls}(X_0)$, induces a locally biLipschitz surjective homeomorphism

$$F : T_{ls}(X_0) \to l^\infty.$$

Proof. Let M_0 be such that $l_{\alpha_n}(X_0) \le M_0$ for each $\alpha_n \in \mathcal{P}$.

Step I: We establish that $F(T_{ls}(X_0)) \subset l^\infty$ which is already proved in [1]. We give another proof in order to facilitate the rest of the argument. By the definition, $X \in T_{ls}(X_0)$ if there is $M > 0$ such that

$$|\log \frac{l_\gamma(X)}{l_\gamma(X_0)}| \le M$$

for each simple closed curve $\gamma \in \mathcal{C}$ on X_0. In particular $\{\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}\}_{n\in\mathbb{N}}$ is a bounded sequence.

It remains to bound the twists. The choice of the twists of X_0 on α_n are determined up to integer multiples of $l_{\alpha_n}(X_0)$. Without loss of generality, we normalize them such that, for each $n \in \mathbb{N}$,

$$0 \le t_{\alpha_n}(X_0) < l_{\alpha_n}(X_0).$$

Given this normalization, it is enough to prove that

$$|t_{\alpha_n}(X)| / \max\{1, |\log l_{\alpha_n}(X_0)|\}$$

is bounded uniformly in $n \in \mathbb{N}$.

Using [3], there exists a surface X' which is K-quasiconformal to X_0 such that $l_{\alpha_n}(X') = l_{\alpha_n}(X)$ for all $n \in \mathbb{N}$, where $K = K(M)$ (cf. [1]). The K-quasiconformal map $f : X_0 \to X'$ maps each pair of pants $P \in \mathcal{P}$ of X_0 onto a geodesic pair of pants of X' such that on each boundary geodesic the map is affine. Divide each geodesic pair of pants into two right angled hexagons by three *seams*, namely three geodesic arcs connecting pairs of boundary curves and orthogonal to them. Each hexagon contains half of each boundary geodesic of the pair of pants which are called *a-sides*. The other three sides of the hexagon which are seams are called *b-sides*. The two hexagons are glued along their b-sides to obtain the pair of pants and the pairwise union of their a-sides forms three geodesic boundaries of the pair of pants. The map $f: X_0 \to X'$ maps a-sides of each hexagon of each pair of pants of X_0 onto a-sides of hexagons of the corresponding pair of pants of X and it is affine on the a-sides. Note that a single $\alpha_n \in \mathcal{P}$ is on the boundary of two pairs of pants P_1^0 and P_2^0 of $\mathcal{P}$ which implies that α_n is divided into a-sides with respect to both P_1^0 and P_2^0. The two divisions of α_n into a-sides do not match in general and the distance between the endpoints of two a-sides coming from two pairs of pants is the twist parameter $t_{\alpha_n}(X_0)$ of X_0 at the closed geodesic α_n for the pants decomposition $\mathcal{P}$. The map $f: X_0 \to X'$ is affine on each α_n, it maps the foots of the seams of the pair of pants P_i^0 to the foots of the corresponding seams of $f(P_i^0) = P_i'$ for $i = 1, 2$ and it does not introduce any full twisting along α_n by its construction (cf. [1], [3]). Then for $K = K(M)$ we have

$$t_{\alpha_n}(X') = \frac{l_{\alpha_n}(X')}{l_{\alpha_n}(X_0)} t_{\alpha_n}(X_0) \leq K t_{\alpha_n}(X_0).$$

Let $t_n = t_{\alpha_n}(X) - t_{\alpha_n}(X')$. Then X is obtained by a(n infinite) multi twist on X' along the family $\mathcal{P} = \{\alpha_n\}$ by the amount $\{t_n\}$. It is enough to prove that $\frac{|t_n|}{\max\{1, |\log l_{\alpha_n}(X_0)\}}$ is bounded in terms of $d_{ls}(X_0, X)$ because $|t_{\alpha_n}(X')| \leq KM_0$. This is proved in [1] using results of Minsky [11] and Choi-Rafi [6]. We give a more direct proof of this result below. Fix a cuff α_n and let $P_1' = f(P_1^0)$ and $P_2' = f(P_2^0)$ be two geodesic pairs of pants with common boundary α_n. Either $P_1' \neq P_2'$ or $P_1' = P_2'$ and we divide the argument into these two cases.

Case 1. Assume that $P_1' \neq P_2'$. There exists a unique geodesic arc $\gamma_n^i \subset P_i'$, for $i = 1, 2$, which starts and ends at α_n that is orthogonal to α_n at both of its endpoints. Let β_n be a closed curve on X' obtained by concatenating γ_n^1 followed by an arc of α_n from an endpoint of γ_n^1 to an endpoint of γ_n^2 in the direction of the left twist along α_n followed by γ_n^2 followed by an arc of α_n connecting other two endpoints of γ_n^1 and γ_n^2 in the direction of the right twist (cf. Figure 1).

We will give an upper bound for $t_{\alpha_n}(X)$ in terms of $l_{\beta_n}(X)$. Fix three consecutive lifts $\tilde{\alpha}_n^j$, for $j = 1, 2, 3$, of α_n under the universal covering $\pi: \mathbb{H}^2 \to X$. Let $\tilde{\beta}_n^*$ be the lift of the geodesic representative β_n^* of β_n that intersects $\tilde{\alpha}_n^j$, for $j = 1, 2, 3$. Moreover, let $\tilde{\gamma}_n^i$ be the lift of γ_n^i that connects $\tilde{\alpha}_n^i$ and $\tilde{\alpha}_n^{i+1}$ (cf. Figure 2). Let $a_1 = \tilde{\gamma}_n^1 \cap \tilde{\alpha}_n^2$ and $a_2 = \tilde{\gamma}_n^2 \cap \tilde{\alpha}_n^2$, and $b = \tilde{\beta}_n^* \cap \tilde{\alpha}_n^2$. The lengths satisfy $l_{\gamma_n^i}(X') = l_{\gamma_n^i}(X)$ because X is obtained from X' by a multi twist along $\{\alpha_n\}$. We either have $d_{hyp}(a_1, b) \geq |t_{\alpha_n}(X)|/2$ or $d_{hyp}(a_2, b) \geq |t_{\alpha_n}(X)|/2$. Consider the case that

$$d_{hyp}(a_2, b) \geq |t_{\alpha_n}(X)|/2 \tag{2}$$

and the other case is analogous. Let $c_2 = \tilde{\gamma}_n^2 \cap \tilde{\alpha}_n^3$ and let $c_1 \in \tilde{\alpha}_n^3$ be the foot of the orthogonal from b to $\tilde{\alpha}_n^3$. Consider the quadrilateral with vertices b, c_1, c_2 and a_2 (cf. Figure 2). We get

$$\sinh d_{hyp}(b, c_1) = \sinh l_{\tilde{\gamma}_n^2}(X) \cosh d_{hyp}(b, a_2). \tag{3}$$

By the Collar lemma [5], there exists $C_1(M) > 0$ such that

$$l_{\tilde{\gamma}_n^i}(X) = l_{\tilde{\gamma}_n^i}(X') \geq C_1(M) \max\{1, |\log l_{\alpha_n}(X_0)|\}. \tag{4}$$

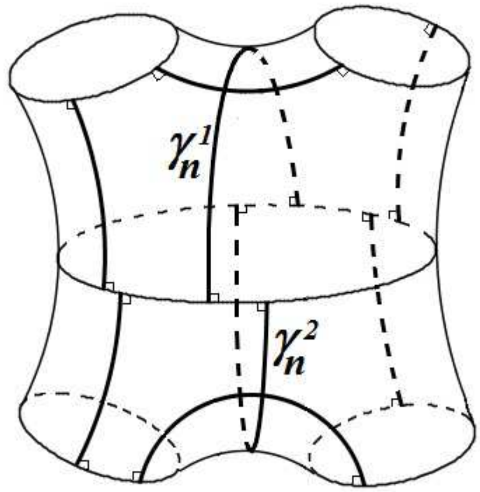

FIGURE 1. The curve β_n.

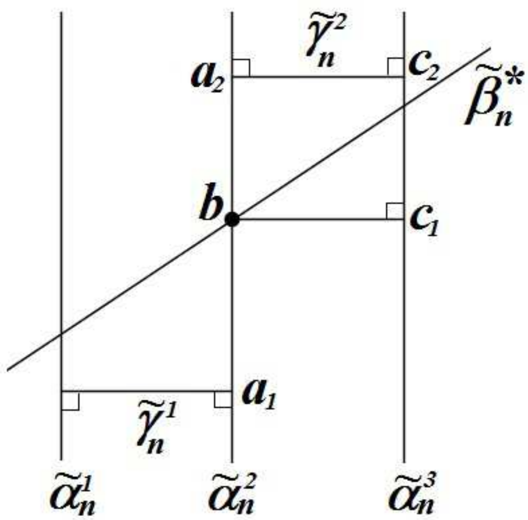

FIGURE 2.

Since $\sinh d_{hyp}(b, c_1) \leq \frac{e^{d_{hyp}(b,c_1)}}{2}$ and $\cosh d_{hyp}(b, a_2) \geq \frac{e^{d_{hyp}(b,a_2)}}{2}$, and by (3) and (4), we have

$$d_{hyp}(b, a_2) \leq C_2(M) \max\{1, |\log l_{\alpha_n}(X_0)|\} + d_{hyp}(b, c_1)$$

which implies

$$(5)\qquad \begin{aligned} d_{hyp}(b,a_2) &\leq C_2(M)\max\{1,|\log l_{\alpha_n}(X_0)|\} + l_{\beta_n}(X) \leq \\ &\leq C_2(M)\max\{1,|\log l_{\alpha_n}(X_0)|\} + e^M l_{\beta_n}(X_0). \end{aligned}$$

Note that by construction

$$l_{\beta_n}(X_0) \leq l_{\gamma_n^1}(X_0) + l_{\gamma_n^2}(X_0) + l_{\alpha_n}(X_0).$$

We estimate $l_{\gamma_n^i}(X_0)$ from the above using right-angled pentagons. Namely each hexagon of P_i^0 contains a half of the arc γ_n^i and γ_n^i intersects the b-side of the hexagons that connects the two boundary geodesics of P_i^0 different from α_n. Then both hexagons of P_i^0 are divided into two right-angled pentagons by γ_n^i. The sides of the obtained pentagons are as follows in the cyclic order: a portion of the a-side on α_n, followed by the half of γ_n^i, followed by a portion of a b-side, followed by an a-side on a boundary curve different from α_n and followed by a b-side. We choose one of the two pentagons such that the portion of the a-side has length at least $\frac{1}{4}l_{\alpha_n}(X_0)$. Since $l_{\alpha_n}(X_0) \leq M_0$ it follows that any a-side of the hexagon has length at most $\frac{1}{2}M_0$ and a hyperbolic formula for the right-angled pentagons gives

$$\cosh\frac{M_0}{2} \geq \sinh\frac{1}{4}l_{\alpha_n}(X_0)\sinh\frac{1}{2}l_{\gamma_n^i}(X_0) \geq \frac{1}{4}l_{\alpha_n}(X_0)\sinh\frac{1}{2}l_{\gamma_n^i}(X_0).$$

The inequality implies that there is $C_3(M_0)$ such that

$$l_{\gamma_n^i}(X_0) \leq C_3(M_0)\max\{1,|\log l_{\alpha_n}(X_0)|\}$$

which in turn implies

$$(6)\qquad l_{\beta_n}(X_0) \leq C_4(M_0)\max\{1,|\log l_{\alpha_n}(X_0)|\}$$

for some constant $C_4(M_0) > 0$.

By (2), (5) and (6) we have

$$(7)\qquad \frac{|t_{\alpha_n}(X)|}{2} \leq d_{hyp}(a_2,b) \leq [C_2(M) + e^M C_4(M_0)]max\{1,|\log l_{\alpha_n}(X_0)|\}$$

which gives

$$(8)\qquad \frac{|t_{\alpha_n}(X)|}{\max\{1,|\log l_{\alpha_n}(X_0)|\}} \leq C_5(M,M_0)$$

and this finishes the proof of $F(T_{ls}(X_0)) \subset l^\infty$ in the case $P_1^0 \neq P_2^0$.

Case 2. Assume that $P_1^0 = P_2^0$. We define γ_n to be the unique geodesic arc in $f(P_1^0) = f(P_2^0) = P_1'$ starting and ending at α_n and orthogonal to α_n at both of its endpoints. Then we define a closed curve β_n to consists of γ_n and an arc of α_n of the size $t_{\alpha_n}(X')$. The above argument applies to this case as well.

Step II: $F : T_{ls}(X_0) \to l^\infty$ **is surjective.** For $a \in l^\infty$ the surface X_a obtained by gluing the pants with prescribed cuffs and twists is complete. The marking map for X_a can be chosen to be a homeomorphism because each twist is realized in an annulus containing a given cuff. Let $X_a' = X_{a'}$ be the surface obtained by a K-quasiconformal map $f : X_0 \to X_a'$ such that $l_{\alpha_n}(X_a') = l_{\alpha_n}(X_a)$ for all $n \in \mathbb{N}$ as before (cf. [3]). Then

$$\Big|\log\frac{l_\beta(X_a')}{l_\beta(X_0)}\Big| \leq M(K) < \infty$$

for $n \in \mathbb{N}$. The surface X_a is obtained by a multi twist around $\mathcal{P} = \{\alpha_n\}$ by the amount $\{t_n = t_{\alpha_n}(X_a) - t_{\alpha_n}(X'_a)\}$. Note that $0 \leq t_{\alpha_n}(X_{a'}) = \frac{l_{\alpha_n}(X_a)}{l_{\alpha_n}(X_0)} t_{\alpha_n}(X_0) < l_{\alpha_n}(X_a)$. By the Collar lemma [5], we have that

$$l_\beta(X_{a'}), l_\beta(X_a) \geq C \sum_n i(\alpha_n, \beta) \max\{1, |\log l_{\alpha_n}(X_0)|\} \tag{9}$$

for each simple closed curve β on X_0. Then

$$\log \frac{l_\beta(X_a)}{l_\beta(X_{a'})} \leq \log \frac{l_\beta(X_{a'}) + \sum_n i(\alpha_n, \beta)|t_n|}{l_\beta(X_{a'})}$$

and

$$|t_n| \leq C' \max\{1, |\log l_{\alpha_n}(X_0)|\}$$

which together with (9) implies that

$$\log \frac{l_\beta(X_a)}{l_\beta(X_{a'})} \leq \frac{C'}{C} = C''$$

for each $\beta \in \mathcal{C}$. In the exactly the same fashion, we obtain

$$\log \frac{l_\beta(X_{a'})}{l_\beta(X_a)} \leq C'''$$

for each $\beta \in \mathcal{C}$. Thus $X_a \in T_{ls}(X_0)$ and $F : T_{ls}(X_0) \to l^\infty$ is onto.

Step III: $F : T_{ls}(X_0) \to l^\infty$ **is localy Lipschitz.** Let $X_1 \in T_{ls}(X_0)$ be fixed and let $X, Y \in B_{\frac{1}{2}}(X_1)$ be two arbitrary points in the ball of radius $\frac{1}{2}$ centered at $X_1 \in T_{ls}(X_0)$. Consequently $d_{ls}(X,Y) < 1$. Note that $|\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)} - \log \frac{l_{\alpha_n}(Y)}{l_{\alpha_n}(X_0)}| = |\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(Y)}| \leq d_{ls}(X,Y)$ for each $n \in \mathbb{N}$. It remains to consider $\frac{|t_{\alpha_n}(X) - t_{\alpha_n}(Y)|}{\max\{1, |\log l_{\alpha_n}(X_0)|\}}$. By [1] and [3], there exists a $[1 + C d_{ls}(X,Y)]$-quasiconformal map $f : X \to X'$ such that $l_{\alpha_n}(X') = l_{\alpha_n}(Y)$ for each $n \in \mathbb{N}$ with $C = C(e^{d_{ls}(X_0,X_1)+1} M_0) > 0$. Let $0 \leq \tilde{t}_{\alpha_n}(X) < l_{\alpha_n}(X)$ be such that there exists an integer $k \in \mathbb{Z}$ with

$$t_{\alpha_n}(X) = k \cdot l_{\alpha_n}(X) + \tilde{t}_{\alpha_n}(X). \tag{10}$$

Note that, for $C' = C'(M_1 + 1, M_0)$, we have

$$|k| \leq C' \frac{|\log l_{\alpha_n}(X_0)|}{l_{\alpha_n}(X_0)} \tag{11}$$

by (8) and (10) because normalized twists $\tilde{t}_{\alpha_n}$ are bounded from the above and

$$e^{-(M_1+1)} l_{\alpha_n}(X_0) \leq l_{\alpha_n}(X) \leq e^{M_1+1} l_{\alpha_n}(X_0).$$

The construction of $f : X \to X'$ from [1] implies that

$$t_{\alpha_n}(X') = k \cdot l_{\alpha_n}(Y) + \frac{l_{\alpha_n}(Y)}{l_{\alpha_n}(X)} \tilde{t}_{\alpha_n}(X).$$

We estimate $|t_{\alpha_n}(X') - t_{\alpha_n}(X)|$. Let $M_1 = d_{ls}(X_0, X_1)$. Then we have

$$\begin{aligned} |t_{\alpha_n}(X') - t_{\alpha_n}(X)| &\leq |k| \cdot |l_{\alpha_n}(Y) - l_{\alpha_n}(X)| + \tilde{t}_{\alpha_n}(X) \Big| \frac{l_{\alpha_n}(Y)}{l_{\alpha_n}(X)} - 1 \Big| \\ &\leq C' \frac{\max\{1, |\log l_{\alpha_n}(X_0)|\}}{l_{\alpha_n}(X_0)} l_{\alpha_n}(X) \Big| \frac{l_{\alpha_n}(Y)}{l_{\alpha_n}(X)} - 1 \Big| + \tilde{t}_{\alpha_n}(X) \Big| \frac{l_{\alpha_n}(Y)}{l_{\alpha_n}(X)} - 1 \Big| \end{aligned} \tag{12}$$

which implies

$$\text{(13)}\quad \frac{|t_{\alpha_n}(X)-t_{\alpha_n}(X')|}{\max\{1,|\log l_{\alpha_n}(X_0)|\}}\leq (C'e^M+M_0e^M)\Big|\frac{l_{\alpha_n}(Y)}{l_{\alpha_n}(X)}-1\Big|\leq C_1\Big|\log\frac{l_{\alpha_n}(Y)}{l_{\alpha_n}(X)}\Big|.$$

Note that $d_{ls}(X,X')\leq Cd_{ls}(X,Y)$ which implies that

$$d_{ls}(X',Y)\leq d_{ls}(X',X)+d_{ls}(X,Y)\leq C_2d_{ls}(X,Y).$$

Let

$$\text{(14)}\qquad t_n=t_{\alpha_n}(Y)-t_{\alpha_n}(X').$$

The surface Y is obtained from X' by a multi twist along $\{\alpha_n\}$ by the amount $\{t_n\}$. As before, we divide the argument into two cases: $P_1'=P_2'$ and $P_1'\neq P_2'$.

Case 1. Given α_n, assume first that the pairs of pants with boundary α_n are equal, namely $P_1'=P_2'$. Let β_n be a closed curve obtained by concatenating the unique arc γ_n in P_1' orthogonal to α_n at both of its endpoints followed by an arc on α_n of the length $\tilde{t}_{\alpha_n}(X')$. Let β_n^* be the geodesic representative of β_n.

Denote by X_t' the hyperbolic surface obtained by twisting the amount $t\cdot t_n$, for $t\in\mathbb{R}$ and t_n defined by (14), along the cuffs α_n on the surface X'. Note that $X_0'=X'$ and that by the definition of t_n we have that $X_1'=Y$.

Recall that (cf. [10], [7])

$$\text{(15)}\qquad \frac{d}{d(t\cdot t_n)}l_{\beta_n^*}(X_t')=\cos\varphi_t^*$$

where $\varphi_t^*\in(0,\pi)$ is the angle between $\tilde{\beta}_n^*$ and $\tilde{\alpha}_n$. Let us fix $\epsilon_0>0$. Note that φ_t^* is either increasing or decreasing from φ_0^* to φ_1^* in t for $0\leq t\leq 1$ (depending whether t_n is positive or negative) due to the fact that the geodesic length along a left earthquake with support α_n is a convex function (cf. [10], [7]).

Assume that $t_n>0$. If $\cos\varphi_0^*\geq\epsilon_0$ (which implies $\cos\varphi_t^*\geq\epsilon_0$ for $0\leq t$) then we set $\beta_n^{**}:=\beta_n^*$. If $\cos\varphi_0^*<\epsilon_0$ then we choose β_n^{**} such that $\cos\varphi_t^{**}>\epsilon_0$ as follows.

Consider universal covering $\pi:\mathbb{H}^2\to X'$ such that one lift $\tilde{\alpha}_n$ of α_n is the y-axis. Further we arrange that two lifts $\tilde{\gamma}_n^{-1}$ and $\tilde{\gamma}_n^1$ of the arc γ_n that are adjacent to the y-axis from the left and the from the right meet the y-axis between i and $e^{l_{\alpha_n}(X')}i$. Let $b<0$ be an endpoint on $\mathbb{R}$ of the hyperbolic geodesic containing $\tilde{\gamma}_n^{-1}$ and let $a>0$ be an endpoint on $\mathbb{R}$ of the geodesic containing $\tilde{\gamma}_n^1$. For any $k\in\mathbb{Z}$, a k full left twists on α_n on the surface X' maps the curve β_n^* to a new curve β_n^{**}. The curve obtained by the concatenating the arc γ_n with the arc which winds around α_n k-times plus the shear amount $\tilde{t}_{\alpha_n}(X')$ is homotopic to β_n^{**}. The lift of the above arc has two orthogonal sub arcs to the y-axis one from the left which is equal to $\tilde{\gamma}_n^{-1}$ which meets y axis at a point $|b|i$ between i and $e^{l_{\alpha_n}(X')}i$, and the other orthogonal arc $\tilde{\gamma}_n^2$ which meets the y-axis at a point $c_2=|a|e^{kl_{\alpha_n}(X')}i$. By the definition of left twists, it follows that one endpoint of a lift $\tilde{\beta}_n^{**}$ of β_n^{**} is between b and 0, and the other endpoint of $\tilde{\beta}_n^{**}$ is between $ae^{kl_{\alpha_n}(X')}$ and ∞. Among all the geodesics whose one endpoint is in the interval $[b,0)$ and the other endpoint is in the interval $[ae^{kl_{\alpha_n}(X')},\infty)$, the geodesic with endpoints b and $ae^{kl_{\alpha_n}(X')}$ subtends the largest angle φ_0 with the y-axis. We have

$$\cos\varphi_0=\frac{ae^{kl_{\alpha_n}(X')}+b}{ae^{kl_{\alpha_n}(X')}-b}.$$

Define

$$k = [\frac{1}{l_{\alpha_n}(X')} \log \frac{1+\epsilon_0}{1-\epsilon_0}] + 2$$

where $[x]$ is the integer part of $x \in \mathbb{R}$. Then we have that

$$\cos \varphi_0 \geq \epsilon_0$$

which implies that

$$\frac{1}{t_n} \frac{d}{dt} l_{\beta_n^{**}}(X_t') = \frac{d}{d(t \cdot t_n)} l_{\beta_n^{**}}(X_t') \geq \epsilon_0$$

for all $t \in [0,1]$.

Note that

$$l_{\beta_n^{**}}(X') \leq k l_{\alpha_n}(X') + C|\log l_{\alpha_n}(X')| \leq C' \max\{1, |\log l_{\alpha_n}(X_0)|\}$$

By the Mean Value Theorem there exists $t^* \in (0,1)$ such that

$$|l_{\beta_n^{**}}(Y) - l_{\beta_n^{**}}(X')| = |\frac{d}{dt} l_{\beta_n^{**}}(X_{t^*}')| \geq \epsilon_0 t_n$$

because $X_1' = Y$. Since $l_{\beta_n^{**}}(X') \leq C' \max\{1, |\log l_{\alpha_n}(X_0)|\}$, the above gives

$$\frac{|t_n|}{\max\{1, |\log l_{\alpha_n}(X_0)|\}} \leq \frac{|t_n|}{l_{\beta_n^{**}}(X')} \leq \frac{C}{\epsilon_0} |\frac{l_{\beta_n^{**}}(Y)}{l_{\beta_n^{**}}(X')} - 1| \leq \frac{C}{\epsilon_0} |\log \frac{l_{\beta_n^{**}}(Y)}{l_{\beta_n^{**}}(X')}|.$$

Assume now that $t_n < 0$. Then we use a similar method by considering $\cos \varphi_t^* \leq -\epsilon_0$ and k full right twist around α_n to replace β_n^* with β_n^{**}. The proof proceeds analogously.

Case 2. The second case is when $P_1' \neq P_2'$. Define a closed curve $\beta_n \subset P_1' \cup P_2' \subset X'$ to consists of the unique arc γ_n^1 in P_1' orthogonal at both of its endpoints to α_n followed by the arc in α_n (in the direction of the left twist) of the size at most $l_{\alpha_n}(X')$ followed by the unique arc $\gamma_n^2 \subset P_2'$ orthogonal to α_n at both of its endpoints followed by an arc on α_n of size at most $l_{\alpha_n}(X')$. For the convenience of the notation, denote by β_n the closed geodesic homotopic to β_n. The arcs γ_n^i, for $i = 1, 2$, have lengths comparable to $\max\{1, |\log l_{\alpha_n}(X_0)|\}$ up to positive multiplicative constants.

Let $\tilde{\alpha}_n^j$, for $j = 1, 2$, be two consecutive lifts of α_n. Two lifts $\tilde{\gamma}_n^{j,k}$, for $k = 1, 2$, of γ_n^j which meet $\tilde{\alpha}_n^j$ can be chosen such that the distance between their foots on $\tilde{\alpha}_n^j$ is at most $l_{\alpha_n}(X')$. Assume that $t_n > 0$. We perform k full left twists along α_n to obtain a new closed curve β_n^{**} from the closed curve β_n^*. When $k = [\frac{1}{l_{\alpha_n}(X')} \log \frac{1+\epsilon_0}{1-\epsilon_0}] + 2$, we get (similar to Case 1) for both angles φ_n^j that the new closed geodesic β_n^{**} subtends with α_n,

$$\cos \varphi_n^j \geq \epsilon_0.$$

Then

$$\frac{d}{dt} l_{\beta_n^{**}}(X_t') = \cos \varphi_n^1 + \cos \varphi_n^2 \geq 2\epsilon_0$$

which gives

$$\frac{|t_n|}{\max\{1, |\log l_{\alpha_n}(X_0)|\}} \leq C |\log \frac{l_{\beta_n^{**}}(Y)}{l_{\beta_n^{**}}(X')}|.$$

When $t_n < 0$, the proof proceeds as before.

Thus we established that the map $F : T_{ls}(X_0) \to l^\infty$ is locally Lipschitz.

Step IV: $F^{-1}: l^\infty \to T_{ls}(X_0)$ is locally Lipschitz. We consider the map $F^{-1}: l^\infty \to T_{ls}(X_0)$ and prove that it is also locally Lipschitz. Let $a_0 \in l^\infty$ be fixed. Denote by X_{a_0} the surface corresponding to a_0, namely $X_{a_0} = F^{-1}(a_0) \in T_{ls}(X_0)$. Let $a, b \in l^\infty$ such that $\|a - a_0\|_\infty < \frac{1}{2}$ and $\|b - a_0\|_\infty < \frac{1}{2}$ which implies $\|a-b\|_\infty < 1$. There exists a $(1+C|\log \frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)}|)$-quasiconformal map $f: X_b \to X_b'$ such that $l_{\alpha_n}(X_b') = l_{\alpha_n}(X_a)$ for all n (cf. [3], [1]). Recall that

$$t_{\alpha_n}(X_b) = kl_{\alpha_n}(X_b) + \tilde{t}_{\alpha_n}(X_b)$$

where $k \in \mathbb{Z}$, $0 \leq \tilde{t}_\alpha(X_b) < l_{\alpha_n}(X_b)$ and

$$|k| \leq \frac{C\max\{1, |\log l_{\alpha_n}(X_0)|\}}{l_{\alpha_n}(X_b)}. \tag{16}$$

By the construction of $f: X_b \to X_b'$, we have

$$t_{\alpha_n}(X_b') = kl_{\alpha_n}(X_a) + \frac{l_{\alpha_n}(X_a)}{l_{\alpha_n}(X_b)}\tilde{t}_{\alpha_n}(X_b).$$

It follows that

$$|t_{\alpha_n}(X_b) - t_{\alpha_n}(X_b')| \leq |k| l_{\alpha_n}(X_a)\Big|\frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)} - 1\Big| + l_{\alpha_n}(X_b)\Big|\frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)} - 1\Big|. \tag{17}$$

Since $a, b \in l^\infty$, it follows that there exists $C > 0$ such that

$$\Big|\frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)} - 1\Big| \leq C\Big|\log\frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)}\Big|. \tag{18}$$

The inequalities (17), (16) and (18) imply

$$|t_{\alpha_n}(X_b) - t_{\alpha_n}(X_b')| \leq C\max\{1, |\log l_{\alpha_n}(X_0)|\}\frac{l_{\alpha_n}(X_a)}{l_{\alpha_n}(X_b)}|\log\frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)}| + C'|\log\frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)}|$$

where $C' = C'(\|a_0\|_\infty + \frac{1}{2})$, and since $|\log\frac{l_{\alpha_n}(X_b)}{l_{\alpha_n}(X_a)}| \leq \|a - b\|_\infty$, we get

$$\frac{|t_{\alpha_n}(X_b) - t_{\alpha_n}(X_b')|}{\max\{1, |\log l_{\alpha_n}(X_0)|\}} \leq C''\|a - b\|_\infty. \tag{19}$$

Since $f: X_b \to X_b'$ is a $(1+C\|a-b\|_\infty)$-quasiconformal, it follows that $d_{ls}(X_b, X_b') \leq C\|a - b\|_\infty$. Moreover, if $X_b' = F^{-1}(b')$ then (19) implies that that $\|b' - b\|_\infty \leq C\|a - b\|_\infty$. Finally, $\|a - b'\|_\infty \leq \|a - b\|_\infty + \|b - b'\|_\infty \leq C\|a - b\|_\infty$.

It remains to estimate the length-spectrum distance between $X_b' = X_{b'}$ and X_a. This part of the argumentt is essentially contained in [1]. Note that X_a is obtained from $X_{b'}$ by multi twist along α_n by the amount $t_n' = t_{\alpha_n}(X_a) - t_{\alpha_n}(X_{b'})$. The estimate (19) and the triangle inequality $\|t_{\alpha_n}(X_a) - t_{\alpha_n}(X_{b'})\|_\infty \leq \|t_{\alpha_n}(X_a) - t_{\alpha_n}(X_b)\|_\infty + \|t_{\alpha_n}(X_b) - t_{\alpha_n}(X_{b'})\|_\infty$ gives that

$$|t_n'| = |t_{\alpha_n}(X_a) - t_{\alpha_n}(X_{b'})| \leq C\|a - b\|_\infty \max\{1, |\log l_{\alpha_n}(X_0)|\}.$$

For any simple closed geodesic β on $X_{b'}$, we estimate $|\log\frac{l_\beta(X_{b'})}{l_\beta(X_a)}|$. We have

$$l_\beta(X_{b'}) \leq l_\beta(X_a) + \sum_{n=1}^\infty i(\beta, \alpha_n)|t_n'| \leq l_\beta(X_a) +$$

$$+ C\|a - b\|_\infty \sum_{n=1}^\infty i(\beta, \alpha_n)\max\{1, |\log l_{\alpha_n}(X_0)|\}$$

and

$$l_\beta(X_a) \geq C' \sum_{n=1}^{\infty} i(\beta, \alpha_n) \max\{1, |\log l_{\alpha_n}(X_a)|\}$$

by the Collar lemma. Since $X_a \in T_{ls}(X_0)$, it follows that there exists $M > 0$ such that $|\log l_{\alpha_n}(X_a)| \geq |\log l_{\alpha_n}(X_0)| - M$. Thus there exists $C'' > 0$ such that

$$\max\{1, |\log l_{\alpha_n}(X_a)|\} \geq C'' \max\{1, |\log l_{\alpha_n}(X_0)|\}.$$

The above inequalities imply that

$$\frac{l_\beta(X_{b'})}{l_\beta(X_a)} \leq 1 + C''' \|a - b\|_\infty$$

and by reversing roles played by X_a and $X_{b'}$ we get

$$\frac{l_\beta(X_a)}{l_\beta(X_{b'})} \leq 1 + C''' \|a - b\|_\infty.$$

This proves that $F^{-1} : l^\infty \to T_{ls}(X_0)$ is Lipschitz. □

Since l^∞ is contractible, we get

Corollary 2.2. *The length spectrum Teichmüller space $T_{ls}(X_0)$ for any hyperbolic surface X_0 with an upper bounded pants decomposition is contractible.*

3. The closure of $T_{qc}(X_0)$ in $T_{ls}(X_0)$

A question of characterizing the closure of the image of $T_{qc}(X_0)$ inside $T_{ls}(X_0)$ was raised in [1]. We use our understanding of the topology on the Fenchel-Nielsen coordinates that makes the map $F : T_{ls}(X_0) \to l^\infty$ into a homeomorphism to give a characterization of the closure of $T_{qc}(X_0)$.

Let $l = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R}\}$ be the space of all sequences of real numbers. We first define $\tilde{F} : T_{ls}(X_0) \to l$ by setting

$$\tilde{F}(X) = \{(x_1, x_2, \ldots) \in l : x_{2n-1} = \log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)},\ x_{2n} = t_{\alpha_n}(X) - t_{\alpha_n}(X_0) \text{ for } n \in \mathbb{N}\}.$$

If α_n is a boundary component we use only the length coordinate.

By [1] or by Theorem 1, $\tilde{F}(T_{ls}) \subset l$ consists of all $\bar{x} = (x_1, x_2, \ldots) \in l$ such that

$$\sup_n \max\{|x_{2n-1}|, \frac{|x_{2n}|}{\max\{1, |\log l_{\alpha_n}(X_0)|\}}\} < \infty.$$

Let $O(1)$ denotes a bounded function and let $O(M) := M \cdot O(1)$ as $M \to \infty$. Moreover, $o(1)$ denotes a function which converges to 0 as $M \to \infty$ and let $o(M) = M \cdot o(1)$. Then $\bar{x} = (x_1, x_2, \ldots)$ are the Fenchel-Nielsen coordinates of $X \in T_{ls}(X_0)$ if and only if

$$|x_{2n-1}| = O(1)$$

and

$$|x_{2n}| = O(\max\{1, |\log l_{\alpha_n}(X_0)|\}).$$

By [2], the image $F(T_{qc}(X_0)) \subset l$ of the quasiconformal Teichmüller space $T_{qc}(X_0)$ consists of all $\bar{x} = (x_1, x_2, \ldots)$ such that

$$\|\bar{x}\|_\infty < \infty,$$

or equivalently

$$|x_n| = O(1).$$

Theorem 3.1. *Let X_0 be a complete hyperbolic surface with an upper bounded pants decomposition $\mathcal{P}=\{\alpha_n\}$. Then $X\in T_{ls}(X_0)$ is in the closure of $T_{qc}(X_0)$ for the metric d_{ls} if and only if*

$$\sup_{\alpha_n\in\mathcal{P}}\Big|\log\frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}\Big|<\infty$$

and

$$|t_{\alpha_n}(X)-t_{\alpha_n}(X_0)|=o(|\log l_{\alpha_n}(X_0)|) \tag{20}$$

as $|\log l_{\alpha_n}(X_0)|\to\infty$.

Proof. We first note that if X_0 has a (geodesic) pants decomposition which is bounded from the above and from the below then (cf. [12], [1]) $T_{qc}(X_0)=T_{ls}(X_0)$. Therefore we assume that there is a pants decomposition of X_0 which is upper bounded with a sequence of cuffs whose lengths go to 0.

Let $X_i\in T_{qc}(X_0)$ such that $X_i\to X$ in the length spectrum metric d_{ls} as $i\to\infty$. Then $d_{ls}(X_0,X)<\infty$, namely $X\in T_{ls}(X_0)$. Let $\{\alpha_{n_k}\}_k$ be the set of all geodesics in $\mathcal{P}$ such that $l_{\alpha_{n_k}}(X_0)\leq\frac{1}{e}$. Then by Theorem 2.1

$$\sup_k\frac{|t_{\alpha_{n_k}}(X_i)-t_{\alpha_{n_k}}(X)|}{|\log l_{\alpha_{n_k}}(X_0)|}\to 0$$

as $i\to\infty$. Thus for any $\epsilon>0$ there exists i_0 such that for all $i>i_0$ we have

$$|t_{\alpha_{n_k}}(X)-t_{\alpha_{n_k}}(X_0)|\leq|t_{\alpha_{n_k}}(X_i)-t_{\alpha_{n_k}}(X_0)|+\epsilon|\log l_{\alpha_{n_k}}(X_0)|.$$

Assume on the contrary that (20) is false. Then there exists $C>0$ and subsequence k_j such that $l_{\alpha_{n_{k_j}}}(X_0)\to 0$ as $j\to\infty$ and

$$|t_{\alpha_{n_{k_j}}}(X)-t_{\alpha_{n_{k_j}}}(X_0)|\geq C|\log l_{\alpha_{n_{k_j}}}(X_0)|.$$

Choose $\epsilon=\frac{C}{2}$. The above two inequalities give for all $i>i_0$

$$|t_{\alpha_{n_{k_j}}}(X_i)-t_{\alpha_{n_{k_j}}}(X_0)|\geq\frac{C}{2}|\log l_{\alpha_{n_{k_j}}}(X_0)|$$

which contradicts $X_i\to X$ as $i\to\infty$. Thus X satisfies (20).

Assume that $X\in T_{ls}(X_0)$ satisfies (20). We need to find a sequence $X_i\in T_{qc}(X_0)$ such that $X_i\to X$ as $i\to\infty$ for the length spectrum metric d_{ls}. For a given $i\in\mathbb{N}$, let $X_i\in T_{ls}(X_0)$ be defined by the Fenchel-Nielsen coordinates

$$l_{\alpha_n}(X_i):=l_{\alpha_n}(X)$$

and

$$t_{\alpha_n}(X_i)-t_{\alpha_n}(X_0):=\mathrm{sgn}[t_{\alpha_n}(X)-t_{\alpha_n}(X_0)]\min\{|t_{\alpha_n}(X)-t_{\alpha_n}(X_0)|,i\}. \tag{21}$$

By [2], we have $X_i\in T_{qc}(X_0)$. Let $M=d_{ls}(X_0,X)$ and choose $\epsilon>0$. Since X satisfies (20), it follows that there exists $\delta>0$ such that

$$\frac{|t_{\alpha_n}(X)-t_{\alpha_n}(X_0)|}{|\log l_{\alpha_n}(X_0)|}<\frac{\epsilon}{2}$$

for all $\alpha_n\in\mathcal{P}$ with $l_{\alpha_n}(X_0)\leq\delta$. Moreover, there exists $C=C(\delta)>0$ such that

$$|t_{\alpha_n}(X)-t_{\alpha_n}(X_0)|\leq C$$

for all $\alpha_n\in\mathcal{P}$ with $l_{\alpha_n}(X_0)>\delta$.

For $l_{\alpha_n}(X_0) \leq \delta$, we have

$$\frac{|t_{\alpha_n}(X) - t_{\alpha_n}(X_i)|}{|\log l_{\alpha_n}(X_0)|} \leq \frac{|t_{\alpha_n}(X) - t_{\alpha_n}(X_0)|}{|\log l_{\alpha_n}(X_0)|} + \frac{|t_{\alpha_n}(X_0) - t_{\alpha_n}(X_i)|}{|\log l_{\alpha_n}(X_0)|} \leq$$
$$\leq 2\frac{|t_{\alpha_n}(X) - t_{\alpha_n}(X_0)|}{|\log l_{\alpha_n}(X_0)|} < \epsilon.$$

For $l_{\alpha_n}(X_0) > \delta$, we have that $t_{\alpha_n}(X_i) = t_{\alpha_n}(X)$ for each $i > C$. Thus $X_i \to X$ as $i \to \infty$ in the length spectrum metric d_{ls}. □

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