BENDINGS BY FINITELY ADDITIVE TRANSVERSE COCYCLES

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Abstract. Let S be any closed hyperbolic surface and let λ be a maximal geodesic lamination on S. The amount of bending of an abstract pleated surface (homeomorphic to S) with the pleating locus λ is completely determined by an $(\mathbb{R}/2\pi\mathbb{Z})$-valued finitely additive transverse cocycle β to the geodesic lamination λ. We give a sufficient condition on β such that the corresponding pleating map $\tilde{f}_\beta : \mathbb{H}^2 \to \mathbb{H}^3$ induces a quasiFuchsian representation of the surface group $\pi_1(S)$. Our condition is genus independent.

1. Introduction

Let S be a closed hyperbolic surface and let λ be a maximal geodesic lamination on S. The universal covering $\tilde{S}$ of S is isometrically identified with the hyperbolic plane $\mathbb{H}^2$. Denote by $\tilde{\lambda}$ the lift of λ to $\mathbb{H}^2$ and denote by $\mathbb{H}^3$ the hyperbolic three space. Each component of $\mathbb{H}^2 - \tilde{\lambda}$, called *plaque*, is an ideal hyperbolic triangle. A *pleated surface* with the *pleating locus* λ is an immersion $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ which conjugates the covering group of $\mathbb{H}^2 \to S$ into a subgroup of the isometries of $\mathbb{H}^3$, and is totally geodesic on plaques and on geodesics of $\tilde{\lambda}$. The pleating map $\tilde{f}$ is an isometry from $\mathbb{H}^2$ onto its image $f(\mathbb{H}^2)$ for the path metric on $f(\mathbb{H}^2)$ induced by the hyperbolic metric of $\mathbb{H}^3$. The pleatings along $\tilde{\lambda}$ give rise to a finitely additive $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle to λ, and conversely any such cocycle defines a pleated surface with the pleating locus λ (cf. Bonahon [3]). The main result in the paper gives a sufficient condition on transverse cocycle such that the pleating map is injective on the boundary.

1.1. **Pleating maps along measured laminations.** Let μ be a measured (geodesic) lamination on the hyperbolic plane $\mathbb{H}^2$. By definition, μ is a collection of positive Borel measures on hyperbolic arcs transverse to the support geodesic lamination $|\mu|$. The collection of measures is invariant under homotopies relative the geodesics in the support $|\mu|$. For example, μ could be the lift of a measured lamination on a closed hyperbolic surface.

Given a closed geodesic arc I transverse to $|\mu|$, denote by $\mu(I) \geq 0$ the total mass of the measure deposited on I. The Thurston norm of μ is given by

$$\|\mu\| = \sup_I \mu(I)$$

where the supremum is over all closed hyperbolic arcs of length 1.

Given a complex number $t \in \mathbb{C}$ there exists a (unique) pleated surface corresponding to the complex measure $t\mu$ with the bending along $|\mu|$ determined by

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the imaginary part of $t\mu$ (and the path hyperbolic metric on the immersed surface determined by the real part of $t\mu$). Then we have the following sufficient condition for the pleating map to be an embedding (cf. [8], [13]).

Theorem. *Let μ be a measured lamination on the hyperbolic plane $\mathbb{H}^2$ with finite Thurston norm. There exist universal $C > 0$ and $\epsilon > 0$ such that whenever $t \in \mathbb{C}$ satisfies*

$$|Im(t)| < \frac{\epsilon}{\|\mu\| e^{C\|\mu\|\cdot|Re(t)|}}$$

the pleating map induced by $t\mu$ is an embedding.

The purpose of this paper is to extend the above theorem to finitely additive transverse cocycles to (maximal) geodesic laminations on closed hyperbolic surfaces. Natural examples of finitely additive real valued transverse cocycles arise from real Fenchel-Nielsen coordinates on pants decomposition of closed surfaces. Namely, a geodesic pairs of pants decomposition of a closed hyperbolic surface can be completed to a maximal geodesic lamination by adding three geodesics to each pair of pants that spiral around cuffs. The obtained maximal geodesic lamination has finitely many leaves (i.e. geodesics) and the components of the complement of the geodesic lamination are ideal hyperbolic triangles. The Fenchel-Nielsen coordinates describe the shape of the pairs of pants (i.e. the cuff lengths) and how two pairs of pants fit together (i.e. the twist parameters) which completely describes the hyperbolic metric on the surface. On the other hand, a finitely additive real valued transverse cocycle to the maximal geodesic lamination describes how complementary ideal triangles fit together to give the hyperbolic metric on the surface (cf. [3]).

In their proof of the surface subgroup conjecture, Kahn and Markovic [10] used a sufficient condition on complex Fenchel-Nielsen coordinates to obtain embedded pleated surfaces. In [14] we obtained a sufficient condition on finitely additive transverse cocycles to finite geodesic laminations that gives embedded pleated surfaces. Our main objective is to extend the scope of this theorem to arbitrary maximal geodesic laminations.

1.2. **Finitely additive cocycles and pleating maps.** Let λ be an arbitrary maximal geodesic lamination on a closed hyperbolic surface S. The plaques of λ are ideal hyperbolic triangles. A *finitely additive real valued transverse cocycle* to λ is an assignment of a real valued finitely additive measure to each hyperbolic arc transverse to λ that is invariant under homotopies relative the leaves of λ. A finitely additive transverse cocycle defines Hölder distribution on all hyperbolic arcs transverse to λ, and conversely any transverse Hölder distribution defines a finitely additive transverse cocycle to λ (cf. Bonahon [2]).

As in the case of finite geodesic laminations, the hyperbolic metric on a closed surface is completely determined by the induced finitely additive transverse cocycle. Moreover, the Teichmüller space $T(S)$ of a closed hyperbolic surface S is parametrized by open cone $\mathcal{C}(\lambda)$ inside the space $\mathcal{H}(\lambda, \mathbb{R})$ of finitely additive real valued transverse cocycles to λ (cf. Thurston [17] and Bonahon [3]). A detailed study of the space $\mathcal{H}(\lambda, \mathbb{R})$ is carried out by Bonahon (cf. [2], [4]).

A *pleated surface* with the *pleating locus* λ is an immersion

$$\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$$

which is an isometry on each complementary region to $\tilde{\lambda}$ and on each geodesic of $\tilde{\lambda}$ such that

$$\tilde{f} G \tilde{f}^{-1} = K,$$

where G is the covering group of $\mathbb{H}^2 \to S$ and K is a subgroup of the isometry group of $\mathbb{H}^3$. A pleated surface along λ induces a $(\mathbb{C}/2\pi i\mathbb{Z})$-valued transverse cocycle to λ where the real part parametrizes the induced path metric and the imaginary part parametrizes the amount of bending along λ. Moreover, the space of all representations of G that realize λ is parametrized by an open subspace $\mathcal{C}(\lambda) + i\mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ of the space $\mathcal{H}(\lambda, \mathbb{C}/2\pi i\mathbb{Z})$ (cf. Bonahon [3]).

1.3. **The main result.** We fix a hyperbolic metric on S and give a sufficient condition on $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle β to the maximal geodesic lamination λ such that the extension to the boundary $\tilde{f}_\beta : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$ of the corresponding pleating map $\tilde{f}_\beta : \mathbb{H}^2 \to \mathbb{H}^3$ is injective.

Let $\{h_1, \ldots, h_m\}$ be a set of closed geodesic arcs on S transversely intersecting λ (with endpoints in the plaques of λ) such that the components of $\lambda - \cup_{i=1}^m h_i$ have finite length. In addition, if h_i's are collapsed to points and homotopic arcs of $\lambda - \cup_{i=1}^m h_i$ are identified, we obtain a trivalent (topological) train track which carries λ.

A *gap* of h_i is a component of $h_i - \lambda$ that does not contain an endpoint of h_i. Given a gap d of h_i, denote by g_d^+ and g_d^- two arcs of $\lambda - \{h_1, \ldots, h_m\}$ that pass through endpoints of d oriented in the same direction (with respect to h_i). If g_d^+ and g_d^- together with the gap d and another (on some arc h_j) gap form a quadrilateral, then we say that g_d^+ and g_d^- are *parallel*.

Since the topological train track (obtained by collapsing h_i's) is trivalent, it follows that each h_i has exactly one gap d such that g_d^+ and g_d^- are not parallel in one direction. Choose an arbitrary point in the interior of d to divide h_i into two arcs h_i', h_i'' with $h_i = h_i' \cup h_i''$. We form a new set of arcs by taking all h_i, h_i', h_i'' and for simplicity of notation denote it by $\{k_1, \ldots, k_n\}$, called a *set of ties for* λ. (An equivalent definition for a set of ties is given using "geometric train tracks", cf. Bonahon [2] and §2).

Fix a set of ties $\{k_1, \ldots, k_n\}$ for λ. Two arcs k_i and k_j are *paired* if there exists an arc in $\lambda - \{k_1, \ldots, k_n\}$ that connects them. Define

$$l^* = \max_{i,j} diam(k_i \bigcup k_j) \tag{1}$$

and

$$l_* = \min_{i,j} dist(k_i, k_j), \tag{2}$$

where the maximum and the minimum are over all paired arcs k_i, k_j, $diam(k_i \bigcup k_j)$ is the diameter of $k_i \bigcup k_j$, and $dist(k_i, k_j)$ is the distance between k_i and k_j.

Moreover, we define

$$w^* = \max_{1 \le i \le n} |k_i|$$

and

$$w_* = \min_{1 \le i \le n} |k_i|,$$

where $|k_i|$ is the length of the arc k_i.

A set of ties $\{k_1, \dots, k_n\}$ for λ is said to be *geometric* if each angle of the intersection between an arc in $\{k_1, \dots, k_n\}$ and a geodesics of λ is in the interval $[\pi/4, 3\pi/4]$, and

$$w^* \leq 1/20. \tag{3}$$

The above quantities l^*, l_*, w^* and w_* give quantitative information about the hyperbolic metric on S and the position of the geodesic lamination λ on S. We use this information in order to give a sufficient condition on the bending cocycles such that the bending map is injective.

We define the *norm* of β for the geometric family of arcs $\{k_1, \dots, k_n\}$ by

$$\|\beta\|_{max} = \max\{|\beta(k_i)| : 1 \leq i \leq n\}. \tag{4}$$

The norm $\|\beta\|_{max}$ is analogous to the Thurston norm of measured laminations.

In general, a finitely additive real measure β on a closed interval I has infinite "variation" in the sense that there exists a sequence I_n of subintervals of I with $|\beta(I_n)| \to \infty$ as $n \to \infty$ (cf. Bonahon [4]). In order to find a sufficient condition for the injectivity of the bending map $\tilde{f}_\beta : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$, we introduce (below) a notion of "variation on large gaps" on the set of ties. (It is important to note that if β does not have small enough variation on large gaps in order to have injectivity of $\tilde{f}_\beta$ then, for $|t| > 0$ small enough, $t\beta$ does have small enough variation on large gaps such that $\tilde{f}_{t\beta}$ is injective.)

Given $\delta > 0$, we introduce the δ-variation of β on k_i as follows. Give an arbitrary orientation to k_i. Let $\{d_j : j = 1, \dots, j_i\}$ be a finite family of gaps of k_i together with the two components of $k_i - \lambda$ containing the endpoints of k_i. Define k_{d_j}, for $j = 1, \dots, j_i$, to be the subarc of k_i whose initial point is the initial point of k_i and whose endpoint is a point in d_j.

Then the *δ-variation of β on k_i* is given by

$$\|\beta\|_{var_\delta, k_i} = \max_{1 \leq j \leq j_i} |\beta(k_{d_j})|, \tag{5}$$

where the set $\{d_j : j = 1, \dots, j_i\}$ is chosen such that the length of $k_i \setminus \cup_{j=1}^{j_i} d_j$ is less than $\delta |k_i|$ ($|k_i|$ denotes the length of k_i).

Moreover, the *δ-variation of β on a geometric family* $\{k_1, \dots, k_n\}$ is given by

$$\|\beta\|_{var_\delta} = \max_{1 \leq i \leq n} \|\beta\|_{var_\delta, k_i}. \tag{6}$$

Our main result is a sufficient condition for the injectivity of the bending map corresponding to a transverse cocycle β in terms of a geometric set of arcs.

Theorem 1.1. *There exist $\epsilon > 0$ and $\delta > 0$ such that for any closed hyperbolic surface S and a maximal geodesic lamination λ on S the following holds. Let $\{k_1, \dots, k_n\}$ be a geometric set of arcs for λ such that*

$$w^* < \frac{e^{-2l^*} \tanh \frac{l_*}{2}}{8\pi}. \tag{7}$$

If an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle β to λ satisfies

$$\|\beta\|_{max} < \epsilon w_* \tag{8}$$

and

$$\|\beta\|_{var_\delta} < \epsilon \tag{9}$$

then the developing map $\tilde{f}_\beta : \mathbb{H}^2 \to \mathbb{H}^3$ *continuously extends to an injective map* $\tilde{f}_\beta : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$.

Remark 1.2. The condition (8) is the standard condition that works for the transverse measures (cf. [8], [13]). The condition (9) is a new condition needed to control the geometry of the realization of the transverse cocycle β due to the fact that the variation of β is unbounded. Note that the condition (8) for a choice of a family of geometric arcs does not imply similar condition on an arbitrary arc of length at most 1. For this reason it is necessary that the arcs k_i's are on a relatively large distance compared to their sizes which is made explicit by (7).

Remark 1.3. The quantities l^* and l_* depend on the lamination λ. The constants ϵ and δ are computed in terms of l^* and l_* in the proof of the theorem. If a geometric set of arcs $\{k_1, \ldots, k_n\}$ does not satisfy (7) then we can divide each arc into several subarcs until the condition is satisfied. If λ contains short closed geodesics then l_* is small for any choice of a geometric set of arcs for λ. A generic geodesic lamination λ contains no closed geodesics and the choice of a geometric set of arcs can be made such that $l^* \geq 1/5$ and $l_* \geq l^*/4$ in which case we can choose $w^* = 4.41719 \times 10^{-10}$ and $\epsilon = \delta = 3.61749 \times 10^{-17}$. We give a table of values for ϵ and δ when $l_* = l^*/4$ for various values of l^* (cf. Table 6.2). It seems that the optimal value is $l^* = 0.0238523$ in which case $\epsilon = \delta = 2.01795 \times 10^{-13}$ and $w^* \leq 1.27126 \times 10^{-11}$.

Let α be an $\mathbb{R}$-valued transverse cocycle to λ which is induced by the hyperbolic metric on S (cf [3]). For $z \in \mathbb{C}$, define the transverse cocycle α_z by

$$\alpha_z(k) = (1 + z)\alpha(k) \mod (2\pi i \mathbb{Z})$$

for each arc k transverse to λ.

The developing shear-bend map $\tilde{f}_z : \mathbb{H}^2 \to \mathbb{H}^3$ (normalized to be the identity on a fixed plaque of λ) corresponding to the transverse cocycle α_z induces a holomorphic family (in z) of representation of $\pi_1(S)$ to $PSL_2(\mathbb{C})$ (cf. [3]). As a corollary to the above theorem, we obtain

Corollary 1.4. *Let* α *be an* $\mathbb{R}$*-valued transverse cocycle to a geodesic lamination* λ *corresponding to a hyperbolic metric on a closed surface* S *and let* $\tilde{f}_z$ *be the shear-bend map for* α_z. *Then there exists* $\epsilon > 0$ *such that the shear-bend map*

$$\tilde{f}_z : \mathbb{H}^2 \to \mathbb{H}^3$$

extends by continuity to a holomorphic motion of $\partial_\infty \mathbb{H}^2$ *in* $\partial_\infty \mathbb{H}^3$ *for the parameter* $\{z \in \mathbb{C} : |z| < \epsilon\}$.

1.4. **Outline of the proof of main theorem.** It is enough to prove that any two points $x, y \in \partial_\infty \mathbb{H}^2$ are mapped onto distinct points in $\partial_\infty \mathbb{H}^3$ under the bending map $\tilde{f}_\beta$. Let g be a geodesic in $\mathbb{H}^2$ whose endpoints are x and y.

If g is a leaf of $\tilde{\lambda}$ then $\tilde{f}_\beta(x) \neq \tilde{f}_\beta(y)$ because the bending map is an isometry on leaves and plaques of $\tilde{\lambda}$.

From now on we assume that g transversely intersects $\tilde{\lambda}$. Then g necessarily intersects plaques of $\tilde{\lambda}$ by the maximality(of $\tilde{\lambda}$).

Consider a fixed geometric set of ties $\{k_1, \ldots, k_n\}$ for λ. If two ties are paired, connect their vertices by hyperbolic arcs to form a quadrilateral, called a *long rectangle*. The two paired ties are *short sides* and the other two sides are *long sides* of the long rectangle. The set of all long rectangles are edges and the set of ties are vertices of a *geometric train track* τ. The geodesic lamination λ is contained in the interior of the union of all edges (i.e. long rectangles) of τ.

Denote by $\tilde{\lambda}$ and $\tilde{\tau}$ the lifts to $\mathbb{H}^2$ of the geodesic lamination λ and the geometric train track τ. Then the geodesic lamination $\tilde{\lambda}$ is contained in the interior of the union of edges (i.e. long rectangles) of $\tilde{\tau}$.

If geodesic g is completely contained in $\tilde{\tau}$ then g is a leaf of $\tilde{\lambda}$. Therefore g contains a point p outside $\tilde{\tau}$. The point p divides the geodesic g into two rays g_1 and g_2 with the common initial point p and endpoints x and y, respectively.

We normalize the bending map $\tilde{f}_\beta$ to be the identity on the plaque of $\tilde{\lambda}$ which contains p. Form two hyperbolic cones $\mathcal{C}(p,g_1,\pi/2)$ and $\mathcal{C}(p,g_2,\pi/2)$ whose central axes are g_1 and g_2 with angle $\pi/2$. A *shadow* of a cone is the set of endpoints (on the boundary $\partial_\infty\mathbb{H}^3$) of the rays contained in the cone. We prove that $\tilde{f}_\beta(x)$ stays in the (open) shadow at $\partial_\infty\mathbb{H}^3$ of the cone $\mathcal{C}(p,g_1,\pi/2)$, and analogously for $\tilde{f}_\beta(y)$. This implies the desired result since the open shadows of $\mathcal{C}(p,g_1,\pi/2)$ and $\mathcal{C}(p,g_2,\pi/2)$ are disjoint.

We consider the ray g_1 and $\tilde{f}_\beta(x)$. The bending map $\tilde{f}_\beta$ is mapping the ray g_1 to a piecewise geodesic in $\mathbb{H}^3$ with the bending points $\tilde{\lambda}\cap g_1$ and the bending amount given by the finitely additive transverse cocycle β. The idea of controlling the position of $\tilde{f}_\beta(x)$ is to divide the ray g_1 into finite arcs such that the finitely additive "β-measure" on each arc is well-behaved with respect to the size of the arc. Consequently the bending map $\tilde{f}_\beta$ moves the endpoints of the arcs and the tangent vectors to g_1 at these endpoints such that $\tilde{f}_\beta(g_1)$ stays inside the cone $\mathcal{C}(p,g_1,\pi/2)$.

In more details, consider the intersections of the ray g_1 with the boundary sides of the long rectangles (i.e edges of $\tilde{\tau}$) in the order from the initial point p. The first point of intersection a_1 of g_1 and (the union of long rectangles of) $\tilde{\tau}$ is on a long side of a long rectangle E_1. If g_1 exits E_1 through other long side of E_1, then denote by b_1 that point. If g_1 exits an adjacent edge E_2 through its long side then denote that point by b_1. If g_1 exits an edge E_3 adjacent to E_2 through its long side then denote the point of exit by b_1. Finally if g_1 exists E_3 through a short side, denote by b_1 the point of exit of g_1 from a short side of E_2.

We obtained the first arc $[a_1,b_1]$, denote by E_{a_1} the edge of $\tilde{\tau}$ which g_1 enters at the point a_1 and denote by E_{b_1} the edge which g_1 leaves at the point b_1. We take a_2 to be the point at which g_1 enters the first edge E_{a_2} after the edge E_{b_1}. We note that it is possible that $b_1=a_2$ if E_{b_1} and E_{a_2} share a short edge. Then b_2 is determined analogously to b_1.

We continue this process to obtain a sequence of disjoint arcs $\{(a_n,b_n)\}$ on g_1 given in the increasing order for the orientation of g_1. If $b_n\neq a_{n+1}$ then the open arc (b_n,a_{n+1}) does not intersect $\tilde{\tau}$ and the closed arc $[b_n,a_{n+1}]$ does not intersect $\tilde{\lambda}$. According to our definition, either $[a_n,b_n]$ connects long sides of (possibly the same) long rectangle(s) and intersects at most three long rectangles, or $[a_n,b_n]$ connects two short sides of a long rectangle while intersecting at most three long rectangles (cf. §6).

Consider a sequence of nested cones $\{\mathcal{C}(a_n,g_1,\pi/2)\supset\mathcal{C}(b_n,g_1,\pi/2)\}$. Let P_{a_n} and P_{b_n} be plaques containing a_n and b_n, respectively. Then $\tilde{f}_\beta|_{P_{a_n}}$ and $\tilde{f}_\beta|_{P_{b_n}}$ are hyperbolic isometries of $\mathbb{H}^3$ such that

$$\tilde{f}_\beta|_{P_{b_n}}=\tilde{f}_\beta|_{P_{a_n}}\circ R_{[a_n,b_n]}$$

where $R_{[a_n,b_n]}$ is a hyperbolic isometry defined using the transverse cocycle β on the arc $[a_n, b_n]$ (cf. Bonahon [3] and §4).

We need to prove that

$$\tilde{f}_\beta|_{P_{a_n}}(\mathcal{C}(a_n, g_1, \pi/2)) \supset \tilde{f}_\beta|_{P_{b_n}}(\mathcal{C}(b_n, g_1, \pi/2))$$

for all n, which is equivalent to

$$\mathcal{C}(a_n, g_1, \pi/2) \supset R_{[a_n,b_n]}(\mathcal{C}(b_n, g_1, \pi/2)) \tag{10}$$

because the maps are hyperbolic isometries.

The core of the proof of the above theorem is bounding the isometry $R_{[a_n,b_n]}$ such that (10) holds for all possible geodesics g in $\mathbb{H}^2$ simultaneously. This is achieved by careful choice of a set of ties as in the theorem. To establish (10), we consider different possibilities for the intersection of the arc $[a_n, b_n]$ with the edges of $\tilde{\tau}$.

Assume that $[a_n, b_n]$ connects two long sides of a single long rectangle in $\tilde{\tau}$. There is a definite lower bound on the length of $[a_n, b_n]$ in terms of w_* and l^* (cf. Lemma 2.2). Lemma 5.2 gives a numerical bound on the distance of $R_{[a_n,b_n]}$ from the identity (which is a constant multiple of the length of $[a_n, b_n]$) such that (10) holds.

Denote by T_g^c the hyperbolic isometry with the axis g and the translation length $c \in \mathbb{C}$. The isometry $R_{[a_n,b_n]}$ is given by (cf. Bonahon [3] and §4).

$$R_{[a_n,b_n]} = \lim_{i\to\infty} B_1 B_2 \cdots B_i R_{P_{b_n}}$$

where

$$B_j = R_{g_{d_j}^{P_{a_n}}}^{\beta(P_{a_n}, P_{d_j})} R_{g_{d_j}^{P_{b_n}}}^{-\beta(P_{a_n}, P_{d_j})}$$

with P_{d_j} being the plaque containing the gap d_j, $\beta(P_{a_n}, P_{d_j})$ being the β-mass of a closed arc with endpoints in P_{a_n} and P_{d_j} for $j = 1, \ldots, i$; and where

$$R_{P_{b_n}} = R_{g_{P_{b_n}}^{P_{a_n}}}^{\beta([a_n,b_n])}.$$

Lemma 5.5 and 5.4 give estimates for B_j and $T_{P_{b_n}}$ which allows us to conclude that $\|R_{[a_n,b_n]} - Id\|$ is bounded by a linear function with variables $|\beta([a_n, b_n])| \leq \|\beta\|_{max}$, $\|\beta\|_{var_\delta}$ and $\delta > 0$ (cf. §6). This allows us to choose universal $\epsilon > 0$ and $\delta > 0$ such that the assumptions of Lemma 5.2 are satisfied and we obtain (10) in this case.

Assume that $[a_n, b_n]$ enters the edge E_1 at a short side, then enters the adjacent edge E_2 at a short side, and it exits E_2 at a long edge. The composition

$$B_1 B_2 \cdots B_i$$

is estimated in the same fashion as above. However, the hyperbolic rotation

$$R_{P_{b_n}}$$

can have arbitrary large angle $\beta([a_n, b_n])$ since β is finitely additive transverse cocycle and $[a_n, b_n]$ does not cross the set of all geodesics following a single edge of $\tilde{\tau}$. In order to have a control on where $\mathcal{C}(b_n, g_1, \pi/2)$ is mapped by $R_{P_{b_n}}$, it is necessary that the angles of intersections between the geodesics of $\tilde{\lambda}$ and $[a_n, b_n]$ are small (cf. Lemma 5.3). The angles are made small enough by the choice of w^* and l^* in the main theorem. Thus we have the inclusion of the cones again. All other cases are dealt by combining the above two case with slightly larger constants which gives the proof of the main theorem (cf. §6).

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2. Geodesic laminations

Given a hyperbolic surface and a (maximal) geodesic lamination on the surface, we define a "metric" train track using rectangles as edges and sides at which two rectangles meet (called short sides of rectangle) as vertices. The geodesic lamination will be contained in the interior of the union of rectangles and the angles at which geodesics of the lamination meet the vertices (i.e. short sides of rectangles) are bounded away from 0 and π. In the lemma below, we find a lower bound on the distance between two opposite sides of a rectangle(the sides that are not short) in terms of the diameters of the rectangle and the lengths of the short sides.

Let S be a closed hyperbolic surface and λ a maximal geodesic lamination on S. Each component of $S \setminus \lambda$, called *plaque* of λ, is an ideal hyperbolic triangle for the path metric of the complement. Let $\{h_1, \dots, h_m\}$ be a set of finite geodesic arcs on S with endpoints in the plaques of λ such that each geodesic of λ is divided into finite length arcs by the set $\cup_{i=1}^m h_i$. The family of arcs $\lambda \setminus \cup_{i=1}^m h_i$ consists of finitely many homotopy classes relative $\{h_1, \dots, h_m\}$ and we assume that after identifying all the arcs of the homotopy classes the obtained "topological train track" has the property that each vertex (corresponding to some h_i) is either trivalent or bivalent. The usual definition of train tracks does not allow bivalent vertices but we do allow them. The reason is that we need more flexibility to obtain a train track which has good geometric properties.

We form a "metric train track" τ as follows. A *gap* of h_i (with respect to λ) is a connected component of $h_i \setminus \lambda$. If h_i corresponds to a trivalent vertex of the corresponding topological train track, we divide arc h_i into two subarcs h_i^1 and h_i^2 with a division point in a gap of h_i such that the endpoints of the arcs of $\lambda \setminus \cup_{i=1}^m h_i$ which belong to different homotopy classes lie in different subarcs. For the convenience of notation, denote by $\{k_1, \dots, k_n\}$ the set of all arcs h_i, h_i^j for $i = 1, \dots, m$ and $j = 1, 2$.

We connect the endpoints of $\{k_1, \dots, k_n\}$ by geodesic arcs inside the plaques of λ (whenever this is possible) to obtain a finite collection of geodesic quadrilaterals whose two sides are among k_i's and the other two sides are obtained by connecting the chosen points on k_i's inside the components of $S \setminus (\lambda \cup \bigcup_{i=1}^n k_i)$. We call these quadrilaterals, somewhat improperly, *long rectangles*. The sides of the rectangles which are among k_i's are said to be *short* and the other two sides are said to be *long*. The finite collection of long rectangles forms a *(metric) train track* τ on S such that the long rectangles are the *edges* of τ and the *switches* of τ are the arcs $\{k_1, \dots, k_n\}$, where we allow switches to be either trivalent or bivalent. The geodesic lamination λ is a subset of the interior of (the union of the edges of) the train track τ and it is said that λ is *carried* by τ. The train tack τ is homotopic to the topological train track. This kind of train tracks were introduced by Bonahon (cf. [2]).

Definition 2.1. Let l^* be the maximum of the diameters of the long rectangles of τ and let l_* be the shortest distance between two short sides of the long rectangles.

Let w_* be the minimum of the lengths of the short sides over all long rectangles of τ and let w^* be the maximum of the lengths of the short sides over all long rectangles of τ.

We impose two conditions on τ. Namely we require that the angles at the vertices of each long rectangle lie in the interval $[\pi/4, 3\pi/4]$ and that

$$w^* \leq \frac{1}{20}.$$

A (metric) train track which satisfies these conditions is said to be *geometric* and the corresponding collection of arcs $\{k_1, k_2, \dots, k_n\}$ is said to be *geometric*. We note that the angles are given by the choice of the arcs $\{h_1, \dots, h_m\}$ above and that w^* can be made small enough by further subdividing the arcs $\{h_1, \dots, h_m\}$, if necessary.

Lemma 2.2. *Let R be a long rectangle of a geometric train track τ with short sides k_1 and k_2. Then the distance d between the long sides of R satisfies*

$$d \geq \frac{1}{20e^{l^*}} \min\{|k_1|, |k_2|\},$$

where $|k_i|$ is the length of k_i.

Proof. Let k_1 and k_2 be the short sides of R, and let l_1 and l_2 be the long sides of R. Denote by h_1 and h_2 orthogonal arcs from $k_1 \cap l_2$ and $k_2 \cap l_2$ onto l_1, respectively. The hyperbolic sine formula, the bounds on the angles at the vertices of R and the mean value theorem give

$$|h_i| \cosh 1 \geq \sinh |h_i| \geq \frac{1}{\sqrt{2}} \sinh |k_i| \geq \frac{1}{\sqrt{2}} |k_i|$$

which gives

$$|h_i| \geq \frac{1}{\sqrt{2}\cosh 1} |k_i|.$$

By possibly decreasing R, we can assume that h_1 and h_2 have the same length $|h_1| = |h_2| \geq \frac{1}{\sqrt{2}\cosh 1} \min\{|k_1|, |k_2|\}$. Let h be the arc which is orthogonal to both l_1 and l_2. An elementary hyperbolic geometry formula applied to the rectangle whose two sides are h_1 and h gives

$$|h| \cosh 1 \geq \sinh |h| \geq \frac{\sinh \frac{\min\{|k_1|, |k_2|\}}{\sqrt{2}\cosh 1}}{\sqrt{\sinh^2 \frac{l^* + 4\min\{|k_1|, |k_2|\}}{2} \cosh^2(\min\{|k_1|, |k_2|\}) + 1}}$$

which in turn gives

$$d \geq |h| \geq \frac{1}{2(\cosh^2 1) e^{l^*+1}} \min\{|k_1|, |k_2|\}.$$

□

3. Transverse cocycles to geodesic laminations

In this section we define finitely additive transverse cocycles to a geodesic lamination λ on the surface S (cf. Bonahon [2]). The transverse cocycles that we consider are $\mathbb{R}$-valued, $(\mathbb{R}/2\pi\mathbb{Z})$-valued and $(\mathbb{C}/2\pi i\mathbb{Z})$-valued.

Definition 3.1. [3] A *real valued transverse cocycle* α to λ is an assignment of a finitely additive real valued measure to each arc k transverse to λ which is invariant under homotopies relative λ. Namely, α assigns a real number $\alpha(k)$ to each closed arc k transverse to λ whose endpoints are in $S-\lambda$ such that if k' is an arc homotopic to k relative λ then $\alpha(k)=\alpha(k')$. Moreover if $k=k_1\cup k_2$, where k_1 and k_2 are transverse arcs to λ with disjoint interiors, then $\alpha(k)=\alpha(k_1)+\alpha(k_2)$.

Remark 3.2. Let α be a finitely additive transverse cocycle to a geodesic lamination λ. Let k be a closed arc transverse to λ such that $|\alpha(k)|\neq 0$ and α is not countably additive on k. Then there exists a sequence of subarcs $\{k_n\}$ of k such that

$$|k_n|\to 0$$

and

$$|\alpha(k_n)|\to\infty$$

as $n\to\infty$. This phenomenon does not appear for the countably additive transverse cocycles and it forces delicate arguments when working with finitely additive cocycles (cf. [2], [3], [4]).

A hyperbolic metric on S induces a real valued transverse cocycle to a maximal geodesic lamination λ as follows (cf. [3]).

Each complementary triangle (i.e. plaque) of λ can be partially foliated by three families of horocyclic arcs with centers at the vertices of the complementary triangle such that the portion which is not foliated is a finite triangle containing the center of the complementary ideal hyperbolic triangle.

Every point of every boundary geodesic of a plaque lies in exactly one horocyclic leaf of the partial foliation except for the vertices of the finite triangle where two horocyclic leaves meet (cf. Bonahon [2]). The partial foliation of the plaques extends to the leaves of λ by the continuity and the surface is foliated except for the finite triangles (with horocyclic boundaries) inside the plaques. Note that the boundary geodesics of the plaques are leaves of λ. However, in general, geodesic laminations can have uncountably many leaves and the extension of the foliation is non trivial (cf. [3]).

Lift a maximal geodesic lamination λ to maximal geodesic lamination $\tilde{\lambda}$ of the hyperbolic plane $\mathbb{H}^2$. Given a closed hyperbolic arc k on S with endpoints in plaques of λ which transversely intersects λ, denote by $\tilde{k}$ its lift to $\mathbb{H}^2$. Let P_1 and P_2 be the plaques of $\tilde{\lambda}$ which contain the endpoints of $\tilde{k}$. Let p_i be the vertex of the central triangle of P_i on the boundary of P_i facing P_{i+1}, for $i=1,2$ (where $i+1$ is taken modulo 2). Let p_1' be the point where the leaf of the (lifted) horocyclic foliation through the point p_1 meets the boundary of P_2 facing P_1. The value of the transverse cocycle to the arc k is the signed distance between p_2 and p_1' when the geodesic through them is oriented as a part of the boundary of P_2 (cf. [3]).

A finitely additive real valued transverse cocycle to a maximal geodesic lamination λ induces a transverse Hölder distribution to λ (i.e. a linear functional on Hölder continuous functions on each transverse arc to λ that is invariant under homotopies relative λ). Conversely, a transverse Hölder distribution to λ induces a real valued finitely additive transverse cocycle to λ (cf. [4]).

Denote by $\mathcal{H}(\lambda,\mathbb{R})$ the space of real valued finitely additive transverse cocycles to λ. The subset of transverse cocycles which arise from hyperbolic metrics on S is an open cone in $\mathcal{H}(\lambda,\mathbb{R})$ denoted by $\mathcal{C}(\lambda)$ (cf. [3]). In fact, $\mathcal{C}(\lambda)$ parametrizes the Teichmüller space $T(S)$ of the closed surface S (cf. [3]). Given a point in $\mathcal{C}(\lambda)$, there is a procedure of recovering the corresponding metric on S by constructing the corresponding representation of the fundamental group $\pi_1(S)$ using the transverse cocycle (cf. [3]).

We will also need finitely additive transverse cocycles which take values in $\mathbb{R}/2\pi\mathbb{Z}$ and in $\mathbb{C}/2\pi i\mathbb{Z}$.

Definition 3.3. [3] An $(\mathbb{R}/2\pi\mathbb{Z})$-*valued transverse cocycle* β for λ is an assignment of $\beta(k) \in \mathbb{R}/2\pi\mathbb{Z}$ to each transverse arc k to λ which is invariant under homotopy relative λ and which is finitely additive. For example, an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle β is obtained by taking a real valued transverse cocycle α and setting $\beta(k) := \alpha(k) \mod (2\pi\mathbb{Z})$.

Similarly, a $(\mathbb{C}/2\pi i\mathbb{Z})$-*valued transverse cocycle* β for λ is an assignment of $\beta(k) \in \mathbb{R}/2\pi\mathbb{Z}$ to each transverse arc k to λ which is invariant under homotopy relative λ and which is finitely additive.

A *pleated surface* with the *pleating locus* λ is a continuous map

$$\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$$

which is an isometry on each leaf and on each plaque of the lift $\tilde{\lambda} = \pi^{-1}(\lambda)$ for the universal covering $\pi : \mathbb{H}^2 \to S$, and which conjugates the covering group G of S into a subgroup fGf^{-1} of $PSL_2(\mathbb{C})$.

Each plaque of $\tilde{\lambda}$ is isometrically mapped by $\tilde{f}$ to an ideal hyperbolic triangle in $\mathbb{H}^3$ and the amount of bending along $\tilde{\lambda}$ is measured by an $(\mathbb{R}/2\pi\mathbb{Z})$-valued finitely additive transverse cocycle to $\tilde{\lambda}$. The pleating map can be recovered from an $(\mathbb{R}/2\pi\mathbb{Z})$-valued finitely additive transverse cocycle by constructing the isometries of $\mathbb{H}^3$ which are equal to the restriction of the pleating map on the given plaque (cf. Bonahon [2] and §4). Moreover, a pleating map with the pleating locus λ induces a $(\mathbb{C}/2\pi i\mathbb{Z})$-valued finitely additive transverse cocycle to $\tilde{\lambda}$ with the real part determining the path metric on $\tilde{f}(\mathbb{H}^2)$(when considered as a subset of the hyperbolic three space $\mathbb{H}^3$) and the imaginary part giving the amount of bending along $\tilde{\lambda}$.

Let α be either an $\mathbb{R}$-valued or an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle for λ and let τ be a geometric train track that carries λ. Given an edge $E \in \tau$, let k_E be a geodesic arc which connects the two long boundary sides of E. Define $\alpha(E) := \alpha(k_E)$. Note that $\alpha(E)$ is independent of the choice of k_E by the invariance of α under homotopy relative λ. The transverse cocycle α is completely determined by the values $\alpha(E)$, $E \in \tau$(cf. [2]).

4. The realizations of $\mathbb{R}$-valued and $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycles

The purpose of this section is to recall the procedure of constructing the realizations of $\mathbb{R}$-valued and $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle to a maximal geodesic lamination on a closed surface S (cf. [3]). In the former case we obtain a representation of $\pi_1(S)$ into $PSL_2(\mathbb{R})$ which gives a hyperbolic metric on S. In the later case, we start from a hyperbolic metric on S and obtain a pleated surface with the pleating locus λ and the bending amount given by the $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle. The core argument in the proof of the main theorem is estimating the pleating map arising from an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle which satisfies certain geometric conditions.

We consider a hyperbolic surface S and a maximal geodesic lamination λ of S. Bonahon [3] defined an injective map from the Teichmüller space $T(S)$ of S into the space $\mathcal{H}(\lambda,\mathbb{R})$ of all $\mathbb{R}$-valued transverse cocycles to a fixed maximal geodesic lamination λ which is a homeomorphism onto an open cone $\mathcal{C}(\lambda)$ of $\mathcal{H}(\lambda,\mathbb{R})$.

Denote by σ_0 a fixed hyperbolic metric on S. Then σ_0 represents the base point in $T(S)$ and let $\alpha_0\in\mathcal{H}(\lambda,\mathbb{R})$ be the corresponding transverse cocycle. Then [3, Proposition 13] any other real-valued cocycle $\alpha\in\mathcal{H}(\lambda,\mathbb{R})$ which is close enough to α_0 is also in the image of $T(S)$ in $\mathcal{H}(\lambda,\mathbb{R})$. Namely, when the difference $\alpha-\alpha_0$ is small in the sense that the norm $\|\alpha-\alpha_0\|_{\max}$ is small, where

$$\|\alpha\|_{\max}:=\max_E|\alpha(E)|$$

and the maximum is over all edges E of a train track τ that carries λ then α determines a point in $T(S)$.

The proof of the above statement is given by constructing the realization of α_1 starting from the realization of α_0. We recall that λ is a maximal geodesic lamination for the metric σ_0. We lift λ to a geodesic lamination $\tilde{\lambda}$ of the universal covering $\mathbb{H}^2$. Components of $\mathbb{H}^2\setminus\tilde{\lambda}$ are called *plaques* of $\tilde{\lambda}$ and they are lifts of plaques (i.e. connected components of $S\setminus\lambda$) of λ. Each plaque of $\tilde{\lambda}$ is an ideal hyperbolic triangle.

Let k be an oriented geodesic arc in $\mathbb{H}^2$ from plaque P to plaque Q of $\tilde{\lambda}$. Denote by $\mathcal{P}_{P,Q}$ be the set of all plaques of $\tilde{\lambda}$ that separate P and Q, and by $\mathcal{P}=\{P_1,P_2,\dots,P_n\}$ a finite subset of $\mathcal{P}_{P,Q}$ such that the indices increase from P to Q. Let g_i^P and g_i^Q be the geodesics on the boundary of the plaque P_i that separate P_i from P and Q, respectively. Let g_Q^P be the geodesic on the boundary of Q that separates P and Q. Define $\alpha=\alpha_1-\alpha_0$. Let $\alpha(P,P_i)$ denote the α-mass of a geodesic arc with endpoints in P and P_i; similar definition for $\alpha(P,Q)$.

Let

$$D_{P_i}=T_{g_i^P}^{\alpha(P,P_i)}T_{g_i^Q}^{-\alpha(P,P_i)}$$

where $T_{g_i^P}^{\alpha(P,P_i)}$, $T_{g_i^Q}^{-\alpha(P,P_i)}$ are hyperbolic translations(in $\mathbb{H}^2$) with axes g_i^P, g_i^Q which are oriented to the left as seen from P, Q and with the translation lengths $\alpha(P,P_i)$, $-\alpha(P,P_i)$. Let

$$T_Q=T_{g_Q^P}^{\alpha(P,Q)}.$$

A finite approximation $\varphi_{\mathcal{P}}$ to the realization of the transverse cocycle α_1 corresponding to a finite set of plaques $\mathcal{P}$ is given by

$$\varphi_{\mathcal{P}}=D_{P_1}D_{P_2}\cdots D_{P_n}T_Q.$$

The realization $\varphi_{P,Q}$ of the transverse cocycle α is given by (cf. [3])

$$\varphi_{P,Q}=\lim_{\mathcal{P}\to\mathcal{P}_{P,Q}}\varphi_{\mathcal{P}}.$$

Let

$$\psi_{\mathcal{P}}=D_{P_1}D_{P_2}\cdots D_{P_n}$$

and let

$$\psi_{P,Q}=\lim_{\mathcal{P}\to\mathcal{P}_{P,Q}}\psi_{\mathcal{P}}.$$

It follows that

$$\varphi_{P,Q} = \psi_{P,Q} \circ T_{g_Q^P}^{\alpha(P,Q)}.$$

The quantity $\psi_{P,Q}$ is the *difference from* T_Q *of the realization* $\varphi_{P,Q}$ of hyperbolic metric σ on S. Bonahon [3] proved that for a fixed surface S and small *norm*

$$\|\alpha_1 - \alpha_0\|_{\max} = \max_{E \in \tau} |\alpha_1(E) - \alpha_0(E)|$$

the difference $\psi_{P,Q}$ always lies in a compact subset of $PSL_2(\mathbb{R})$.

Our main interest are bending pleated surfaces. The bending of (abstract) pleated surfaces with the pleating locus λ is completely determined by an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycles for the geodesic lamination λ on S and each $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle to λ is realized by an abstract pleated surface with pleating locus λ (cf. [3]). Denote by $\mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ the space of all $(\mathbb{R}/2\pi\mathbb{Z})$-valued finitely additive transverse cocycles to the lamination λ. The space of all abstract pleated surfaces with the pleating locus λ is parametrized by $\mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$, where $\mathcal{C}(\lambda)$ is the open cone in $\mathcal{H}(\lambda, \mathbb{R})$ which parametrizes the Teichmüller space $T(S)$ (cf. [3]).

Let $\beta \in \mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ and denote by R_g^a hyperbolic rotation in $\mathbb{H}^3$ with the axis g and the rotation angle $a \in \mathbb{R}$. Let $\mathcal{P}_{P,Q}$ be the set of all plaques of $\tilde{\lambda}$ separating plaques P and Q, and let $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ be a finite subset of $\mathcal{P}_{P,Q}$.

Define

$$B_{P_i} = R_{g_i^P}^{\beta(P,P_i)} R_{g_i^Q}^{-\beta(P,P_i)}$$

where $P_i \in \mathcal{P}$, $\beta(P, P_i)$ the β-mass of a geodesic arc connecting P and P_i, g_i^P the geodesic on the boundary of P_i facing P, and g_i^Q the geodesic on the boundary of P_i facing Q. Define

$$R_Q = R_{g_Q^P}^{\beta(P,Q)}.$$

A finite approximation $\varphi_{\mathcal{P}}$ of the realization of the bending cocycle β is defined by

$$\varphi_{\mathcal{P}} = B_{P_1} B_{P_2} \cdots B_{P_n} R_Q$$

and the realization $\mathcal{P}_{P,Q}$ of β is defined by (cf. [3])

$$\varphi_{P,Q} = \lim_{\mathcal{P} \to \mathcal{P}_{P,Q}} \varphi_{\mathcal{P}}$$

Note that the realization exists for all $\beta \in \mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ because of the compactness of $\mathbb{R}/2\pi\mathbb{R}$. The *difference from* R_Q *of the realization* of β is given by

$$\psi_{P,Q} = \lim_{\mathcal{P} \to \mathcal{P}_{P,Q}} \psi_{\mathcal{P}} \tag{11}$$

where

$$\psi_{\mathcal{P}} = B_{P_1} B_{P_2} \cdots B_{P_n}. \tag{12}$$

The bending map $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ is defined by fixing a plaque P and setting $\tilde{f}|_Q = \varphi_{P,Q}$ for any plaque Q. Note that $\tilde{f}|_P = id$.

Let $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map for the bending pleated surface defined by β, where $\mathbb{H}^2$ is identified with the (xz)-half-plane in the upper half-space $\mathbb{H}^3 = \{(z,t) : z \in \mathbb{C}, t > 0\}$. Then $\tilde{f}$ does not necessarily extend to an injective map from $\partial_\infty \mathbb{H}^2$ into $\partial_\infty \mathbb{H}^3$. The core argument in the proof of the main theorem establishes that the geometric conditions on β guarantee injectivity of the continuous extension $\tilde{f}:\partial_\infty\mathbb{H}^2\to\partial_\infty\mathbb{H}^3$ (which is achieved in §6).

5. The nested cones

In this section we prove several lemmas needed in the proof of the main theorem in §6. As discussed in Introduction, the main argument establishes the injectivity of the bending map $\tilde{f}:\mathbb{H}^2\to\mathbb{H}^3$. Namely, given $x,y\in\partial_\infty\mathbb{H}^2$ with $x\neq y$, we need to prove that $\tilde{f}(x)\neq\tilde{f}(y)$. We consider the geodesic g in $\mathbb{H}^2$ whose ideal endpoints are x and y, and prove that the image of g under the bending map $\tilde{f}$ behaves well enough to have distinct endpoints in $\partial_\infty\mathbb{H}^3$ (cf. §6).

Let $p\in g$ be a point in a plaque of $\tilde{\lambda}$, and let g_1,g_2 be the two geodesic rays that g is divided into by the point p. We consider two hyperbolic cones both having vertex p, angle $\pi/2$ and axes g_1 and g_2. The open shadows of the two cones are disjoint (in fact, each open shadow is a component of the complement of the boundary circle of an embedded hyperbolic plane in $\mathbb{H}^3$ containing point p and orthogonal to g), and x belongs to one and y belongs to the other shadow. In order to prove that $\tilde{f}(x)\neq\tilde{f}(y)$, we normalize $\tilde{f}$ to be the identity on the plaque containing p and prove that the image $\tilde{f}(g_i)$, for $i=1,2$, stays in the corresponding hyperbolic cone.

To do so, we divide g_i into disjoint subarcs and consider a sequence of nested cones at the endpoints of these subarcs with the same angle $\pi/2$. The goal is to prove that the image under $\tilde{f}$ of the nested sequence of cones remains nested. It is enough to consider the consecutive hyperbolic cones and prove they stay embedded.

There are essentially two different phenomenon that can occur for the transverse cocycle β along the sequence of arcs on g_i. Each arc either intersects an edge E of the geometric train track such that it connects the long sides of the long rectangle E(in which case the arc intersects exactly the set of geodesics of $\tilde{\lambda}$ traversing E) or it connects two short sides of a long rectangle E while intersecting only a portion of geodesic that traverse E.

In Lemma 5.2, we prove that if an isometry A of $\mathbb{H}^3$ is close to the identity then the image of the inside cone stays in the outside cone as long as the distance between the vertices of the two cones is comparable to the size of $\|A-id\|$. Lemma 5.1 is used in the proof of Lemma 5.2.

In the former case, the realization of the cocycle β is on the distance from the identity comparable to the quantities $\|\beta\|_{\max}$ and $\|\beta\|_{var_\delta}$ by Lemma 5.4 and 5.5. Since the arc connects two long sides of a long rectangle, Lemma 2.1 applies to conclude that there is a lower bound on the length of the arc. Then Lemma 5.2 implies the desired nesting of the cones.

In the later case, the β-mass of the arc can be arbitrary large. In order to estimate the realization $\varphi_{P,Q}$ where P and Q are plaques containing the endpoints of the arc, we recall $\varphi_{P,Q}=\psi_{P,Q}R_Q$. The isometry $\psi_{P,Q}$ is approximated by $\psi_{\mathcal{P}}=B_1B_2\cdots B_n$ and the above argument proves that $\psi_{P,Q}$ is close enough to the identity when $\|\beta\|_{max}$ and $\|\beta\|_{var_\delta}$ are small enough. The rotation R_Q does not have small angle since the β-mass can be large on the arc and Lemma 5.4 does not apply. Instead, since the distance between the short sides of each long rectangle is long, Lemma 5.3 gives the nesting of the cones in this case. We note that we do not have this phenomenon when the transverse cocycle is countably additive (i.e. it is a measure).

Let $g \subset \mathbb{H}^3$ be a geodesic ray with initial point p_0, and let p be a point on the ray g. For $0 < \theta < \pi$, the *cone* $\mathcal{C}(p, g, \theta)$ *with vertex* p*, axis* g *and angle* θ is the set of all $w \in \mathbb{H}^3$ such that the angle at p between the positive direction of g and the geodesic ray from p through w is less than θ. A non-zero vector $(p, v) \in T^1(\mathbb{H}^3)$ uniquely determines a geodesic ray g with the basepoint p tangent to v. By definition $\mathcal{C}(p, v, \theta)$ is $\mathcal{C}(p, g, \theta)$. The *shadow of the cone* $\mathcal{C}(p, g, \theta)$ is the set $\partial_\infty \mathcal{C}(p, g, \theta)$ of endpoints at $\partial_\infty \mathbb{H}^3$ of all geodesic rays starting at p and inside $\mathcal{C}(p, g, \theta)$.

For $d > 0$, let $p_d \in g$ be the point on g which is on the distance d from $p_0 = p$. Let $\eta > 0$ be the maximal angle such that $\mathcal{C}(p_d, g, \eta) \subset \mathcal{C}(p_0, g, \theta)$. Then $\eta = \eta(d, \theta)$ is a continuous function of d and θ. For a fixed $0 < \theta < 2\pi$, we have $\eta(d, \theta) > \theta$ and $\eta(d, \theta) \to \theta$ as $d \to 0$ (cf. [9]).

We use quaternions to represent the upper half-space model $\mathbb{H}^3 = \{z + tj : z \in \mathbb{C}, t > 0\}$ of the hyperbolic three-space $\mathbb{H}^3$ (cf. Beardon [1]). The space of isometries of $\mathbb{H}^3$ is identified with $PSL_2(\mathbb{C})$ which is equipped with the norm

$$\|A\| = \max\{|a| + |b|, |c| + |d|\}$$

where $A(z) = \frac{az+b}{cz+d} \in PSL_2(\mathbb{C})$ and $ad - bc = 1$. The Poincaré extension to $\mathbb{H}^3$ of the action of $A \in PSL_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ is computed in [1] to be

$$A(z + tj) = \frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2 + tj}{|cz + d|^2 + |c|^2 t^2}.$$

An isometry of $\mathbb{H}^3$ which is close to the identity moves points on a bounded distance from $j \in \mathbb{H}^3$ by a small amount and the tangent vectors are rotated by a small angle with respect to the Euclidean parallel transport in $\mathbb{R}^3$. In Lemma 5.1, we give a quantitative statement of the above fact including the situation when the points are not on the bounded distances from $j \in \mathbb{H}^3$.

Given $p = z + tj \in \mathbb{H}^3$, we define

$$ht(p) = t$$

and

$$Z(p) = z.$$

Denote by $T_p\mathbb{H}^3$ the tangent space at the point $p \in \mathbb{H}^3$. The unit tangent space $T^1_p\mathbb{H}^3$ at the point p is the quotient $(T_p\mathbb{H}^3 - \{0\})/\mathbb{R}^+$ of non-zero tangent vectors $T_p\mathbb{H}^3 - \{0\}$ by the positive real numbers $\mathbb{R}^+$. If $(p, u) \in T_p\mathbb{H}^3$, then $(p, u)/\sim = \{(p, tu) | t \in \mathbb{R}^+\}$ is the corresponding unit tangent vector. For simplicity of notation, we denote by (p, u) the unit tangent vector corresponding to (p, u). Denote by $T^1\mathbb{H}^3$ the unit tangent space to $\mathbb{H}^3$.

Given $(p, v), (q, w) \in T^1\mathbb{H}^3$, define the distance on $T\mathbb{H}^3$ by

$$D_{T^1\mathbb{H}^3}((p, u), (q, v)) = \max\{|ht(p) - ht(q)|, |Z(p) - Z(q)|, |\angle(u, v)|\}$$

where $\angle(u, v)$ is the angle between the vectors u and the euclidean parallel translate of v at the point p.

If an isometry of $\mathbb{H}^3$ is close to the identity, then the unit tangent vectors in a compact subset of $T^1\mathbb{H}^3$ are moved by a small amount. In the lemma below, we give a quantitative statement which will be used in the proof of Lemma 5.2.

Lemma 5.1. *Let $m_0 > 0$ and $C > 0$ be fixed. Define $\eta'(m_0, C) = \min\{\frac{1}{10(C+1)m_0}, \frac{1}{4}\}$. If $0 \leq \eta < \eta'$, $0 < m \leq m_0$, $A \in PSL_2(\mathbb{C})$ with*

$$\|A - Id\| < \eta m$$

and $(p, u) \in T^1\mathbb{H}^3$ such that

$$D_{T^1\mathbb{H}^3}((p, u), (e^{-m}j, -j)) < C\eta m$$

then

$$D_{T^1\mathbb{H}^3}(A(p, u), (e^{-m}j, -j)) < C_1\eta m$$

where $C_1 = 2\pi(60C + 9)$.

Proof. Let $ht(p) = h$ and $Z(p) = z$. Let $p_1 = A(p)$, and $ht(p_1) = h_1$ and $Z(p_1) = z_1$. For $A \in PSL_2(\mathbb{C})$, the Poincaré extension (cf. [1]) of A is given by the formula

$$A(P) = \frac{(az + b)\overline{(cz + d)} + a\bar{c}h^2}{|cz + d|^2 + |c|^2h^2} + \frac{h}{|cz + d|^2 + |c|^2h^2}j.$$

By the assumption, $|Z(p) - Z(e^{-m}j)| = |Z(p)| = |z| < C\eta m$ and $|ht(p) - ht(e^{-m}j)| = |h - e^{-m}| < C\eta m$. We set

$$\eta'(m_0, C) = \min\Big\{\frac{1}{10(C + 1)m_0}, \frac{1}{4}\Big\}.$$

If $\eta \leq \eta'(m_0, C)$ then $C\eta m \leq \frac{1}{10}$ and $\eta m \leq \frac{1}{10}$. Moreover, $\|A - Id\| < \eta m$ implies that $|b|, |c|, |a - 1|, |d - 1| < \eta m$.

For $p_1 = A(p)$, the above inequalities together with some elementary computations give

$$|ht(p_1) - e^{-m}| \leq (2C + 1)\eta m$$

and

$$|Z(p_1)| \leq 4(C + 1)\eta m.$$

Let $v = v_1 + v_2i - j$ and $w = A'(p)v = w_1 + w_2i + w_3j$. The assumption $|\angle(v, -j)| < C\eta m$ implies that $|v_1|, |v_2| < \frac{\pi}{2}C\eta m$. Note that

$$w_k = \frac{\partial A_k}{\partial x}v_1 + \frac{\partial A_k}{\partial y}v_2 + \frac{\partial A_k}{\partial h}v_3$$

for $k = 1, 2, 3$, where $A = A_1 + A_2i + A_3j$.

Some more elementary (and long) computations give

$$|\frac{\partial A_1}{\partial x}|, |\frac{\partial A_1}{\partial y}|, |\frac{\partial A_2}{\partial x}|, |\frac{\partial A_2}{\partial y}| \leq 15,$$

$$|\frac{\partial A_1}{\partial h}|, |\frac{\partial A_2}{\partial h}| \leq 9\eta m,$$

$$|\frac{\partial A_3}{\partial x}|, |\frac{\partial A_3}{\partial y}| \leq 12\eta m,$$

and

$$|\frac{\partial A_3}{\partial h}| \geq 1 - 6\eta m$$

at the point p.

The above estimates provide

$$|w_1|, |w_2| \leq (60C + 9)\eta m$$

and

$$|w_3| \geq 1-(4C+6)\eta m.$$

Then

$$|\angle(w,-j)| \leq \frac{\pi(60C+9)}{1-(4C+6)\eta m}\eta m.$$

Since $\eta \leq \frac{1}{(4C+6)m_0}$ we obtain

$$|\angle(w,-j)| \leq 2\pi(60C+9)\eta m. \tag{13}$$

□

Consider cone $\mathcal{C}(j,-j,\frac{\pi}{2})$ with the vertex j and the central geodesic ray in the direction of the tangent vector $-j$. Then cone $\mathcal{C}(e^{-m}j,-j,\frac{\pi}{2})$ is a subset of the above cone. If (p,u) is close enough to $(e^{-m}j,-j)$ and if $A\in PSL_2(\mathbb{R})$ is close enough to the identity, then the cone $\mathcal{C}(A(p),A'(p)u,\frac{\pi}{2})$ is a subset of $\mathcal{C}(j,-j,\frac{\pi}{2})$.

The following lemma establishes a sufficient quantitative bound on the size of a hyperbolic isometry $A\in PSL_2(\mathbb{C})$ and on the distance of (p,u) to $(e^{-m}j,-j)$ such that $\mathcal{C}(j,-j,\frac{\pi}{2})\supset\mathcal{C}(A(p),A'(p)u,\frac{\pi}{2})$.

Lemma 5.2. *Fix $m_0>0$ and $C>0$. Let $\eta''(m_0,C)=\min\{\frac{e^{-m_0}}{32(C+1)m_0},\frac{e^{-m_0}}{66\pi(60C+9)}\}$. Then for each $0\leq\eta<\eta''(m_0,C)$, for each $0<m\leq m_0$ and for each $A\in PSL_2(\mathbb{C})$ with*

$$\|A-Id\|<\eta m$$

we have

$$\partial_\infty\mathcal{C}(j,-j,\frac{\pi}{2})\supset\overline{\partial_\infty\mathcal{C}(A(p),A'(p)u,\frac{\pi}{2})},$$

where $(p,u)\in T^1\mathbb{H}^3$ is such that

$$D_{T^1\mathbb{H}^3}((p,u),(e^{-m}j,-j))<C\eta m.$$

Proof. Let $p_1=A(p)$; write $h_1=ht(P_1)$ and $z_1=Z(P_1)$. By the proof of Lemma 5.1 we have

$$e^{-m}-(2C+1)\eta m\leq h_1\leq e^{-m}+(2C+1)\eta m$$

and

$$|z_1|\leq 4(C+1)\eta m.$$

Let $y=|z_1|$. Note that z_1 is the foot in $\mathbb{C}$ of the vertical line through p_1 perpendicular to $\mathbb{C}$. Let $x\in\mathbb{C}$ be the center of the euclidean hemisphere in $\mathbb{H}^3$ which passes through p_1 and is tangent to the unit radius euclidean hemisphere centered at $0\in\mathbb{C}$. Let φ be the angle at p_1 between the radius of the hemisphere centered at x and the vertical line through p_1 (cf. Figure 1).

From Figure 1, we have

$$(1-x)^2=(x-y)^2+h_1^2\leq x^2+h_1^2$$

which implies

$$x\geq\frac{1-h_1^2}{2}\geq\frac{1-h_1}{2}\geq\frac{1-e^{-m}-(2C+1)\eta m}{2}\geq\frac{1-(2C+1)\eta}{2}m.$$

If η satisfies

$$\eta\leq\frac{1}{2(2C+1)},$$

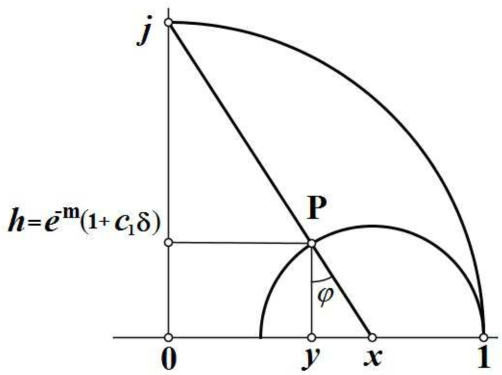

FIGURE 1.

then

$$x \geq \frac{1}{4}m. \tag{14}$$

This gives

$$\tan\varphi = \frac{x-y}{h_1} \geq \frac{\frac{1}{4}m - 4(C+1)\eta m}{e^{-m} + (2C+1)\eta m} \geq \frac{1}{16}m$$

when $\eta \leq \frac{1}{32m_0(C+1)}$.

Let φ_0 be the angle at p_1 in Figure 1 when $m = m_0$. Then we have

$$\varphi \geq \frac{\varphi_0}{\tan\varphi_0}\tan\varphi \geq \frac{\varphi_0}{\tan\varphi_0}\frac{1}{16}m. \tag{15}$$

We need an upper bound on $\frac{\tan\varphi_0}{\varphi_0}$. Let x_0 and y_0 be the values of x and y when $m = m_0$. Similar to the above, we have

$$(1-x_0)^2 = (x_0-y_0)^2 + h^2 \geq (x_0-y_0)^2$$

which gives

$$x_0 \leq \frac{1+y_0}{2} \leq \frac{11}{20}$$

because $y_0 \leq C\eta m \leq \frac{1}{10}$. Then

$$\tan\varphi_0 = \frac{x_0-y_0}{h} \leq \frac{x_0}{h} \leq \frac{\frac{11}{20}}{e^{-m_0} - (2C+1)\eta m_0} \leq \frac{11}{10}e^{m_0}$$

for $\eta \leq \frac{e^{-m_0}}{2m_0(2C+1)}$. Since

$$\frac{\tan\varphi_0}{\varphi_0} \leq \frac{1}{\cos\varphi_0} = \sqrt{\tan^2\varphi_0 + 1} \leq 2e^{m_0}$$

by (15) we obtain

$$\varphi \geq \frac{e^{-m_0}}{32}m. \tag{16}$$

Let $\varphi_0^* = \frac{e^{-m_0}}{33}m$. It follows that

$$\overline{\partial_\infty \mathcal{C}(p_1, -j, \frac{\pi}{2} + \varphi_0^* m)} \subset \partial_\infty \mathcal{C}(j, -j, \frac{\pi}{2}).$$

By Lemma 5.1, $|\angle(w,-j)|\leq 2\pi(60C+9)\eta m$ when $0\leq\eta\leq\eta'(m_0,C)$. Thus we get

$$\overline{\partial_\infty\mathcal{C}(p_1,w,\frac{\pi}{2})}\subset\partial_\infty\mathcal{C}(p_1,-j,\frac{\pi}{2}+\varphi_0^*m)$$

when $\eta\leq\frac{e^{-m_0}}{66\pi(60C+9)}$ which implies the desired nesting of the cones. By putting the bounds on η together, we define $\eta''=\min\{\frac{e^{-m_0}}{32(C+1)m_0},\frac{e^{-m_0}}{66\pi(60C+9)}\}$. □

The above lemma established nesting of a cone and the image of another cone under an isometry A when the isometry is close enough to the identity. In the following lemma the only type of an isometry that we consider is a hyperbolic rotation of $\mathbb{H}^3$ around a geodesic h intersecting the central geodesic ray of the outside cone. Unlike above, the rotation is not necessarily close to the identity since we allow the rotation angle to be arbitrary. The additional condition on the rotation is that its axis h subtends a small angle with the central axis of the outside cone which implies the nesting.

Lemma 5.3. *Let $m_0>0$ be fixed. Let g be the positive z-axis in $\mathbb{H}^3$ and let h be the geodesic in the xz-plane that intersects g at the point $e^{-m}j\in\mathbb{H}^3$ subtending the angle θ. Then for any m with $0\leq m\leq m_0$, any θ with $0\leq\theta<\frac{e^{-m_0}}{16}$, and any $(p,u)\in T^1\mathbb{H}^3$ with*

$$D_{T^1\mathbb{H}^3}((p,u),(e^{-m}j,-j))=\delta<\frac{1}{4}$$

we have

$$D_{T^1\mathbb{H}^3}(R_h^\varphi(p,u),(e^{-m}j,-j))\leq 20\delta+40\sqrt{2}e^{m_0}\theta$$

for any $\varphi\in\mathbb{R}$, where R_h^φ is the rotation of $\mathbb{H}^3$ with the angle φ around the axis h.

Proof. Let $b<0$ and $a>0$ be the endpoints of h. Since the angle between g and h is θ, it follows $a=-b\tan^2\theta$. Moreover h intersects g in the geodesic arc $[j,e^{-m_0}]$ which gives $e^{-m_0}\leq\sqrt{-2ab}\leq 1$. Consequently

$$\sqrt{2}e^{-m_0}\frac{1}{\theta}\leq|b|\leq\frac{\pi}{\sqrt{2}}\frac{1}{\theta}\tag{17}$$

and

$$\frac{e^{-2m_0}}{\sqrt{2}\pi}\theta\leq a\leq\frac{e^{m_0}}{2\sqrt{2}}\theta.\tag{18}$$

Note that

$$R_h^\varphi(z)=\frac{\frac{a-e^{i\varphi}b}{e^{i\varphi/2}(a-b)}z+\frac{ab(e^{i\varphi}-1)}{e^{i\varphi/2}(a-b)}}{\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}z+\frac{e^{i\varphi}a-b}{e^{i\varphi/2}(a-b)}}$$

and let $R_h^\varphi(z+tj)$ be the extension of R_h^φ to $\mathbb{H}^3$. Then

$$Z(R_h^\varphi(z+tj))=\frac{\left[\frac{a-e^{i\varphi}b}{e^{i\varphi/2}(a-b)}z+\frac{ab(e^{i\varphi}-1)}{e^{i\varphi/2}(a-b)}\right]\overline{\left[\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}z+\frac{e^{i\varphi}a-b}{e^{i\varphi/2}(a-b)}\right]}}{\left|\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}z+\frac{e^{i\varphi}a-b}{e^{i\varphi/2}(a-b)}\right|^2+\left|\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}\right|^2t^2}+$$
$$+\frac{\frac{a-e^{i\varphi}b}{e^{i\varphi/2}(a-b)}\frac{ab(e^{-i\varphi}-1)}{e^{-i\varphi/2}(a-b)}t^2}{\left|\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}z+\frac{e^{i\varphi}a-b}{e^{i\varphi/2}(a-b)}\right|^2+\left|\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}\right|^2t^2}$$

and

$$ht(R_h^\theta(p,u))=\frac{t}{\left|\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}z+\frac{e^{i\varphi}a-b}{e^{i\varphi/2}(a-b)}\right|^2+\left|\frac{1-e^{i\varphi}}{e^{i\varphi/2}(a-b)}\right|^2t^2}.$$

The bounds on δ and θ give that $a < \frac{1}{8}$, $|z| < \frac{1}{4}$ and $|b| > 4$. Similar to the proof of Lemma 5.1 we obtain the desired estimates. $\square$

In the following lemma we estimate the size of a hyperbolic rotation in terms of the distance of its axis to $j \in \mathbb{H}^3$ and the rotation angle.

Lemma 5.4. *Let A be a rotation in $\mathbb{H}^3$ by an angle ϵ around geodesic $l_{a,b}$ with endpoints $a, b \in \bar{\mathbb{R}} = \partial_\infty \mathbb{H}^2 \subset \partial_\infty \mathbb{H}^3$. If the geodesic with endpoints a and b intersects the ball of radius $m_0 > 0$ centered at $j \in \mathbb{H}^3$, then*

$$\|A - Id\| \leq (1 + e^{2m_0}) \frac{|\epsilon|}{2}.$$

Proof. We prove the lemma when both a and b are finite points. When a or b is ∞, the proof is left to the reader.

Note that

$$A(z) = \frac{\frac{a - be^{i\epsilon}}{(a-b)e^{i\frac{\epsilon}{2}}} z + \frac{ab(e^{i\epsilon} - 1)}{(a-b)e^{i\frac{\epsilon}{2}}}}{\frac{1 - e^{i\epsilon}}{(a-b)e^{i\frac{\epsilon}{2}}} z + \frac{ae^{i\epsilon} - b}{(a-b)e^{i\frac{\epsilon}{2}}}}.$$

Assume that $|a| \leq |b|$. Since $l_{a,b}$ intersects the ball of radius m_0 centered at $j \in \mathbb{H}^3$, it follows that $|a| \leq e^{m_0}$ and $|a - b| \geq 2e^{m_0}$. Then

$$\Big|\frac{a - be^{i\epsilon}}{(a - b)e^{i\frac{\epsilon}{2}}} - 1\Big| = \frac{|1 - e^{i\frac{\epsilon}{2}}| \cdot |a + be^{i\frac{\epsilon}{2}}|}{|a - b|} \leq (1 + \frac{2|a|}{|a - b|})\frac{\epsilon}{2} \leq (1 + e^{2m_0})\frac{\epsilon}{2}.$$

Further, we claim that

$$\Big|\frac{ab(e^{i\epsilon} - 1)}{a - b}\Big| \leq \frac{|ab|}{|a - b|}\epsilon \leq \frac{e^{2m_0}}{4}\epsilon.$$

To see this, note that $j \in \mathbb{H}^3$ is on the distance at most m_0 from the geodesic $l_{a,b}$. Then [1, formula (7.20.4)] gives

$$\sinh m_0 \geq \frac{\cosh^2 m_0 + ab}{|(b - a)| cosh m_0}$$

which implies the above. In a similar fashion, we obtain

$$\Big|\frac{ae^{i\epsilon} - b}{(a - b)e^{i\frac{\epsilon}{2}}} - 1\Big| \leq (1 + e^{2m_0})\frac{\epsilon}{2}$$

and

$$\Big|\frac{1 - e^{i\epsilon}}{(a - b)e^{i\frac{\epsilon}{2}}}\Big| \leq e^{m_0}\epsilon.$$

$\square$

The following lemma is well-known [3] and we estimate the constant involved.

Lemma 5.5. *Let g_1 and g_2 be geodesics in $\mathbb{H}^2 \subset \mathbb{H}^3$ that have a common endpoint and that intersect the ball $B_{m_0}(i)$ of radius $m_0 > 0$ centered at $i \in \mathbb{H}^2$. Let s be a geodesic arc that connects g_1 and g_2 inside $B_{m_0}(i)$. Then*

$$\|R_{g_1}^{\epsilon} \circ R_{g_2}^{-\epsilon} - Id\| \leq 2e^{m_0}(1 + e^{m_0})|s| \cdot |\epsilon|,$$

where $|s|$ is the hyperbolic length of the geodesic arc s.

Proof. Let $A(z) = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$ be a rotation of $\mathbb{H}^2$ around i. Note that $\|A\| \leq \sqrt{2}$. For the given embedding of $\mathbb{H}^2$ into $\mathbb{H}^3$, we have that $i \in \mathbb{H}^2$ is identified with $j \in \mathbb{H}^3$ and that the mapping A acts as a rotation in $\mathbb{H}^3$ around the geodesic with endpoints $i, -i \in \partial_\infty \mathbb{H}^3 = \bar{\mathbb{C}}$. The geodesic of the rotation passes through $j \in \mathbb{H}^3$ and $\mathbb{H}^2$ is orthogonal to this geodesic(thus setwise preserved by A).

Let $t \in \mathbb{R} \cup \{\infty\}$ be the common endpoint of the geodesics g_1 and g_2. We choose the rotation A such that $A(t) = \infty$. Then $A \circ R^{\epsilon}_{g_j} \circ A^{-1} = R^{\epsilon}_{g_j'}$, where $g_j' = A(g_j)$ for $j = 1, 2$. Note that

$$\|R^{\epsilon}_{g_1} \circ R^{-\epsilon}_{g_2} - Id\| \leq \|A^{-1}\| \cdot \|R^{\epsilon}_{g_1'} \circ R^{-\epsilon}_{g_2'} - Id\| \cdot \|A\| = 2\|R^{\epsilon}_{g_1'} \circ R^{-\epsilon}_{g_2'} - Id\|.$$

Let a and b be the endpoints of g_1' and g_2', respectively. A short computation gives

$$\|R^{\epsilon}_{g_1'} \circ R^{-\epsilon}_{g_2'} - Id\| = 2|a-b| \cdot |\sin\frac{\epsilon}{2}| \leq |a-b| \cdot |\epsilon|.$$

Let $P_1 \in g_1'$ and $P_2 \in g_2'$ be the endpoints of s. Without loss of generality, assume that $ht(P_1) \geq ht(P_2)$. Let l' be the arc issued from P_2 that is orthogonal to g_1'. Let $x = ht(P_2)$. A direct computations gives

$$|l'| = \log\Big[\frac{|a-b|}{x} + \sqrt{1 + \Big(\frac{|a-b|}{x}\Big)^2}\Big] \tag{19}$$

Note that $h = \frac{|a-b|}{x}$ is the length of the horocyclic arc centered at ∞ between g_1' and g_2' at the height x. Let h_0 be the maximum of the lengths of horocyclic arcs with the center at ∞ and inside the ball of radius m_0 centered at $i \in \mathbb{H}^2$. Then we obtain

$$|l'| \geq \log(1+h) \geq \frac{1}{1+h_0} h$$

which implies

$$|a-b| \leq x(1+h_0)|l'| \leq e^{m_0}(1+e^{m_0})|l'|$$

and the lemma follows. □

6. Injectivity on the boundary

Recall that the embedding of $\mathbb{H}^2$ in $\mathbb{H}^3$ given by the mapping $z = x + yi \mapsto x + yj$, for $y > 0$ and $x \in \mathbb{R}$. In this section, we prove that the bending map of a pleated surface realizing a transverse cocycle $\beta \in \mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ induces an injective map from $\partial_\infty \mathbb{H}^2$ into $\partial_\infty \mathbb{H}^3$ under geometric conditions on the bending transverse cocycle β given in Theorem 1.1.

6.1. **Outline of the proof.** For any $x, y \in \partial_\infty \mathbb{H}^2$ with $x \neq y$ we need to prove that $\tilde{f}(x) \neq \tilde{f}(y)$. Let g be a hyperbolic geodesic in $\mathbb{H}^2 \subset \mathbb{H}^3$ whose endpoints are x and y. If g is a leaf of $\tilde{\lambda}$ then $\tilde{f}(g) \subset \mathbb{H}^3$ is a geodesic. Thus $\tilde{f}(x) \neq \tilde{f}(y)$ in the case when g is a leaf of $\tilde{\lambda}$.

The main work is in proving $\tilde{f}(x) \neq \tilde{f}(y)$ when g is not a leaf of $\tilde{\lambda}$. Let p be a point on g contained in a plaque of $\tilde{\lambda}$. The geodesic g is divided by p into two geodesic rays g_1 and g_2. We form two hyperbolic cones $\mathcal{C}(p, g_1, \pi/2)$ and $\mathcal{C}(p, g_2, \pi/2)$ whose shadows on $\partial_\infty \mathbb{H}^3$ are disjoint and contain x and y, respectively. The idea is to prove that $\tilde{f}(g_i)$ stays in the cone $\mathcal{C}(p, g_i, \pi/2)$, for $i = 1, 2$. It is enough to restrict ones attention to $\tilde{f}(g_1)$ and the proof for the other ray is analogous.

We start by considering all the intersection points of g_1 with the boundary sides of the long rectangles of $\tilde{\tau}$. We form a division of g_1 into arcs $\{(a_n, b_n)\}_n$ such that $a_n < b_n \leq a_{n+1} < b_{n+1}$ for all a_n, where a_n, b_n are (some of the) points of the intersection of g_1 with the boundary sides of long rectangles and the arc (b_n, a_{n+1}), if $b_n < a_{n+1}$, is outside of the geometric train track $\tilde{\tau}$. Moreover, each (a_n, b_n) is chosen such that either it connects two long sides of a long rectangle; or it connects a long side of one long rectangle with a long side of its immediate neighbor rectangle; or it connects one long side of a long rectangle with another long side of a rectangle while passing through another rectangle; or it connects a short side of one rectangle to the other short side of the same rectangle; or it connects a short side of one rectangle with a long side of the adjacent rectangle. In short, any arc (a_n, b_n) intersects at most three long rectangles; and it either intersects the family of all geodesics crossing a long rectangle E while connecting the long sides of E, or it intersects the family of arcs intersecting a subarc of a short side of rectangle E while connecting the short sides of E.

We simplify the considerations by assuming that (a_n, b_n) either connects two long sides or it connects two short sides of a single long rectangle E. The arguments for these two cases also prove the theorem in other case with slightly smaller ϵ and δ.

Assume first that (a_n, b_n) connects two long sides of a long rectangle E. Let P and Q be plaques of $\tilde{\lambda}$ which contain a_n and b_n. Recall that the realization $\varphi_{P,Q}$ is given by

$$\varphi_{P,Q} = \psi_{P,Q} R_Q$$

where

$$\psi_{P,Q} = \lim_{\mathcal{P} \to \mathcal{P}_{P,Q}} B_1 B_2 \cdots B_n$$

for $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$.

Recall that

$$B_{P_i} = R_{g_i^P}^{\beta(P,P_i)} R_{g_i^Q}^{-\beta(P,P_i)}$$

where $P_i \in \mathcal{P}$, $\beta(P, P_i)$ the β-mass of a geodesic arc connecting P and P_i, g_i^P the geodesic on the boundary of P_i facing P, and g_i^Q the geodesic on the boundary of P_i facing Q. Moreover

$$R_Q = R_{g_Q^P}^{\beta(P,Q)}.$$

Using the fact that $\|\beta\|_{max}$ is small, we get that R_Q is close to the identity by Lemma 5.4. Moreover, since $\|\beta\|_{var_\delta}$ is small when δ is fixed, a repeated use of Lemma 5.5 gives that $B_1 B_2 \cdots B_n$ is close to the identity independently of n which implies that $\varphi_{P,Q}$ is close to the identity for $\epsilon > 0$ and $\delta > 0$ small enough. Then Lemma 5.2 guarantees that the image (under normalized $\tilde{f}$ which is the identity on P) of the cone at b_n is contained in the cone at a_n.

Assume next that the arc (a_n, b_n) connects two short sides of a long rectangle E. In this case $|\beta([a_n, b_n])|$ might not be small. The above estimates on $\psi_{P,Q}$ still hold by the same method using the fact that $\|\beta\|_{var_\delta}$ is small when δ is fixed and small enough. However, the rotation R_Q might not have small angle which makes R_Q bounded away from the identity. This is a new phenomenon which does not appear in the case when the bendings are by measured (i.e. countably additive) transverse cocycles. Since the arc (a_n, b_n) connects two short sides of a long rectangle, it follows that the angles of the intersections of the arc (a_n, b_n) with the leaves of $\tilde{\lambda}$ are small. Then Lemma 5.3 implies that the cone at b_n is mapped by R_Q to a nearby cone. Further, the nearby cone is mapped by $\psi_{P,Q}$ to a cone contained in the cone at a_n by Lemma 5.2. This finishes the proof of the nesting of cones in both cases and the proof of the theorem.

6.2. **The first step in proof of Theorem 1.1.** Let $\{k_1,\dots,k_n\}$ be a set of geometric arcs for the geodesic lamination λ satisfying all the properties given in §2. Let τ be the corresponding geometric train track. Let $\tilde{\lambda}$ and $\tilde{\tau}$ be lifts to $\mathbb{H}^2$ of the geodesic lamination λ and the geometric train track τ. For simplicity, we denote by k_j any lift of an arc k_j and by E any lift of an edge E.

Let g be an arbitrary geodesic in $\mathbb{H}^2$. Our goal is to show that the endpoints of g in $\partial_\infty\mathbb{H}^2$ are mapped to distinct points in $\partial_\infty\mathbb{H}^3$ under the bending map $\tilde{f}$ corresponding to the transverse cocycle β. If g is contained in $\tilde{\tau}$ then it coincides with a geodesic of $\tilde{\lambda}$. Since the bending map $\tilde{f}$ sends a geodesic in $\tilde{\lambda}$ to a geodesic in $\partial_\infty\mathbb{H}^3$, it follows that the endpoints of g are mapped to distinct points.

The main case to consider is when g transversely intersects $\tilde{\lambda}$. Then there exists $p\in g\cap(\mathbb{H}^2-\tilde{\tau})$ because if g is completely contained in $\tilde{\tau}$ then it is a geodesic of $\tilde{\lambda}$. The point p divides the geodesic g into two geodesic rays g_1 and g_2 with endpoints x and y, respectively. We consider the geodesic ray g_1 and similar conclusions hold for g_2.

First divide the geodesic ray g_1 into subarcs using the points of intersections of g_1 with the boundary sides of the edges of $\tilde{\tau}$ (i.e. boundary sides of long rectangles) as follows. The point p is the initial point of g_1. The first point a_1 of the intersection of g_1 with $\tilde{\tau}$ is at a long side of an edge E of $\tilde{\tau}$. We consider the next point p_1 of the intersection of g_1 with a boundary side of an edge E of $\tilde{\tau}$. If p_1 is on the long side of E then we set $b_1=p_1$ and $[a_1,b_1]$ is the first subarc in the division of g_1. If p_1 is on the short side of the edge E then we consider the next point q_1 of the intersection of g_1 with boundary sides of the edges of $\tilde{\tau}$. If q_1 is on a long side of an edge then we set $b_1=q_1$. If q_1 is on a short side and the next point of the intersection of g_1 with the boundary sides of $\tilde{\tau}$ is also on a short side, then we set $q_1=b_1$. If q_1 is on a short side and the next point of the intersection r_1 is on long side, then we set $b_1=r_1$. Note that the arc $[a_1,b_1]$ intersects interiors of at most three edges of $\tilde{\tau}$.

Assume that we have defined first n arcs $\{[a_1,b_1],\dots,[a_n,b_n]\}$ and we proceed to define $(n+1)$-st arc. If b_n is on the boundary of two edges of $\tilde{\tau}$, then we set $a_{n+1}=b_n$; otherwise we let a_{n+1} to be the first intersection point of g_1 with the boundary sides of the edges of $\tilde{\tau}$ that comes after b_n. Then b_{n+1} is chosen in a same fashion as b_1 above. We continue this process indefinitely. Thus we obtain a family of arcs $\{[a_n,b_n]\}_{n\in\mathbb{N}}$. If a_n does not belong to a plaque of $\tilde{\lambda}$ then we replace it with a nearby point on g_1 which belongs to a plaque and call the new point a_n again. Do the same for b_n. This situation occurs when a_n or b_n belong to the intersections of a short side of an edge of $\tilde{\tau}$ and the geodesic lamination $\tilde{\lambda}$. The complement in g_1 of the union of arcs $[a_n,b_n]$ does not intersect $\tilde{\lambda}$.

We consider a sequence of nested cones $\mathcal{C}(a_n,g_1,\frac{\pi}{2})\supset\mathcal{C}(b_n,g_1,\frac{\pi}{2})$ for $n\in\mathbb{N}$. When the bending map $\tilde{f}$ is normalized to be the identity at the plaque containing a_n, it is enough to prove that

$$\overline{\partial_\infty\tilde{f}(\mathcal{C}(b_n,g_1,\frac{\pi}{2}))}\subset\partial_\infty\mathcal{C}(a_n,g_1,\frac{\pi}{2}).$$

Because this property is geometric, it is independent under the post-compositions by the isometries of the bending map and it can be repeated along the sequence of arcs $\{[a_n, b_n]\}$. Thus we obtain that the sequence of the images under the bending map of the shadows of the cones is nested and in particular, the image under $\tilde{f}$ of the endpoint of g_1 is contained in $\partial_\infty \mathcal{C}(p, g_1, \frac{\pi}{2})$. Similarly, the image under $\tilde{f}$ of the endpoint of g_2 is contained in $\partial_\infty \mathcal{C}(p, g_2, \frac{\pi}{2})$. Since $\partial_\infty \mathcal{C}(p, g_1, \frac{\pi}{2}) \cap \partial_\infty \mathcal{C}(p, g_2, \frac{\pi}{2}) = \emptyset$, the bending map $\tilde{f}$ sends the endpoints of g into distinct points of $\partial_\infty \mathbb{H}^3$.

To finish the proof, it remains to show that

$$\overline{\partial_\infty \tilde{f}(\mathcal{C}(b_n, g_1, \frac{\pi}{2}))} \subset \partial_\infty \mathcal{C}(a_n, g_1, \frac{\pi}{2}) \tag{20}$$

where the bending map $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ is normalized to be the identity at the point a_n. The rest of the proof is divided into cases depending on the combinatorics of the intersection of $[a_n, b_n]$ with the edges of $\tilde{\tau}$.

6.3. **Case I:** $[a_n, b_n]$ **connects two long sides of an edge** E**.** Recall that l^* is the maximum of the diameters of the edges of the train track $\tilde{\tau}$. Then each $[a_n, b_n]$ has length less than or equal to $3l^*$ since it intersects at most three edges of $\tilde{\tau}$.

Assume that $[a_n, b_n]$ intersects interior of a single edge E of $\tilde{\tau}$ and that it connects the two long sides of E. Let P and Q be the plaques that contain a_n and b_n, respectively. By pre-composing the bending map with an isometry of $\mathbb{H}^2$, we can assume that $a_n = i \in \mathbb{H}^2$. Note that $\mathbb{H}^2$ is identified with $\{(z, t) : Im(z) = 0, t > 0\} \subset \mathbb{H}^3$ and under this identification $i \in \mathbb{H}^2$ corresponds to $j \in \mathbb{H}^3$.

Let $s(E)$ be a short side of E (which means that it is a lift to $\mathbb{H}^2$ of an arc in $\{k_1, \dots, k_n\}$, cf. §2). Then $|s(E)| \leq w^* \leq \frac{1}{20}$. Note that $s(E)$ is contained in ball of radius l^* centered at $a_n = i \in \mathbb{H}^2$ because the diameter of E is at most l^*.

Recall that

$$\psi_{P,Q} = \lim_{\mathcal{P}_l \to \mathcal{P}_{P,Q}} \psi_l$$

where $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ is a set of plaques between P and Q in the given order,

$$\psi_l = B_1 B_2 \dots B_n$$

and

$$B_i = R^{\beta(P,P_i)}_{g^P_{P_i}} R^{-\beta(P,P_i)}_{g^Q_{P_i}}$$

where $g^P_{P_i}$ is the geodesic on the boundary of P_i which separates P_i from P; similar for $g^Q_{P_i}$; and R^a_g is a hyperbolic rotation with the axis g and the rotation angle a.

Since points of $s(E)$ are on the distance at most l^* from $i \in \mathbb{H}^2$, it follows by Lemma 5.5 that

$$\|B_i - Id\| \leq 2e^{l^*}(1 + e^{l^*})|\beta(P, P_i)| \cdot |d_i| \tag{21}$$

where $|d_i|$ is the length of the gap $d_i = E \cap P_i$ and $\beta(P, P_i)$ is taken to be in the interval $(-\pi, \pi]$.

We estimate the norm of $\prod_{i=1}^l B_i$ for arbitrary l. By (21), we have that

$$\|\prod_{i=1}^l B_i\| \leq \prod_{i=1}^l \|B_i\| \leq \prod_d (1 + 2\pi e^{l^*}(1 + e^{l^*})|d|)$$

where the last product is over all gaps d of $s(E)$ with respect to $\tilde{\lambda}$. Then

$$\log \prod_{i=1}^{l} \|B_i\| \leq \sum_d \log(1 + 2\pi e^{l^*}(1 + e^{l^*})|d|) \leq$$
$$\leq \sum_d 2\pi e^{l^*}(1 + e^{l^*})|d| \leq 2\pi e^{l^*}(1 + e^{l^*})|s(E)|.$$

Since $|s(E)| \leq \frac{1}{20}$, we have

$$\|\prod_{i=1}^{l} B_i\| \leq e^{2\pi e^{l^*}(1+e^{l^*})|s(E)|} \leq C(l^*)$$

where

$$C(l^*) = e^{e^{l^*}}. \tag{22}$$

This implies

$$\|\psi_{P,Q} - Id\| \leq C(l^*) \sum_d \|B_d - Id\|,$$

where the sum is over all gaps d of $s(E)$ (i.e. components of $s(E) \setminus \tilde{\lambda}$) except the two components which contain the endpoints of $s(E)$; P_d is the plaque which contains d; and $B_d = R^{\beta(P,P_d)}_{g^P_{P_d}} \circ R^{-\beta(P,P_d)}_{g^Q_{P_d}}$.

We divide $\sum_d \|B_d - Id\|$ into two sums as follows. The first sum $\sum'$ is over finitely many gaps $\{d_i : i = 1, \ldots, n_{s(E)}\}$ of $s(E)$ used in the definition of $\|\beta\|_{var_\delta, s(E)}$ and the second sum $\sum''$ is over the remaining (infinitely many) gaps of $s(E)$.

Each term of $\sum'$ corresponding to a gap d of $s(E)$ is bounded from above by $2e^{l^*}(1 + e^{l^*})|d| \cdot \|\beta\|_{var_\delta}$ by (21). Thus

$$\sum{}' \leq 2e^{l^*}(1 + e^{l^*})|s(E)| \cdot \|\beta\|_{var_\delta}.$$

The term of the second sum $\sum''$ which corresponds to a gap d of $s(E)$ is bounded by $2\pi e^{l^*}(1+e^{l^*})|d|$ by (21). Since the sum of the lengths of all d in $\sum''$ is less than $\delta|s(E)|$, we obtain

$$\sum{}'' \leq 2\pi e^{l^*}(1 + e^{l^*})\delta|s(E)|.$$

Taking the two estimates together, we have

$$\|\psi_{P,Q} - Id\| \leq C'(l^*)(\|\beta\|_{var_\delta} + \delta)|s(E)|$$

where

$$C'(l^*) = 2\pi e^{l^*}(1 + e^{l^*})e^{e^{2l^*}}.$$

By Lemma 5.4 and by $\beta(P,Q) = \beta(s(E))$, we get that

$$\|R_Q - Id\| \leq (1 + e^{2l^*})|\beta(s(E))|/2 \leq \frac{1 + e^{2l^*}}{2}\|\beta\|_{max}$$

where $R_Q = R^{\beta(P,Q)}_{g^P_Q}$ as in the definition of $\varphi_{P,Q}$.

Thus

$$\|R_Q\| \leq 1 + \frac{1 + e^{2l^*}}{2}\|\beta\|_{max} \leq \frac{3 + e^{2l^*}}{2}$$

if we restrict to β with $\|\beta\|_{max} \leq 1$.

Consequently, we obtain

$$\|\varphi_{P,Q} - Id\| \leq C'(l^*)\frac{3+e^{2l^*}}{2}(\|\beta\|_{var_\delta} + \delta)|s(E)| + \frac{1+e^{2l^*}}{2}\|\beta\|_{max} \leq \tag{23}$$
$$\leq C''(l^*)(\|\beta\|_{var_\delta} + \delta)|s(E)| + \frac{1+e^{2l^*}}{2}\|\beta\|_{max}$$

where

$$C''(l^*) = C'(l^*)\frac{3+e^{2l^*}}{2} = \pi(1+e^{l^*})(3+e^{2l^*})e^{l^*+e^{2l^*}}.$$

The inequality (23) holds for both short sides $s_1(E)$ and $s_2(E)$ of the edge E. Without loss of generality we assume that $|s_1(E)| = \min\{|s_1(E)|, |s_2(E)|\}$. Lemma 2.2 implies that $|[a_n, b_n]| \geq \frac{1}{20e^{l^*}}|s_1(E)|$, where $|[a_n, b_n]|$ is the length of the hyperbolic arc $[a_n, b_n]$. Let $\eta''(l^*, 0)$ be the constant from Lemma 5.2 for the given l^* and $C = 0$. Lemma 5.2 implies the desired nesting of the cones if the right side of (23) is less than $\eta''(l^*, 0)|[a_n, b_n]|$. To achieve this, it is enough to set

$$\epsilon = \frac{1}{2}\min\{\frac{\eta''(l^*,0)}{60e^{l^*}C''(l^*)}, \frac{2\eta''(l^*,0)}{3(1+e^{2l^*})}\} = \frac{\eta''(l^*,0)}{120e^{l^*}C''(l^*)}$$

and

$$\delta = \frac{\eta''(l^*,0)}{120e^{l^*}C''(l^*)}$$

for $\eta''(l^*, 0)$ given by Lemma 5.2. The nesting of the cones at a_n and b_n is guaranteed by Lemma 5.2 because $|[a_n, b_n]| \leq l^*$.

6.4. **Case 2: $[a_n, b_n]$ connects long side of an edge to a long side of an adjacent edge.** Assume that $[a_n, b_n]$ enters an edge E_1 through a long side, then enters an edge E_2 through a short side in common with E_1 and exists E_2 through a long side of E_2.

Since the train track τ is bivalent we have that the set of geodesics of $\tilde{\lambda}$ which intersect the arc $[a_n, b_n]$ is either the set of geodesics which traverses the edge E_1 or the set of geodesics which traverses the edge E_2. For definiteness, assume that we are in the former case.

Let $s(E_1)$ be the short side of E_1 that contains one short side of E_2 and let $c_n = [a_n, b_n] \cap s(E_1)$. Normalize such that $a_n = i \in \mathbb{H}^2$. Let $s(E_1)^1$ and $s(E_1)^2$ be the two arcs obtained by dividing $s(E_1)$ with the point c_n such that the arcs $[a_n, c_n]$ and $s(E_1)^1$ have endpoints on the same long side l_1 of E_1, and that the arcs $[c_n, b_n]$ and $s(E_1)^2$ have endpoints on the same long side l_2 of E_2. Let h_1 and h_2 be two arcs from c_n orthogonal to l_1 and l_2, respectively.

It is immediate that $|[a_n, c_n]| \geq |h_1|$ and $|[c_n, b_n]| \geq |h_2|$. The hyperbolic sine rule and the fact that $|s(E_1)| \leq \frac{1}{20}$ give $|s(E_1)^1| \leq \cosh\frac{1}{20}|h_1| \leq \cosh\frac{1}{20}|[a_n, c_n]|$ and $|s(E_1)^2| \leq \cosh\frac{1}{20}|h_2| \leq \cosh\frac{1}{20}|[c_n, b_n]|$. We obtain

$$|s(E_1)| \leq \cosh\frac{1}{20}|[a_n, b_n]|.$$

Similar to Case 1, we get

$$\|\varphi_{P,Q} - Id\| \leq C''(2l^*)(\|\beta\|_{var_\delta} + \delta)|s(E_1)| + \frac{1+e^{4l^*}}{2}\|\beta\|_{max}$$

where we use $2l^*$ instead of l^* because the diameter of $E_1 \cup E_2$ is $2l^*$.

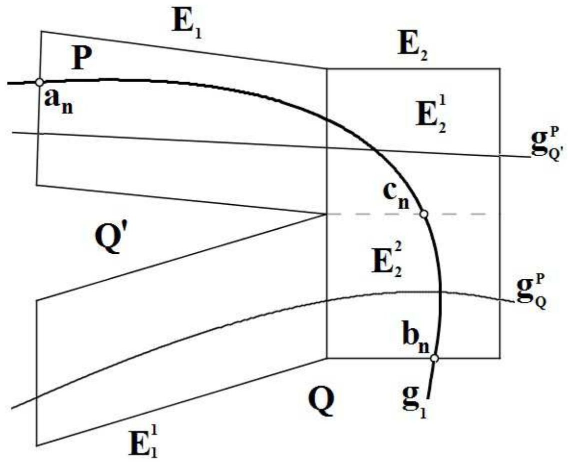

FIGURE 2.

The nesting of the cones at a_n and b_n follows as in Case 1 with the constants

$$\epsilon = \delta = \frac{\eta''(2l^*,0)}{120e^{2l^*}C''(2l^*)}$$

for $\eta''(2l^*,0)$ given by Lemma 5.2. The later case is dealt with in the same fashion with the same constants.

6.5. **Case 3:** $[a_n, b_n]$ **connects two short sides of an edge.** Assume that $[a_n, b_n]$ enters a short side of an edge E_1, then it enters a short side of an edge E_2 which is in common with E_1 and it exists a long side of E_2. See Figure 2 and Figure 3 for different possibilities of the relative positions of E_1, E_2 and g_1. Let P and Q be the plaques that contain a_n and b_n, respectively.

For the position in Figure 2 we argue as follows. Let E_1^1 be the incoming edge which meets E_2 at the same short side as E_1. Let Q' be the plaque that separates the geodesics of $\tilde{\lambda}$ that traverse E_1 from the geodesics of $\tilde{\lambda}$ that traverse E_1^1. Then we have

$$\varphi_{P,Q} = \varphi_{P,Q'} \circ \varphi_{Q',Q}.$$

Note that $\beta(Q', Q) = \beta(E_1^1)$ which implies that $|\beta(Q', Q)| \leq \|\beta\|_{max}$.

By reasoning as in Case 1, we obtain

$$\|\varphi_{Q',Q} - Id\| \leq C''(2l^*)(\|\beta\|_{var_\delta} + \delta)|s(E_1^1)| + \frac{1+e^{4l^*}}{2}\|\beta\|_{max}$$

where $s(E_1^1)$ is the short side of E_1^1 contained in a short side of E_2. Recall that we normalized such that $a_n = j$ and $b_n = e^{-m}j$ for $0 \leq m \leq 2l^*$. Since $[a_n, b_n]$ connects the short sides of the edge E_1, it follows that $|[a_n, b_n]| \geq l_* > 0$.

We apply Lemma 5.1 to $(e^{-m}j, -j) \in T^1\mathbb{H}^3$ with the constants $2l^*$, $C = 0$ and $m \geq l_* > 0$. We get

$$D_{T^1\mathbb{H}^3}(\varphi_{Q',Q}(e^{-m}j, -j), (e^{-m}j, -j)) \leq 18\pi[C''(2l^*)(\|\beta\|_{var_\delta} + \delta)|s(E_1^1)| + \\ + \frac{1+e^{4l^*}}{2}\|\beta\|_{max}] < \eta'(2l^*, 0) \tag{24}$$

whenever

$$\frac{C''(2l^*)(\|\beta\|_{var_\delta}+\delta)|s(E_1^1)|+\frac{1+e^{4l^*}}{2}\|\beta\|_{max}}{l_*}<\frac{\eta'(2l^*,0)}{18\pi}. \tag{25}$$

Recall that

$$\varphi_{P,Q'}=\psi_{P,Q'}\circ R_{g_{Q'}^P}^{\beta(P,Q')}$$

where, in general, $\beta(P,Q')\neq\beta(E_1)$ since a_n belongs to a short side of E_1. In fact $|\beta(E_1)|$ might not be even close to 0.

Let c_n be the point of the intersection between $[a_n,b_n]$ and the common boundary $s(E_1^1)$ of E_1 and E_2. It follows that the arc $[a_n,c_n]$ and the subarc of any geodesic of $\tilde{\lambda}$ that traverses E_1 are remaining close for the length l_*. This implies that they intersect at small angles. We give a numerical statement.

Lemma 6.1. *Let E be an edge of a train track such that the shortest geodesic connecting the short sides has length at least $l>0$ and the maximum of the lengths of the short sides is at most x. Let a_1 and a_2 be geodesic arcs in E connecting the short sides intersecting at an angle ϕ. Then*

$$\phi\leq\frac{\pi}{2}(\coth\frac{l}{2})x.$$

Proof. Let $A=a_1\cap a_2$. Then A divides a_1 into two sub arcs a_1' and a_2'. Without loss of generality, we can assume that the length of a_1' is at least $\frac{1}{2}l$. Then the length of the subarc a_2' of the arc a_2 which connects the short side of E which contains an endpoint of a_1' to the point A is at least $\frac{1}{2}l-x$. Let h be the geodesic arc issued from the endpoint of a_1' on a short side of E orthogonal to a_2'. The length of h is at most x. We obtained a right angled triangle with one angle φ whose opposite side has length at most x, and the side opposite to the right angle has the length at most $\frac{1}{2}l$. The hyperbolic sine rule gives

$$\sin\phi=\frac{\sinh x}{\sinh\frac{1}{2}l}$$

which implies

$$\phi\leq\frac{\pi}{2}(\coth\frac{l}{2})x.$$

□

We apply $R_{g_{Q'}^P}^{\beta(P,Q')}$ to $\varphi_{Q',Q}(e^{-m}j,-j)$. By Lemma 6.1, the angle of intersection ϕ between $g_{Q'}^P$ and g_1 satisfies

$$\phi\leq\frac{\pi}{2}(\coth\frac{l_*}{2})w^*. \tag{26}$$

By Lemma 5.3, we have

$$\begin{aligned}D_{T^1\mathbb{H}^3}([R_{g_{Q'}^P}^{\beta(P,Q')}\circ\varphi_{Q',Q}](e^{-m}j,-j),(e^{-m}j,-j))\leq\\ 20D_{T^1\mathbb{H}^3}(\varphi_{Q',Q}(e^{-m}j,-j),(e^{-m}j,-j))+40\sqrt{2}e^{2l^*}\phi\end{aligned} \tag{27}$$

when $D_{T^1\mathbb{H}^3}(\varphi_{Q',Q}(e^{-m}j,-j),(e^{-m}j,-j))<\frac{1}{4}$ and $\phi<\frac{e^{-2l^*}}{16}$. The former condition is satisfied because $D_{T^1\mathbb{H}^3}(\varphi_{Q',Q}(e^{-m}j,-j),(e^{-m}j,-j))<\eta'(2l^*,0)\leq\frac{1}{4}$. To achieve the later condition we require that $\frac{\pi}{2}(\coth\frac{l_*}{2})w^* < \frac{e^{-2l^*}}{16}$ which implies that

$$w^* < \frac{e^{-2l^*}\tanh\frac{l_*}{2}}{8\pi}. \tag{28}$$

By (24), (26) and (27) we have

$$D_{T^1\mathbb{H}^3}([R^{\beta(P,Q')}_{g^P_{Q'}}\circ\varphi_{Q',Q}](e^{-m}j,-j),(e^{-m}j,-j))\leq$$
$$\frac{560\pi}{l_*}[C''(2l^*)(\|\beta\|_{var_\delta}+\delta)|s(E_1^1)|+\frac{1+e^{4l^*}}{2}\|\beta\|_{max}]l_*+$$
$$+\frac{20\sqrt{2}\pi e^{2l^*}\coth\frac{l_*}{2}}{l_*}w^*l_*$$

Let $\eta''(2l^*,1)$ be the constant from Lemma 5.2. If

$$\delta,\|\beta\|_{var_\delta}\leq\frac{l_*}{4\cdot 560\pi|k_1|C''(2l^*)}\eta''(2l^*,1), \tag{29}$$

$$\|\beta\|_{max}\leq\frac{l_*}{4\cdot 560\pi 3e^{4l^*}}\eta''(2l^*,1) \tag{30}$$

and

$$w^*\leq\frac{l_*}{20\sqrt{2}\pi e^{2l^*}\coth\frac{l_*}{2}}\eta''(2l^*,1), \tag{31}$$

then

$$D_{T^1\mathbb{H}^3}([R^{\beta(P,Q')}_{g^P_{Q'}}\circ\varphi_{Q',Q}](e^{-m}j,-j),(e^{-m}j,-j))\leq\eta''(2l^*,1)l_*. \tag{32}$$

By Case 1, we immediately obtain the estimate

$$\|\psi_{P,Q'}-Id\|\leq C'(2l^*)(\|\beta\|_{var_\delta}+\delta)|s(E_1^1)|.$$

It follows that

$$\|\psi_{P,Q'}-Id\|\leq\eta''(2l^*,1)$$

if

$$\delta,\|\beta\|_{var_\delta}\leq\frac{l_*}{2C'(2l^*)|s(E_1^1)|}$$

which is satisfied because $C'(2l^*)\leq C''(2l^*)$ and by (29). Therefore, Lemma 5.2 and $w^*\leq 1/20$ implies the nesting of the cones if

$$\epsilon=\frac{l_*\eta''(2l^*,1)}{130\pi C''(2l^*)}\leq\min\{\frac{l_*\eta''(2l^*,1)}{4\cdot 560\pi C''(2l^*)|s(E_1^1)|},\frac{l_*\eta''(2l^*,1)}{4\cdot 560\pi\frac{1+e^{4l^*}}{2}}\}$$

and

$$\delta\leq\frac{l_*\eta''(2l^*,1)}{130\pi C''(2l^*)}$$

and (31) holds.

We consider the positions in Figure 3. The top left position in Figure 3 is a subcase of the position in Figure 3 where we set $\varphi_{Q'Q}=Id$ and the nesting follows for the same choices of ϵ, δ and w^*. The top right position is exactly dealt as with the top left position. The bottom position in Figure 3 is exactly equal to the position in Figure 2 and the nesting is achieved by choosing the same constants.

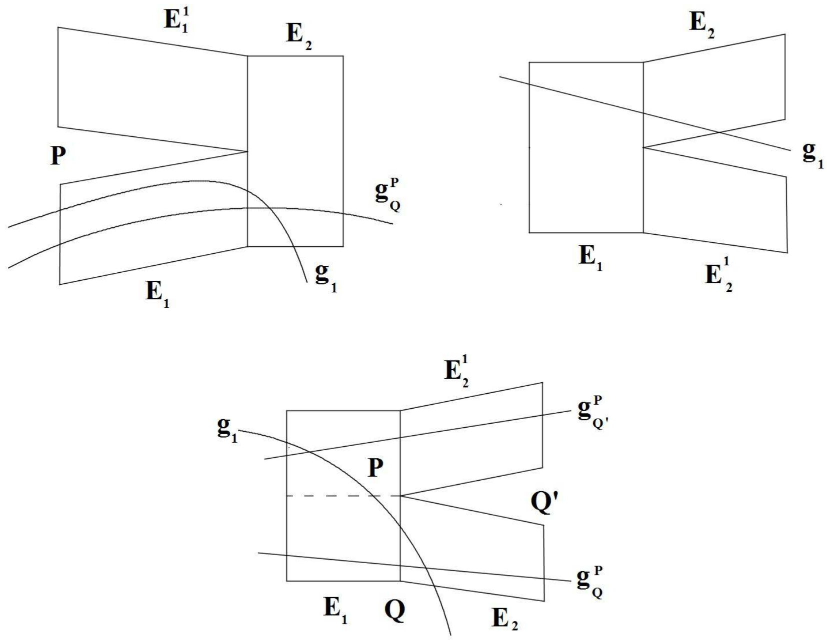

FIGURE 3.

6.6. **All other cases for** $[a_n, b_n]$**.**

Case 4. Assume that $[a_n, b_n]$ enters E on a short edge and that it exists E on the opposite short edge. The argument in this case is contained in the second part of Case 3 and the bounds are the same.

Case 5. Assume that $[a_n, b_n]$ enters an edge E_1 of $\tilde{\tau}$ through a long side, enters another edge E_2 through a short side in common with E_1 and then enters an edge E_3 through a short side in common with E_2, and then exists E_3 through a long side. Let P and Q be the plaques of $\tilde{\tau}$ which contain a_n and b_n, respectively. Note that the arc $[a_n, b_n]$ has length at least m_* because it traverses the edge E_2. Moreover, since the arc $[a_n, b_n]$ connects two long sides of different edges of $\tilde{\tau}$ it follows that the set of geodesics of $\tilde{\tau}$ that intersect $[a_n, b_n]$ is disjoint union of at most three sets of geodesics each of them traversing an edge of $\tilde{\tau}$. The situation in Figure 4 illustrates the case when this union consists of the geodesics traversing the edge above E_2, the edge E_2 and the edge below E_2. Note that the short sides of these three geodesics are on the distance at most $3l^*$ from $a_n = i$. Other possibilities can be easily checked by drawing pictures. It always happen that the set of geodesics of $\tilde{\lambda}$ intersecting $[a_n, b_n]$ is the disjoint union of at most three sets of geodesics traversing three edges of $\tilde{\tau}$ whose short sides are on the distance at most $3l^*$ from a_n. Therefore $\varphi_{P,Q}$ is the composition of at most three Möbis maps $\varphi_{E_i'}$, for $i = 1, 2, 3$, each corresponding to an edge E_i' of $\tilde{\tau}$.

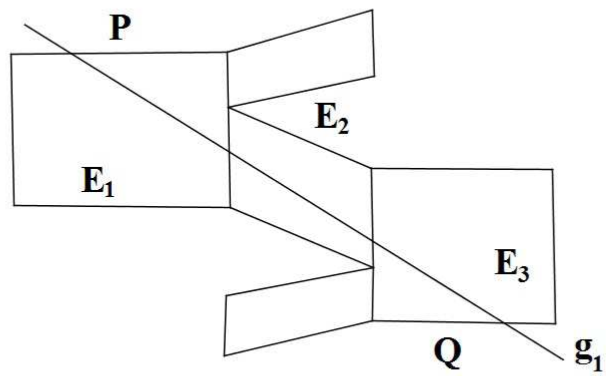

FIGURE 4.

We use the argument from Case 1 to estimate $\|\varphi_{E_i'}-Id\|$. Namely, it is enough to replace l^* with $3l^*$ to obtain

$$\|\varphi_{E_i'}-Id\|\leq C''(3l^*)(\|\beta\|_{var_\delta}+\delta)|s(E_i')|+\frac{1+e^{6l^*}}{2}\|\beta\|_{max}. \tag{33}$$

Consequently, we have

$$\begin{aligned}\|\varphi_{P,Q}-Id\|\leq 3\big(1+C''(3l^*)(\|\beta\|_{var_\delta}+\delta)|s(E_i')|+\frac{1+e^{6l^*}}{2}\|\beta\|_{max}\big)^2\times\\ \times\big(C''(3l^*)(\|\beta\|_{var_\delta}+\delta)|s(E_i')|+\frac{1+e^{6l^*}}{2}\|\beta\|_{max}\big).\end{aligned} \tag{34}$$

Assume that $\delta,\|\beta\|_{var_\delta}\leq\frac{1}{2}$ and $\|\beta\|_{max}\leq 1$. Then $1+C''(3l^*)(\|\beta\|_{var_\delta}+\delta)|s(E_i')|+\frac{1+e^{6l^*}}{2}\|\beta\|_{max}\leq 1+C''(3l^*)+\frac{1+e^{6l^*}}{2}$ because $|s(E_i')|\leq\frac{1}{20}$. We choose

$$\epsilon=\delta=\frac{m_*\eta''(3l^*,0)}{18C''(3l^*)(1+C''(3l^*)+\frac{1+e^{6l^*}}{2})^2}.$$

6.7. **The end of the proof.** We established that the cones are nested along the sequence $\{[a_n,b_n]\}_{n\in\mathbb{N}}$. Thus the bending map $\tilde{f}$ is injective on $\partial_\infty\mathbb{H}^2$ as claimed. We choose ϵ and δ to be the minimum over all cases and the nesting is guaranteed always. *This ends the proof of Theorem 1.1.*

Remark 6.2. The size of ϵ, δ and w^* depends on the above constants l^* and l_* (cf. Table 6.2). The minimum l_* of the distances between short sides of the edges of $\tilde{\tau}$ can be arbitrary small. In fact, when there are short closed geodesics contained in the geodesic lamination λ then the train track τ cannot be modified such that l_* is bigger than a universal positive constant. This fact forces us to include l_* as a part of the geometric information for the geodesic lamination λ.

If λ does not contain closed geodesics then there exists a choice of a geometric train track τ which carries λ such that $l_*\geq 1/20$ and $l^*=1/5$. In this case we

m_0	$\epsilon = \delta$	w^*
.000001	2.20317×10^{-17}	2.45816×10^{-20}
.00001	2.20241×10^{-16}	2.45807×10^{-18}
.0005	1.08066×10^{-14}	6.13315×10^{-15}
.001	2.1201×10^{-14}	2.44836×10^{-14}
.0015	3.1194×10^{-14}	5.4978×10^{-14}
.002	4.07961×10^{-14}	9.75434×10^{-14}
.0025	5.00174×10^{-14}	1.52107×10^{-13}
.003	5.8868×10^{-14}	2.18597×10^{-13}
.005	9.07579×10^{-14}	6.02374×10^{-13}
.01	1.4901×10^{-13}	2.36178×10^{-12}
.05	1.33635×10^{-13}	5.03139×10^{-11}
.1	2.06663×10^{-14}	1.64768×10^{-10}
.25	3.41015×10^{-19}	5.6501×10^{-10}
.5	9.94507×10^{-43}	8.30612×10^{-10}
1	$5.6123380 \times 10^{-550}$	4.479×10^{-10}
2	$1.90389 \times 10^{-212091}$	3.23146×10^{-11}

TABLE 1. Values of ϵ, δ and k^* for the given m_0.

explicitly compute the constants in Theorem 1.1 to be

$$w^* = 4.41719 \times 10^{-10}$$

and

$$\epsilon = \delta = 3.61749 \times 10^{-17}.$$

7. Holomorphic motions and shear-bend cocycle

Given a fixed hyperbolic surface S and a maximal geodesic lamination λ on S, we defined a geometric train track τ that carries λ (cf. §6). Then we found universal $\epsilon > 0$ and $\delta > 0$ such that when an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle β satisfies $\|\beta\|_{max} < \epsilon w_*$ and $\|\beta\|_{var_\delta} < \epsilon$ then the bending map $\tilde{f}$ with the bending cocycle β extends by continuity to an injection $\tilde{f} : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$.

We are considering holomorphic motions in this section, and injectivity of a family of maps is an essential part of the definition. To establish injectivity of a family of bending maps, we use the sufficient condition on β obtained in the previous section. When the hyperbolic metric on S is slightly changed, the metric quantities of the geometric train track τ are slightly changed. This fact is used to prove that there is an open neighborhood of any $\mathbb{R}$-valued transverse cocycle representing a hyperbolic metric on S in the space of $(\mathbb{C}/2\pi i\mathbb{Z})$-valued transverse cocycles whose points induce injective pleating maps.

Let $K \subset \hat{\mathbb{C}}$ and let $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$. A *holomorphic motion* of a set K is a map

$$f : K \times \mathbb{D} \to \hat{\mathbb{C}}$$

such that

$$f(\cdot, w) : K \to \hat{\mathbb{C}}$$

is injective for each $w \in \mathbb{D}$, $f(z,0) = z$ for all $z \in K$, and

$$f(z,\cdot) : \mathbb{D} \to \hat{\mathbb{C}}$$

is holomorphic in $w \in \mathbb{D}$ for each $z \in K$ (see [12]). The variable $w \in \mathbb{D}$ is called the *parameter* of the holomorphic motion of K. It is also possible to define holomorphic motions over simply connected regions of $\mathbb{C}$ when we specify the point where the motion is the identity.

The lambda lemma states that a holomorphic motion of K extends to a holomorphic motion of the closure $\bar{K}$ of K (see [12]). Slodkowski [15] proved that a holomorphic motion of a closed set K which contains at least three points extends to a holomorphic motion of $\hat{\mathbb{C}}$. In fact, if a holomorphic motion of K is invariant under a subgroup G of $PSL_2(\mathbb{C})$ then the extension of the holomorphic motion can be chosen to be G-equivariant on $\hat{\mathbb{C}}$ [6].

A *shear-bend transverse cocycle* β for a geodesic lamination λ on a closed hyperbolic surface S assigns to each arc k transverse to λ (with endpoints of k in the plaques of λ) a number $\beta(k) \in \mathbb{C}/2\pi i\mathbb{Z}$ such that if $k = k_1 \cup k_2$ and k_1, k_2 have disjoint interiors then $\beta(k) = \beta(k_1) + \beta(k_2)$. Denote by $\mathcal{H}(\lambda, \mathbb{C}/2\pi\mathbb{Z})$ the space of all shear-bend transverse cocycles for λ. Bonahon [3] proved that the space of all representations of the fundamental group $\pi_1(S)$ of S in $PSL_2(\mathbb{C})$ which realize λ is homeomorphic to an open subset of $\mathcal{H}(\lambda, \mathbb{C}/2\pi i\mathbb{Z})$, where the real part is restricted to belong to the image of $T(S)$ in $\mathcal{H}(\lambda, \mathbb{R})$ and there is no restrictions on the imaginary part.

Let $\alpha \in \mathcal{H}(\lambda, \mathbb{R})$ be in the image of the Teichmüller space $T(S)$. For $w = u+iv \in \mathbb{C}$, we define $\beta_w(k) = (w\alpha(k)) \pmod{2\pi \mathrm{i}\mathbb{Z}}$ for each arc k transverse to λ. Then $\beta_w \in \mathcal{H}(\lambda, \mathbb{C}/2\pi i\mathbb{Z})$. Let $f_w : \mathbb{H}^2 \to \mathbb{H}^3$ be the shear-bend map corresponding to β_τ as in [3].

Theorem 7.1. *Let $\alpha \in \mathcal{H}(\lambda, \mathbb{R})$ be in the image of $T(S)$ and let $f_{(1+w)}$ be the shear-bend map for $\beta_{(1+w)} \in \mathcal{H}(\lambda, \mathbb{C}/2\pi i\mathbb{Z})$. Then there exists $r > 0$ such that the shear-bend map*

$$f_{(1+w)} : \mathbb{H}^2 \to \mathbb{H}^3$$

extends by continuity to a holomorphic motion of $\partial_\infty \mathbb{H}^2$ in $\partial_\infty \mathbb{H}^3$ for the parameter $\{w \in \mathbb{C} : |w| < r\}$.

Proof. For $w = u + iv$, consider the hyperbolic surface S_{1+u} obtained by shearing along the real part of β_{1+w} which is $(1 + u)\alpha \in \mathcal{H}(\lambda, \mathbb{R})$. By [3], the image of $T(S)$ is a cone in $\mathcal{H}(\lambda, \mathbb{R})$ and therefore S_{1+u} exists for r small enough. Note that S_1 is the original hyperbolic surface S.

Let τ be the train track that carries λ used in the proof of Theorem 1.1. We choose τ such that $w^* = \frac{1}{2} \cdot \frac{e^{-2l^*} \tanh \frac{l_*}{2}}{8\pi}$. For $|u|$ small enough, the endpoints of the switches of τ under the shear map $f_{(1+u)}$ are close to the switches of τ. By connecting the switches with geodesics for the hyperbolic metric of $(1 + u)\alpha$ we construct a train track τ_{1+u} which is homotopic to τ. For $|u|$ small enough, we have the constants $l^*(\tau_{1+u})$, $l_*(\tau_{1+u})$ and $w^*(\tau_{1+u})$ are as close as we need to the original constants l^*, l_* and w^* of the train track $\tau = \tau_1$. The constants $w^*(\tau_{1+u})$ and $\epsilon(\tau_{1+u}) = \delta(\tau_{1+u})$ from the proof of Theorem 1.1 depend continuously on $l^*(\tau_{1+u})$, $l_*(\tau_{1+u})$ and $w^*(\tau_{1+u})$. Thus they depend continuously on u and are bounded away from 0 for u small enough. Then the proof of Theorem 1.1 applies to each β_{1+w} to obtain an injective map map for $|w|$ small enough where the bound on $|Im(w)|$ is obtained from Theorem 1.1. It is clear that when $w = 0$, we have $f_{(1+0)} = id$.

Finally, we fix $z \in \partial_\infty \mathbb{H}^2$ and consider $w \mapsto f_{(1+w)}(z)$. Bonahon [3] proved that the shear-bend map is holomorphic in the transverse cocyle when restricted to a single plaque of $\tilde{\lambda}'$. Since the endpoints of the plaques are dense in $\partial_\infty \mathbb{H}$, it follows that $w \mapsto f_{(1+w)}(z)$ is holomorphic in w for z in a dense subset of $\partial_\infty \mathbb{H}$. By the lambda lemma, this is enough to claim that $f_{(1+w)}$ extends to a holomorphic motion of $\partial_\infty \mathbb{H}^2$ for w in the described neighborhood of $0 \in \mathbb{C}$. □

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