INFINITESIMAL LIOUVILLE DISTRIBUTIONS FOR TEICHMÜLLER SPACE

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1. Introduction

This paper is part of a project which undertakes to study Teichmüller spaces of hyperbolic surfaces with infinite area using Liouville currents. Liouville currents have been a useful tool in the theory of Teichmüller spaces of hyperbolic surfaces with finite area. They were introduced by Bonahon [6] to give a more natural description of Thurston's boundary to the Teichmüller space of a finite surface. Given a Liouville current, it is possible to recover almost all information on the corresponding point in the Teichmüller space. For instance, Otal [13] uses Liouville currents to show that the marked length spectrum of a negatively curved surface determines its metric up to isotopy.

While the results of Bonahon [6] offer a better understanding of the geometry of Teichmüller spaces of finite surfaces, they have yet to be extended to Teichmüller spaces of infinite surfaces. The straightforward application of Liouville currents to infinite surfaces is not possible because Teichmüller spaces of infinite-type surfaces are infinite-dimensional.

In this paper, we analyze variations of Liouville currents as we vary points in the Teichmüller space. It turns out that derivatives of Liouville currents are Hölder distributions. In order to prove this we introduce a new topology on the space of Hölder distributions. When restricted to measures, the new topology is different from the standard weak* topology used by Bonahon [6]. In a related paper [15], the author uses the new topology to propose a Thurston-type boundary for the Teichmüller space of an infinite surface. The existence of such boundary is known only for Teichmüller spaces of finite surfaces [9].

Consider an infinite surface X, in other words X is a Riemann surface whose universal covering $\tilde{X}$ is conformally equivalent to the hyperbolic plane $\mathbb{H}^2$ and which has infinite hyperbolic area. An important example is $X = \mathbb{H}^2$. A Liouville current is a covering group invariant measure on the space $G(\tilde{X})$ of geodesics in the universal covering $\tilde{X}$. The Liouville map

$$L : \mathcal{T}(X) \to \{\text{measures on } G(\tilde{X})\}$$

is defined by $L : m \mapsto L_m$ where $m \in \mathcal{T}(X)$ and L_m is the Liouville current corresponding to the class of hyperbolic metrics m. The topology on the Teichmüller space $\mathcal{T}(X)$ comes from its structure as a complex Banach manifold. In the paper [15], one of the key points was to identify a natural topology on the space of

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measures on $G(\tilde{X})$ for which the Liouville map is a topological embedding. When X is a finite surface, such a topology is the classical weak* topology, but its definition is more elaborate for an infinite surface. The construction of the natural topology on the space of measures on $G(\tilde{X})$ is directly inspired by the results of the current paper.

We investigate the differentiability of the Liouville map L. The space of geodesics $G(\tilde{X})$ is a topological object associated to $\mathcal{T}(X)$. It has no differentiable structure, but it has a preferred class of Hölder equivalent metrics. The class of metrics allows us to define Hölder continuous functions on $G(\tilde{X})$. The space of Hölder distributions $\mathcal{H}(\tilde{X})$ is the dual to the space of Hölder continuous functions with compact supports in $G(\tilde{X})$. The space of all measures on $G(\tilde{X})$ embeds, via integration, into $\mathcal{H}(\tilde{X})$. The space $\mathcal{H}(X)$ consists of all Hölder distributions in $\mathcal{H}(\tilde{X})$ which are invariant under the covering group.

We introduce a topological vector space structure on $\mathcal{H}(\tilde{X})$. We fix one metric in the preferred class of Hölder equivalent metrics on $G(\tilde{X})$. For the fixed metric and for the given ν, $0<\nu\leq 1$, we introduce the space $\mathcal{H}^{\nu}(\tilde{X})$ of Hölder distributions on ν-Hölder continuous functions with compact supports in $G(\tilde{X})$. The space $\mathcal{H}^{\nu}(\tilde{X})$ is a Banach space for ν-norm $\|\cdot\|_{\nu}$, given by

$$\|W\|_{\nu}=\sup_{(\varphi,Q)\in test(\nu)}|W(\varphi)|$$

where $W\in\mathcal{H}^{\nu}(\tilde{X})$ and $test(\nu)$ is the space of ν-test functions. By our definitions $\mathcal{H}(\tilde{X})=\cap_{0<\nu\leq 1}\mathcal{H}^{\nu}(\tilde{X})$ and consequently $\mathcal{H}(\tilde{X})\subset\mathcal{H}^{\nu}(\tilde{X})$ for each ν, $0<\nu\leq 1$. Thus, the space $\mathcal{H}(\tilde{X})$ inherits the family of ν-norms $\|\cdot\|_{\nu}$, $0<\nu\leq 1$, and $\mathcal{H}(\tilde{X})$ is incomplete for any $\|\cdot\|_{\nu}$. However, if we endow $\mathcal{H}(\tilde{X})$ with the topology coming from the family of ν-norms $\|\cdot\|_{\nu}$, $0<\nu\leq 1$, then $\mathcal{H}(\tilde{X})$ is a Fréchet space. In other words, $\mathcal{H}(\tilde{X})$ is a metrizable, complete topological vector space with a translation invariant metric and with a convex local base. The space $\mathcal{H}(X)$ as a subspace of $\mathcal{H}(\tilde{X})$ is also a Fréchet space.

We show the differentiability of L in the Fréchet sense.

Theorem 1. *The Liouville map $L:\mathcal{T}(X)\to\mathcal{H}(X)$ has a tangent map at each $m_0\in\mathcal{T}(X)$. Namely, there is a continuous linear map*

$$T_{m_0}L:T_{m_0}\mathcal{T}(X)\to\mathcal{H}(X)$$

such that, if $\mathcal{B}:A\to\mathcal{T}(X)$ is a chart locally modelling $\mathcal{T}(X)$ on a Banach space and if $\mathcal{B}(q_0)=m_0$, then

$$L\circ\mathcal{B}(q_0+h)=L\circ\mathcal{B}(q_0)+T_{m_0}L\circ T_{q_0}\mathcal{B}(h)+o(h)$$

with $\lim_{h\to 0}\frac{o(h)}{\|h\|}=0$ in $\mathcal{H}(X)$. The tangent map varies continuously with $m_0\in\mathcal{T}(X)$.

In particular, if $t\mapsto m_t$, $t\in(-\epsilon,\epsilon)$, is a differentiable curve in $\mathcal{T}(X)$ with the tangent vector $v\in T_{m_0}\mathcal{T}(X)$ at $t=0$, and if φ is a Hölder continuous function with compact support, then the derivative $\frac{d}{dt}\int_{G(\tilde{X})}\varphi dL_{m_t}|_{t=0}$ exists and it is equal to $T_{m_0}L(v)$. The Hölder continuity of φ is here crucial.

We also establish an explicit formula for the tangent map $T_{m_0}L$. To do so, we fix an identification $(\tilde{X}, m_0)\cong\mathbb{H}^2$. Then $G(\tilde{X})\cong\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}-\Delta$ where Δ is the diagonal of $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$. We realize the tangent vector $v\in T_{m_0}\mathcal{T}(X)$ by a Beltrami differential λ on $(\tilde{X}, m_0)\cong\mathbb{H}^2$.

Theorem 2. *If $t\mapsto m_t$, $t\in(-\epsilon,\epsilon)$, is a differentiable curve in $\mathcal{T}(X)$ with the tangent vector $v\in T_{m_0}\mathcal{T}(X)$ at $t=0$, and if $\varphi: G(\tilde{X})\to\mathbb{R}$ is a Hölder continuous function with compact support then*

$$T_{m_0}L(v)(\varphi)=\frac{d}{dt}L_{m_t}(\varphi)\Big|_{t=0}=-\frac{2}{\pi}Re\int_{\mathbb{H}^2}\lambda(\zeta)\Big[\int_{G(\mathbb{H}^2)}\frac{\varphi(x,y)}{(\zeta-x)^2(\zeta-y)^2}dxdy\Big]d\xi d\eta,$$

where $\zeta=\xi+i\eta$ and the vector $v\in T_{m_0}\mathcal{T}(X)$ is realized by the Beltrami differential λ on $\mathbb{H}^2\cong(\tilde{X},m_0)$.

For finite surfaces, Theorem 1 was proved by Bonahon and Sözen [8]. They also gave a representation of the tangent map in terms of shear coordinates for $\mathcal{T}(X)$. A key ingredient in their proof is a Poincaré series argument which does not extend to infinite surfaces. We consequently have to develop new techniques to deal with infinite surfaces. Our method has the advantage of simultaneously proving the differentiability for all surfaces, finite and infinite. Also, our approach is coordinate independent.

The paper is organized as follows. In section 2 we give necessary background on Teichmüller theory. In section 3 we introduce the space of Hölder distributions and give it a Fréchet space structure. In section 4 we define the Liouville map. In section 5 we prove Theorem 1 and Theorem 2.

2. Teichmüller Theory

Let X denote a Riemann surface with the universal covering $\tilde{X}$ conformally equivalent to the hyperbolic plane $\mathbb{H}^2$. A Riemann surface is of finite type if it is compact, or if it is compact with finitely many points removed, and it is of infinite type otherwise. The Teichmüller theory is different in the finite and in the infinite case. In the finite case the Teichmüller space is finite-dimensional, and in the non-finite case the Teichmüller space is infinite-dimensional. Our methods work in both cases.

We fix an isometric identification of the universal covering $\tilde{X}$ with the hyperbolic plane $\mathbb{H}^2$. The fundamental group $\pi_1(X)$ is then identified with a Fuchsian group Γ, and the identification $\tilde{X}\cong\mathbb{H}^2$ induces an isometry $X\cong\mathbb{H}^2/\Gamma$. The *Teichmüller space* $\mathcal{T}(X)$ of the Riemann surface X is the space of equivalence classes $[f]$ of quasiconformal maps $f:\mathbb{H}^2\to\mathbb{H}^2$ such that $f\Gamma f^{-1}=\Gamma_f$ is a Fuchsian group. The boundary of the hyperbolic plane $\partial_\infty\mathbb{H}^2$ is identified with $\widehat{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$. Two such maps f_1 and f_2 are equivalent if $f_1|_{\widehat{\mathbb{R}}}=\Theta\circ f_2|_{\widehat{\mathbb{R}}}$ for some Möbius map Θ. A quasiconformal map f which conjugates Γ to another Fuchsian group is said to be Γ-*invariant*.

A *Beltrami differential* μ on an open set $\Omega\subset\mathbb{C}$ is a measurable function on Ω with $\|\mu\|_{\infty}<\infty$. A *Beltrami coefficient* μ on an open set $\Omega\subset\mathbb{C}$ is a measurable function on Ω with $\|\mu\|_{\infty}<1$. To any quasiconformal map f of $\mathbb{H}^2$ is associated a Beltrami coefficient μ defined by the *Beltrami equation* $f_{\bar{z}}=\mu f_z$. If f is Γ-invariant then

$$\mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma(z)}=\mu(z) \tag{1}$$

for all $\gamma\in\Gamma$ and for all $z\in\mathbb{H}^2$. Conversely, given a Beltrami coefficient μ which satisfies (1), there exists a unique quasiconformal map $f^{\mu}:\mathbb{H}^2\to\mathbb{H}^2$ which is onto, and which satisfies the Beltrami equation with the given μ, and which fixes 0, 1 and ∞. Such quasiconformal map f^{μ} conjugates Γ onto another Fuchsian group $\Gamma_{f^{\mu}}$. Any other quasiconformal map $\mathbb{H}^2\to\mathbb{H}^2$ with the same Beltrami coefficient μ but which does not fix 0, 1 and ∞ differs from f^{μ} by postcomposition with a Möbius map. More details on quasiconformal maps can be found in [1], [11] and [12], among others.

Let $M(\Gamma)$ be the space of all Beltrami coefficients on $\mathbb{H}^2$ which satisfy (1). Solving the Beltrami equation for each $\mu\in M(\Gamma)$ and taking the equivalence class of the solution, we obtain a continuous map from $M(\Gamma)$ onto $\mathcal{T}(X)$. The solution to the Beltrami equation in the most general form was given by Ahlfors and Bers (see [3] and [1]). For more on the Teichmüller theory see, for example, [1], [11], [2], etc.

The Teichmüller space $\mathcal{T}(X)$ is a complex Banach manifold. The manifold structure was given by Bers [5] in the following way.

The space $B(\Gamma)$ of holomorphic quadratic differentials for Γ consists of all holomorphic functions q in the lower half plane $\mathbb{H}^2_-$ which satisfy $q\circ\gamma(z)\gamma'(z)^2=q(z)$ for all $z\in\mathbb{H}^2_-$ and for all $\gamma\in\Gamma$, and define $\|q\|_{\mathcal{B}(\Gamma)}:=\|q(z)y^2\|_{\infty}<\infty$ where $y=Im(z)$.

Fix $\mu\in M(\Gamma)$. Define a Beltrami coefficient $\tilde{\mu}$ on $\mathbb{C}$ by $\tilde{\mu}(z)=\mu(z)$ for $z\in\mathbb{H}^2$ and $\tilde{\mu}(z)=0$ for $z\in\mathbb{H}^2_-$. Solve the Beltrami equation for $\tilde{\mu}$ in $\mathbb{C}$. The solution f_{μ} which fixes 0, 1 and ∞ is conformal in the lower half plane $\mathbb{H}^2_-$ because its Beltrami coefficient is 0 on $\mathbb{H}^2_-$. We form the Schwarzian derivative of f_{μ}

$$S(f_{\mu})(z)=q_{\mu}(z):=\frac{f_{\mu}'''(z)}{f_{\mu}'(z)}-\frac{3}{2}\Big(\frac{f_{\mu}''(z)}{f_{\mu}'(z)}\Big)^2$$

for $z\in\mathbb{H}^2_-$.

Bers defined map $\tilde{\mathcal{B}}^{-1}:M(\Gamma)\to B(\Gamma)$ by $\tilde{\mathcal{B}}^{-1}(\mu)=q_{\mu}$, and showed that $\tilde{\mathcal{B}}^{-1}(\mu_1)=\tilde{\mathcal{B}}^{-1}(\mu_2)$ if and only if $[f^{\mu_1}]=[f^{\mu_2}]$. Thus $\tilde{\mathcal{B}}^{-1}$ factors through a map $\mathcal{B}^{-1}$ from $\mathcal{T}(X)$ to $B(\Gamma)$, which is an embedding onto an open bounded subset A of $B(\Gamma)$. The map $\mathcal{B}^{-1}$ is consistent with a Banach manifold structure on $\mathcal{T}(X)$, namely $\mathcal{B}^{-1}$ is a single chart for $\mathcal{T}(X)$.

Ahlfors and Weil defined a mapping from $B(\Gamma)$ to Beltrami differentials on $\mathbb{H}^2$ which satisfy (1) by

$$\mathcal{AW}(q)=-2y^2q(\bar{z}) \tag{2}$$

for $q\in B(\Gamma)$. Beltrami differentials λ in the image of $\mathcal{AW}$ are called *harmonic* Beltrami differentials. Note that

$$\|q\|_{B(\Gamma)}=\frac{1}{2}\|\lambda\|_{\infty} \tag{3}$$

for $\lambda = \mathcal{AW}(q)$. By (3), if $\|q\|_{B(\Gamma)} < \frac{1}{2}$ then $\mathcal{AW}(q) = \lambda \in M(\Gamma)$, in other words λ is a Beltrami coefficient on $\mathbb{H}^2$ which satisfy (1). Ahlfors and Weil proved (see [4], [1] or [11]) that

$$S(f_\lambda) = q$$

for all q with $\|q\|_{B(\Gamma)} < 2$ where $\lambda = \mathcal{AW}(q)$. Thus the inverse image $\mathcal{B}$ of $\mathcal{B}^{-1}$ on the neighborhood $\{q; \|q\|_{B(\Gamma)} < \frac{1}{2}\}$ of $0 \in B(\Gamma)$ is given by

$$\mathcal{B}(q) = [f^\lambda] \tag{4}$$

where $\lambda = \mathcal{AW}(q)$.

To describe the tangent space of $\mathcal{T}(X)$, we consider a one parameter family of Beltrami coefficients $\mu + t\lambda \in M(\Gamma)$. The equivalence classes $[f^{\mu+t\lambda}]$ of the one parameter family of solutions $f^{\mu+t\lambda}$ to the Beltrami equations with coefficients $\mu + t\lambda$ gives a path in $\mathcal{T}(X)$. For a fixed $z \in \mathbb{C}$, the map $f^{\mu+t\lambda}(z)$ is differentiable in t. Its derivative $\dot{f}^\mu[\lambda](z) := \frac{d}{dt} f^{\mu+t\lambda}(z)|_{t=0}$ represents a tangent vector at the point $[f^\mu] \in \mathcal{T}(X)$.

A *Beltrami differential* λ *on* $f^\mu(X)$ is a measurable function on $\mathbb{H}^2$ with $\|\lambda\|_\infty < \infty$ which satisfies

$$\lambda \circ \gamma(z) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \lambda(z)$$

for all $\gamma \in \Gamma_{f^\mu}$ and for all $z \in \mathbb{H}^2$. The space of tangent vectors at $[f^\mu] \in \mathcal{T}(X)$ can be identified with the space of equivalence classes of Beltrami differentials on $f^\mu(\mathbb{H}^2) = \mathbb{H}^2$. Two Beltrami differentials λ_1 and λ_2 are equivalent ($\lambda_1 \sim \lambda_2$) if

$$\dot{f}^\mu[\lambda_1](z) = \dot{f}^\mu[\lambda_2](z)$$

for all $z \in \widehat{\mathbb{R}}$. We denote by $\lambda/\sim$ the equivalence class of a Beltrami differential λ.

The image of the path $[f^{\mu+t\lambda}]$ in the chart $\mathcal{B}^{-1}$ is a differentiable path through the point $\mathcal{B}^{-1}([f^\mu]) = q_\mu$. The tangent vector at q_μ to the path $\mathcal{B}^{-1}([f^{\mu+t\lambda}])$ is a holomorphic quadratic differential $q_\mu^\lambda \in B(\Gamma)$ and it maps to the tangent vector $\lambda/\sim$ at $[f^\mu]$ under the tangent map $T_{q_\mu}\mathcal{B}$.

By (2) and (4) we get

$$\mathcal{B}(tq) = [f^{t\lambda}]$$

for $\lambda = \mathcal{AW}(q)$. Thus the tangent map $T_0\mathcal{B}$ at the base point $0 \in B(\Gamma)$ is given by

$$T_0\mathcal{B}(q) = \mathcal{AW}(q)/\sim \tag{5}$$

for $q \in B(\Gamma)$.

Any K-quasiconformal mapping f of $\widehat{\mathbb{C}}$ is $\frac{1}{K}$-Hölder continuous, i.e.

$$k(f(z_1), f(z_2)) \le C[k(z_1, z_2)]^{\frac{1}{K}} \tag{6}$$

for every $z_1, z_2 \in \widehat{\mathbb{C}}$, where C is a constant and k is a Riemannian metric on the sphere $\widehat{\mathbb{C}}$. We say that a family of K-quasiconformal maps is *uniformly Hölder continuous* if (6) holds for all mappings in the family with the fixed constant $C > 0$ and with the fixed metric k. There is a useful criteria for a family of K-quasiconformal mappings to be uniformly Hölder continuous. Namely, a family of K-quasiconformal mappings is uniformly Hölder continuous if and only if there exist three points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ and constant $C_1 > 0$ such that $k(f(z_i), f(z_j)) \ge C_1$ for all mappings f in the family (see [12]).

3. Hölder Distributions

We define an angle metric on the boundary $\partial_\infty \tilde{X} \cong \widehat{\mathbb{R}}$ of the universal cover $\tilde{X} \cong \mathbb{H}^2$. Fix a base point $\tilde{x}_1 \in \tilde{X}$. The angle distance between $\tilde{b}_1, \tilde{b}_2 \in \partial_\infty \tilde{X} \cong \widehat{\mathbb{R}}$ is the angle at $\tilde{x}_1$ between the hyperbolic geodesic rays connecting $\tilde{x}_1$ to $\tilde{b}_1$ and $\tilde{b}_2$.

In general, given any set M with two metrics ρ_1 and ρ_2 on it, we say that ρ_1 is *Hölder equivalent* to ρ_2 if there exists $C > 0$ and ν, $0 < \nu \leq 1$, such that

$$\frac{1}{C}\rho_1(x,y)^{\frac{1}{\nu}} \leq \rho_2(x,y) \leq C\rho_1(x,y)^{\nu}$$

for all $x, y \in M$. If $\nu = 1$ in the above inequality, then we say that ρ_1 is *Lipschitz equivalent* to ρ_2.

The angle metric on the boundary $\partial_\infty \tilde{X}$ depends on the choice of $\tilde{x}_1 \in \tilde{X}$. Two different choices give Lipschitz equivalent metrics. But this is not the end of ambiguities. Since we are interested in the Teichmüller space $\mathcal{T}(X)$ we also change the hyperbolic metric on X such that the identity map on X is quasiconformal. We can identify boundaries of universal coverings for different hyperbolic metrics m_1 and m_2, $\partial_\infty(\tilde{X}, m_1) \cong \partial_\infty(\tilde{X}, m_2) \equiv \widehat{\mathbb{R}}$. The identifying map is Hölder bi-continuous with respect to the angle metrics for different hyperbolic metrics. This follows from the fact that quasiconformal maps of $\mathbb{H}^2$ extend to quasisymmetric maps of $\widehat{\mathbb{R}}$. Both classes of maps are Hölder continuous (see [1] or [12]). For this reason, we consider $\tilde{X} \cong \mathbb{H}^2$ with class of metrics on $\partial_\infty(\tilde{X}) \cong \partial_\infty \mathbb{H}^2 = \widehat{\mathbb{R}}$ which are Hölder equivalent to the standard angle metric, where the standard angle metric on $\widehat{\mathbb{R}}$ is the one corresponding to the choice of base point $i \in \mathbb{H}^2$.

By the above identifications, the space of oriented geodesics $G(\tilde{X})$ in $\tilde{X}$ is identified with $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}} - \Delta \cong G(\mathbb{H}^2)$. Let d be the product metric on $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}} - \Delta$ coming from the standard angle metric on $\widehat{\mathbb{R}}$. We consider the class of all metrics d_1 which are Hölder equivalent to d, i.e. for each d_1 there exists $C > 0$ and ν, $0 < \nu \leq 1$, such that

$$\frac{1}{C}d_1((x,y),(x_1,y_1))^{\frac{1}{\nu}} \leq d((x,y),(x_1,y_1)) \leq Cd_1((x,y),(x_1,y_1))^{\nu}$$

for all $(x,y),(x_1,y_1) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} - \Delta$.

A function $\varphi : G(\tilde{X}) \to \mathbb{R}$ is Hölder continuous with respect to metric d if there exists $C > 0$ and ν, $0 < \nu \leq 1$, such that

$$|\varphi(x,y) - \varphi(x_1,y_1)| \leq Cd((x,y),(x_1,y_1))^{\nu}$$

for all $(x,y),(x_1,y_1) \in G(\tilde{X})$. If we want to specify the Hölder exponent of φ then we say that φ is ν-Hölder continuous function.

For the standard metric d in the above class, we consider the space $H(\tilde{X})$ of all Hölder continuous functions $\varphi : G(\tilde{X}) \to \mathbb{R}$ with compact support. Since in our class of metrics any other metric d_1 is Hölder equivalent to d, it follows that any $\varphi \in H(\tilde{X})$ is also Hölder continuous for any d_1 in our class. Thus the space $H(\tilde{X})$ is an invariant for the above class of metrics. The idea of considering such space and linear functionals on it is due to Bonahon [7].

We fix the standard metric d. If φ is a ν-Hölder continuous function with compact support, then the ν-norm of φ is given by

$$\|\varphi\|_\nu = \max\{\max_{(x,y)} |\varphi((x,y))|,$$
$$\sup_{(x,y)\neq(x_1,y_1)} |\varphi(x,y)-\varphi(x_1,y_1)|d((x,y),(x_1,y_1))^{-\nu}\}$$

where $(x,y),(x_1,y_1)\in\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}-\Delta$.

The set $test(\nu)$ consists of all pairs (φ,Q) where

$$Q=[a,b]\times[c,d]$$

with

$$\frac{(a-c)(b-d)}{(a-d)(b-c)}=2$$

and φ is a ν-Hölder continuous function with respect to d whose support is in Q such that

$$\|\varphi\circ\Theta_Q\|_\nu\leq 1,$$

where Θ_Q is a Möbius transformation which maps -2, -1, 1 and 2 onto a, b, c and d, respectively. Note that the existence of such Θ_Q follows from the condition $\frac{(a-c)(b-d)}{(a-d)(b-c)}=2$.

We are ready to define our main object $\mathcal{H}(\tilde{X})$ and a family of ν-norms on it using $test(\nu)$. The set $\mathcal{H}(\tilde{X})$ consists of all real linear functionals W on the vector space $H(\tilde{X})$ such that $\sup_{(\varphi,Q)\in test(\nu)}|W(\varphi)|<\infty$ for each ν with $0<\nu\leq 1$.

Further, the set $\mathcal{H}(X)$ consists of all Γ-*invariant* $W\in\mathcal{H}(\tilde{X})$, namely $W(\varphi\circ\gamma)=W(\varphi)$ for all $\gamma\in\Gamma$ and for all $\varphi\in H(\tilde{X})$.

We fix ν and denote by $H^\nu(\tilde{X})$ the vector space of all functions in $H(\tilde{X})$ which are ν-Hölder continuous. The space $\mathcal{H}^\nu(\tilde{X})$ consists of all real linear functionals W on $H^\nu(\tilde{X})$ such that $\sup_{(\varphi,Q)\in test(\nu)}|W(\varphi)|<\infty$. We introduce the ν-norm on $\mathcal{H}^\nu(\tilde{X})$ by

$$\|W\|_\nu=\sup_{(\varphi,Q)\in test(\nu)}|W(\varphi)| \tag{7}$$

for $W\in\mathcal{H}^\nu(\tilde{X})$. The ν-norm is a norm on $\mathcal{H}^\nu(\tilde{X})$ because any $\varphi\in H^\nu(\tilde{X})$ can be written as a finite linear combination of functions in $test(\nu)$. It is not hard to see that $\mathcal{H}^\nu(\tilde{X})$ is a Banach space for the ν-norm.

The space $\mathcal{H}^\nu(X)$ consists of all functionals in $\mathcal{H}^\nu(\tilde{X})$ which are Γ-invariant and $\mathcal{H}^\nu(X)$ is also a Banach space for the ν-norm.

Assume $\nu^{'}>\nu$. Any $\nu^{'}$-Hölder continuous function with compact support is ν-Hölder continuous. Thus we have the inclusion $H^{\nu^{'}}(\tilde{X})\subset H^\nu(\tilde{X})$. For $\varphi\in H^{\nu^{'}}(\tilde{X})$, we get

$$\|\varphi\|_\nu\leq D^{\nu^{'}-\nu}\|\varphi\|_{\nu^{'}} \tag{8}$$

where D is the diameter of the support of φ.

If $(\varphi,Q)\in test(\nu^{'})$ then the diameter of the support of $\varphi\circ\Theta_Q$ is $\frac{\pi}{2}$ and by (8)

$$test(\nu^{'})\subset\left(\frac{\pi}{2}\right)^{\nu^{'}-\nu}test(\nu). \tag{9}$$

Thus all linear functionals $W \in \mathcal{H}^{\nu}(\tilde{X})$ restrict to functionals in $\mathcal{H}^{\nu^{'}}(\tilde{X})$. Using convolutions it is easy to show that $H^{\nu^{'}}(\tilde{X})$ is dense in $H^{\nu}(\tilde{X})$ for the ν-norm. Consequently, the map $\mathcal{H}^{\nu}(\tilde{X}) \to \mathcal{H}^{\nu^{'}}(\tilde{X})$ obtained by the restriction of $W \in \mathcal{H}^{\nu}(\tilde{X})$ to its action on $H^{\nu^{'}}(\tilde{X})$ is one to one. We get the inclusion $\mathcal{H}^{\nu}(\tilde{X}) \subset \mathcal{H}^{\nu^{'}}(\tilde{X})$. Then $\mathcal{H}(\tilde{X}) = \cap_{0<\nu\leq 1}\mathcal{H}^{\nu}(\tilde{X})$ and $\mathcal{H}(X) = \cap_{0<\nu\leq 1}\mathcal{H}^{\nu}(X)$.

The restriction of $W \in \mathcal{H}(\tilde{X})$ to the space $H^{\nu}(\tilde{X})$ gives inclusion $\mathcal{H}(\tilde{X}) \subset \mathcal{H}^{\nu}(\tilde{X})$. By the existence of this inclusion, the space $\mathcal{H}(\tilde{X})$ has the ν-norm on it. The topology on $\mathcal{H}(\tilde{X})$ is the coarsest topology which makes ν-norms, $0 < \nu \leq 1$, continuous. The vector space $\mathcal{H}(\tilde{X})$ is a complete topological vector space for this topology. The topology on $\mathcal{H}(X)$ is the subspace topology with respect to $\mathcal{H}(\tilde{X})$ and $\mathcal{H}(X)$ is a complete topological vector space.

By (9), for $\nu^{'} > \nu$, we conclude that

$$\{W \in \mathcal{H}(X); \|W\|_{\nu^{'}} < \epsilon\} \supset \{W \in \mathcal{H}(X); \|W\|_{\nu} < \left(\frac{\pi}{2}\right)^{-\nu^{'}+\nu}\epsilon\}.$$

Thus, the family of $\frac{1}{n}$-norms, for $n = 1, 2, 3, \ldots$, gives the same topology on $\mathcal{H}(X)$ as the family of all ν-norms with $0 < \nu \leq 1$. This implies that $\mathcal{H}(X)$ is metrizable, complete topological vector space. Further, the ν-norms give convex local basis. Thus $\mathcal{H}(X)$ is a Fréchet space (see [14]).

A positive Radon measure α on $G(\tilde{X})$ defines a real linear functional on $H(\tilde{X})$. For $\varphi \in H(\tilde{X})$, we set $\alpha(\varphi) = \int_{G(\tilde{X})} \varphi d\alpha$. We say that a measure α is *bounded* if $\sup \alpha([a,b] \times [c,d]) < \infty$ where the supremum is over all $[a,b] \times [c,d]$ with $\frac{(a-c)(b-d)}{(a-d)(b-c)} = 2$. If α is bounded then $\alpha \in \mathcal{H}(\tilde{X})$. In this case

$$\|\alpha\|_{\nu} \leq \sup \alpha([a,b] \times [c,d])$$

where the supremum is over all $[a,b] \times [c,d]$ with $\frac{(a-c)(b-d)}{(a-d)(b-c)} = 2$. If, in addition, α is Γ-invariant then $\alpha \in \mathcal{H}(X)$.

4. The Liouville Map

For $[f] \in \mathcal{T}(X)$, we define the Liouville measure of $[a,b] \times [c,d]$ as

$$L_{[f]}([a,b] \times [c,d]) = \log \frac{(f(a) - f(c))(f(b) - f(d))}{(f(a) - f(d))(f(b) - f(c))}.$$

By continuity, we can set $L_{[f]}(\{a\} \times [c,d]) = L_{[f]}([a,b] \times \{c\}) = 0$. The quantity $L_{[f]}$ can be extended to a positive Radon measure on $G(\tilde{X})$. To see this, we recall that an elementary computation (see Bonahon [6]) gives

$$L_{[id]}([a,b] \times [c,d]) = \int_{[a,b]\times[c,d]} \frac{dxdy}{(x-y)^2}.$$

We define the extension of $L_{[id]}$ to be the measure with the density $\frac{dxdy}{(x-y)^2}$. Since $L_{[f]}([a,b] \times [c,d]) = L_{[id]}([f(a), f(b)] \times [f(c), f(d)])$ we define

$$L_{[f]}(A) = \int_{f(A)} \frac{dxdy}{(x-y)^2}$$

for any Borel set $A \subset G(\tilde{X})$.

Further, the measure $L_{[f]}$ is bounded because f is quasiconformal (see [1] or [12]), and it is Γ-invariant because f is Γ-invariant. We define the Liouville map

$$L : \mathcal{T}(X) \to \mathcal{H}(X)$$

by

$$L([f]) = L_{[f]}.$$

Recall that the cross-ratio of a quadruple (a, b, c, d) of points in $\widehat{\mathbb{R}}$ is given by

$$cr(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

Note that, for the base point $[id]$ of $\mathcal{T}(X)$, $L_{[id]}$ is the logarithm of the cross-ratio.

5. Differentiability

The Liouville map $L : \mathcal{T}(X) \to \mathcal{H}(X)$ maps the possibly infinite-dimensional Banach manifold $\mathcal{T}(X)$ to the topological vector space $\mathcal{H}(X)$, equipped with the family of ν-norms (7) for $0 < \nu \leq 1$. For a fixed ν, we consider the Banach space $\mathcal{H}^\nu(X)$, which contains $\mathcal{H}(X)$, and we use the same letter L to denote the extension $L : \mathcal{T}(X) \to \mathcal{H}^\nu(X)$. We consider the differentiability of L for each ν with $0 < \nu \leq 1$.

As we noted in Section 2, the Bers map $\mathcal{B} : A \to \mathcal{T}(X)$, where A is a bounded neighborhood of 0 in $B(\Gamma)$, gives a global chart for $\mathcal{T}(X)$. We want to construct a linear map $T_{[f]}L : T_{[f]}\mathcal{T}(X) \to \mathcal{H}^\nu(X)$ such that, if $q = \mathcal{B}^{-1}([f]) \in A$, the map $T_{[f]}L \circ T_q\mathcal{B} : B(\Gamma) \to \mathcal{H}^\nu(X)$ is continuous and

$$\lim_{q_1 \to q} \frac{\|L \circ \mathcal{B}(q_1) - L \circ \mathcal{B}(q) - T_{[f]}L \circ T_q\mathcal{B}(q_1 - q)\|_\nu}{\|q_1 - q\|_{B(\Gamma)}} = 0. \tag{10}$$

We begin with the following:

Special Case: $[f] = [id]$

Let $[f^{t\lambda}]$ be a variation at $[id]$, where λ is a harmonic Beltrami differential. Let $\alpha_t = L([f^{t\lambda}])$. To prove the special case, it suffices to prove that $T_{[id]}L$ is linear, and $T_{[id]}L \circ T_0\mathcal{B}$ is continuous, and

$$\lim_{q \to 0} \frac{\|L \circ \mathcal{B}(q) - L \circ \mathcal{B}(0) - T_{[id]}L \circ T_0\mathcal{B}(q)\|_\nu}{\|q\|_{B(\Gamma)}} = 0. \tag{11}$$

By (3) and by (5), to show that $T_{[id]}L \circ T_0\mathcal{B}$ is continuous it is enough to show that

$$\|T_{[id]}L(\lambda/\sim)\|_\nu \leq C\|\lambda\|_\infty \tag{12}$$

for fixed constant C and for any harmonic Beltrami differential λ.

Let $q_{t\lambda} = \mathcal{B}^{-1}(t\lambda)$ with λ harmonic. Since λ is harmonic, $t\lambda$ is harmonic and consequently, $\mathcal{AW}(q_{t\lambda}) = t\lambda$. Then $T_0\mathcal{B}(q_{t\lambda}) = t\lambda/\sim$ by (5). By (3) and above we get

$$\|t\lambda\|_\infty = 2\|q_{t\lambda}\|_{B(\Gamma)}. \tag{13}$$

We replace $\|q_{t\lambda}\|_{B(\Gamma)}$ with $|t| \cdot \|\lambda\|_\infty$ in (11). By the definition of map L, by (13) and by (5), to show (11) it suffices to show that

$$\lim_{t \to 0} \sup_{\varphi \in test(\nu)} \left| \frac{\int \varphi d\alpha_t - \int \varphi d\alpha_0 - tT_{[id]}L(\lambda/\sim)(\varphi)}{t\|\lambda\|_\infty} \right| = 0 \tag{14}$$

uniformly in λ as long as $\|\lambda\|_\infty$ is bounded.

To begin, we prove the differentiability of $\int \varphi d\alpha_t$ in t.

Lemma 5.1. *Let $[f^{t\lambda}] \in \mathcal{T}(X)$ be a variation at $[id]$, where λ is a harmonic Beltrami differential. Let $\alpha_t = L([f^{t\lambda}])$. Then there is a $W \in \mathcal{H}^\nu(X)$ such that*

$$\frac{d}{dt}\int \varphi d\alpha_t|_{t=0} = W(\varphi)$$

for all $(\varphi, Q) \in test(\nu)$.

Proof. Fix $(\varphi, Q) \in test(\nu)$. Let Θ_t be the Möbius mapping such that $\Theta_t \circ f^{t\lambda} \circ \Theta_Q$ fixes 0, 1 and ∞.

On $[-2,-1] \times [1,2]$ the standard metric d_1 is Lipschitz equivalent to the metric in which the distance between (x, y) and (x_1, y_1) is equal to $\max\{|x-x_1|, |y-y_1|\}$. Thus, in our arguments, we can replace d with such.

We remind the reader that f^μ stands for the unique quasiconformal map $\mathbb{H}^2 \to \mathbb{H}^2$ with Beltrami coefficient μ and which fixes 0, 1 and ∞. If we replace $f^{t\lambda}$ with $\Theta_t \circ f^{t\lambda} \circ \Theta_Q$ the Beltrami coefficient of $\Theta_t \circ f^{t\lambda} \circ \Theta_Q$ is $t\lambda^{'}$, where $\lambda^{'}(\zeta) := \lambda \circ \Theta_Q(\zeta)\frac{\overline{\Theta_Q^{'}(\zeta)}}{\Theta_Q^{'}(\zeta)}$ for $\zeta \in \mathbb{H}^2$ by the chain rule. Then $\|\lambda\|_\infty = \|\lambda^{'}\|_\infty$ and $\Theta_t \circ f^{t\lambda} \circ \Theta_Q = f^{t\lambda^{'}}$ by uniqueness of the solution to the Beltrami equation.

Let us divide $[-2,-1]$ into 2^n equal size intervals $[a_{i-1}, a_i]$, for $i = 1, 2, 3, \dots, 2^n$. Namely $a_0 = -2$, $a_{2^n} = -1$ and $a_i = -2 + \frac{i}{2^n}$. Divide $[1, 2]$ into 2^n equal size intervals $[c_{j-1}, c_j]$, for $j = 1, 2, 3, \dots, 2^n$. Here $c_0 = 1$, $c_{2^n} = 2$ and $c_j = 1 + \frac{j}{2^n}$. This defines 4^n boxes $E_{i,j} = [a_{i-1}, a_i] \times [c_{j-1}, c_j]$ for $i, j = 1, 2, 3, \dots, 2^n$. The union of all the $E_{i,j}$ is equal to $[-2,-1] \times [1,2]$, and each pairwise intersection of two $E_{i,j}$ either is empty or is $\{a_i\} \times [c_{j-1}, c_j]$ or is $[a_{i-1}, a_i] \times \{c_j\}$. In particular, the intersection of two distinct $E_{i,j}$ has zero mass for the Liouville measure $L_{[f^{t\lambda'}]}$.

Define a step function approximation (φ_n, Q) to the function (φ, Q) by setting $\varphi_n \circ \Theta_Q = \sum_{i,j=1}^{2^n} p_{i,j}\chi_{E_{i,j}}$, where $p_{i,j} = (\varphi \circ \Theta_Q)(x_i, y_j)$ for an arbitrary geodesic $(x_i, y_j) \in E_{i,j}$, and where $\chi_{E_{i,j}}$ denotes the characteristic function of $E_{i,j}$. Note that φ_n is not defined on a set of measure zero (the pairwise intersections of the $E_{i,j}$), which is not important for the integration. Let the measure $\beta_t = (\Theta_Q)_*(\alpha_t)$ be the pull back of α_t by Θ_Q. Then by change of variable we get $\int \varphi d\alpha_t - \int \varphi_n d\alpha_t = \int \varphi \circ \Theta_Q d\beta_t - \int \varphi_n \circ \Theta_Q d\beta_t$.

We want to prove that $\frac{d}{dt}\int \varphi d\alpha_t$ exists. Recall that the derivative $\frac{d}{dt}f^{t\lambda^{'}}(z)$ exists, for t such that $\|t\lambda^{'}\|_\infty < 1$ and for each fixed $z \in \mathbb{C}$. By the definition of β_t and by the invariance of the cross ratio with respect to Möbius transformations,

$$\beta_t(E_{i,j}) = \log \frac{[f^{t\lambda^{'}}(a_{i-1}) - f^{t\lambda^{'}}(c_{j-1})][f^{t\lambda^{'}}(a_i) - f^{t\lambda^{'}}(c_j)]}{[f^{t\lambda'}(a_{i-1}) - f^{t\lambda'}(c_j)][f^{t\lambda'}(a_i) - f^{t\lambda'}(c_{j-1})]}.$$

Thus the derivative $\frac{d}{dt}\beta_t(E_{i,j})$ exists because $\beta_t(E_{i,j})$ is the composition of differentiable functions. Consequently, each $\int \varphi_n d\alpha_t$ is differentiable and

$$\frac{d}{dt}\int \varphi_n d\alpha_t = \sum_{i,j=1}^{2^n} p_{i,j}\frac{d}{dt}\beta_t(E_{i,j}).$$

We form the series

$$\int \varphi_1 d\alpha_t + \sum_{n=1}^{\infty} \Big[\int \varphi_{n+1} d\alpha_t - \int \varphi_n d\alpha_t \Big]$$

and note that its n-th partial sum is $\int \varphi_n d\alpha_t$. Using the above change of variable and the fact that φ is ν-Hölder continuous we obtain

$$\begin{aligned} &\Big| \int \varphi d\alpha_t - \int \varphi_n d\alpha_t \Big| \leq \\ &\sum_{i,j=1}^{2^n} \sup_{(x,y)\in E_{i,j}} |\varphi \circ \Theta_Q(x,y) - p_{i,j}| \beta_t(E_{i,j}) \leq \\ &\frac{1}{2^{n\nu}} \beta_t([-2,-1]\times[1,2]) = \frac{1}{2^{n\nu}} \alpha_t([a,b]\times[c,d]). \end{aligned} \tag{15}$$

Thus $\int \varphi_n d\alpha_t$ converges to $\int \varphi d\alpha_t$ as $n \to \infty$. By a familiar theorem from calculus, to show that $\frac{d}{dt} \int \varphi d\alpha_t|_{t=0}$ exists it is enough to show that the series

$$\frac{d}{dt}\int \varphi_1 d\alpha_t + \sum_{n=1}^{\infty} \Big[\frac{d}{dt}\int \varphi_{n+1} d\alpha_t - \frac{d}{dt}\int \varphi_n d\alpha_t \Big] \tag{16}$$

converges uniformly for t in a small neighborhood of 0.

For this purpose we use the infinitesimal Teichmüller theory. Define $\dot{f}^{\mu}[\lambda](z) = \frac{d}{dt} f^{\mu+t\lambda}(z)|_{t=0}$ and $\dot{f}[\lambda](z) = \frac{d}{dt} f^{t\lambda}(z)|_{t=0}$. We recall a formula of Ahlfors [2]

$$\dot{f}^{\mu}[\lambda] = \dot{f}[L^{\mu}\lambda] \circ f^{\mu} \tag{17}$$

where

$$L^{\mu}\lambda = \Big\{ \lambda \frac{(f^{\mu}_z)^2}{|f^{\mu}_z|^2 - |f^{\mu}_{\bar{z}}|^2} \Big\} \circ (f^{\mu})^{-1}.$$

From Wolpert's formula [16] for the variation of the cross ratio we get

$$\begin{aligned} &\frac{d}{dt}\beta_t(E_{i,j})|_{t=0} = \\ &-\frac{2}{\pi} \mathrm{Re} \int_{\mathbb{H}^2} \lambda(\zeta) \frac{(a_{i-1}-a_i)(c_{j-1}-c_j)}{(\zeta-a_{i-1})(\zeta-a_i)(\zeta-c_{j-1})(\zeta-c_j)} d\xi d\eta \end{aligned} \tag{18}$$

where $\zeta = \xi + i\eta$.

Let $a_i^t = f^{t\lambda^{'}}(a_i)$ and $c_j^t = f^{t\lambda^{'}}(c_j)$. By (17) and (18) we get

$$\begin{aligned} &\frac{d}{dt}\beta_t(E_{i,j}) = \\ &-\frac{2}{\pi} \mathrm{Re} \int_{\mathbb{H}^2} L^{t\lambda^{'}} \lambda^{'}(\zeta) \frac{(a^t_{i-1}-a^t_i)(c^t_{j-1}-c^t_j)}{(\zeta-a^t_{i-1})(\zeta-a^t_i)(\zeta-c^t_{j-1})(\zeta-c^t_j)} d\xi d\eta. \end{aligned} \tag{19}$$

We estimate $|\frac{d}{dt}\int \varphi_{n+1} d\alpha_t - \frac{d}{dt}\int \varphi_n d\alpha_t|$. By (19) and by the definition of φ_n we get

$$\begin{aligned} \frac{d}{dt}\int \varphi_n d\alpha_t = &-\frac{2}{\pi}\mathrm{Re}\int_{\mathbb{H}^2} L^{t\lambda^{'}}\lambda^{'}(\zeta) \times \\ &\sum_{i,j=1}^{2^n} \frac{p_{i,j}(a^t_{i-1}-a^t_i)(c^t_{j-1}-c^t_j)}{(\zeta-a^t_{i-1})(\zeta-a^t_i)(\zeta-c^t_{j-1})(\zeta-c^t_j)} d\xi d\eta. \end{aligned} \tag{20}$$

To obtain φ_{n+1} from φ_n we divide each of the intervals $[a_{i-1}, a_i]$ and $[c_{j-1}, c_j]$ into two equal subintervals. For the convenience of the notation we write $E_{i,j} = \cup_{k=1}^4 E_{ik,jk}$ where $E_{ik,jk} = [a_{(i-1)k}, a_{ik}] \times [c_{(j-1)k}, c_{jk}]$ and where a_{ik} equals a_i or a_{i-1} or the midpoint of $[a_{i-1}, a_i]$ and, similarly for c_{jk}. We fix $(x_{ik}, y_{jk}) \in E_{ik,jk}$ and define $p_{ik,jk} = \varphi(x_{ik}, y_{jk})$. Then $\varphi_{n+1} \circ \Theta_Q = \sum_{i,j=1}^{2^n} \sum_{k=1}^4 p_{ik,jk}\chi_{E_{ik,jk}}$.

Since φ is ν-Hölder continuous we get $|p_{i,j} - p_{ik,jk}| \leq \frac{1}{2^{n\nu}} = \frac{1}{4^{\frac{n\nu}{2}}}$. Then by (20) we get

$$\begin{aligned}
&\Big|\frac{d}{dt}\int \varphi_n d\alpha_t - \frac{d}{dt}\int \varphi_{n+1} d\alpha_t\Big| \leq \frac{4^n}{4^{\frac{n\nu}{2}}}\|L^{t\lambda'}\lambda'\|_\infty \times \\
&\max_{i,j,k}\Big\{|(a^t_{(i-1)k} - a^t_{ik})(c^t_{(j-1)k} - c^t_{jk})|\Big\}\times \\
&\max_{i,j,k}\Big\{\int_{\mathbb{H}^2} \frac{d\xi d\eta}{|\zeta - a^t_{(i-1)k}||\zeta - a^t_{ik}||\zeta - c^t_{(j-1)k}||\zeta - c^t_{jk}|}\Big\}.
\end{aligned} \tag{21}$$

For t small, the family $f^{t\lambda'}$ has constant of quasiconformality close to 1. Also the family $f^{t\lambda'}$ fixes 0, 1 and ∞ by the definition. Thus the family $f^{t\lambda'}$ is uniformly Hölder continuous with the Hölder exponent ω close to 1. Then

$$|a^t_{(i-1)k} - a^t_{ik}||c^t_{(j-1)k} - c^t_{jk}| \leq \frac{C_1}{4^{n\omega}} \tag{22}$$

for fixed constant C_1.

By an elementary integration (see Gardiner-Lakic [11, section 3.4]) we get

$$\int_{\mathbb{H}^2} \frac{d\xi d\eta}{|\zeta - a^t_{(i-1)k}||\zeta - a^t_{ik}||\zeta - c^t_{(j-1)k}||\zeta - c^t_{jk}|} \leq C_2 + C_3 n \tag{23}$$

for fixed constants C_2 and C_3. In more details, we divide the domain of the integration $\mathbb{H}^2$ into four sets: $A_1 = \{\zeta \in \mathbb{H}^2; |\zeta| > R\}$, $A_2 = \{\zeta \in \mathbb{H}^2; |\zeta - a^t_{ik}| \leq 1\}$, $A_3 = \{\zeta \in \mathbb{H}^2; |\zeta - c^t_{jk}| \leq 1\}$ and $A_4 = \mathbb{H}^2 - (A_1 \cup A_2 \cup A_3)$, for R large. The integral over A_1 is of the order of R^{-2}. After using the substitution $(a^t_{ik} - a^t_{(i-1)k})z = \zeta - a^t_{ik}$, the integral over A_2 becomes

$$\int_{|z| \leq \frac{1}{|a^t_{ik} - a^t_{(i-1)k}|}} \frac{dxdy}{|z(z-1)|}.$$

Using the uniform Hölder continuity of the family $f^{t\lambda'}$, the above integral is less that or equal to $\tilde{C}_2 + \tilde{C}_3 n$. A similar inequality holds for the integral over A_3. The integral over A_4 is of the order of R^2. For fixed $R > 0$, we get (23).

We choose t small enough such that $\frac{\nu}{2} + \omega - 1 > 0$. By (21), (22) and (23) we get

$$\sum_{n=1}^{\infty}\Big|\frac{d}{dt}\int \varphi_n d\alpha_t - \frac{d}{dt}\int \varphi_{n+1} d\alpha_t\Big| \leq C\|L^{t\lambda'}\lambda'\|_\infty \sum_{n=1}^{\infty} \frac{n}{4^{(\frac{\nu}{2}+\omega-1)n}} \tag{24}$$

for fixed constant $C > 0$. It follows that the series (16) converges uniformly and absolutely for small t, and its sum is bounded by a constant independent of $(\varphi, Q) \in test(\nu)$. Thus the derivative $\frac{d}{dt}\int \varphi d\alpha_t|_{t=0} =: W(\varphi)$ exists and $W \in \mathcal{H}^\nu(\tilde{X})$. Because α_t is Γ-invariant, it follows that W is Γ-invariant. Thus $W \in \mathcal{H}^\nu(X)$. □

In the following Lemma, we keep the notation $W(\varphi) = \frac{d}{dt}\int \varphi d\alpha_t|_{t=0}$, and λ denotes a harmonic Beltrami differential.

Lemma 5.2. *The map $T_{[id]}\mathcal{T}(X) \to \mathcal{H}^{\nu}(X)$ defined by $T_{[id]}L(\lambda/\sim) = W$ is linear and continuous. Further,*

$$\lim_{t\to 0} \sup_{\varphi\in test(\nu)} \Big| \frac{\int \varphi d\alpha_t - \int \varphi d\alpha_0 - tW(\varphi)}{t\|\lambda\|_\infty} \Big| = 0 \tag{25}$$

uniformly in λ as long as $\|\lambda\|_\infty$ is bounded.

Proof. In the inequality (24), we put $t = 0$ to obtain

$$|T_{[id]}L(\lambda/\sim)(\varphi)| = |W(\varphi)| \leq C\|\lambda\|_\infty$$

for fixed constant $C > 0$ and for all $(\varphi, Q) \in test(\nu)$. Thus $T_{[id]}L$ is bounded.

The Liouville map L can be approximated by a sequence of maps L_n given by

$$L_n([f])(\varphi) = \int \varphi_n d\alpha$$

where $\alpha = L([f])$ and (φ_n, Q) is a step function approximation to (φ, Q) as in Lemma 5.1. Then the maps L_n are differentiable maps of $\mathcal{T}(X)$ into $\mathcal{H}^{\nu}(X)$.

The tangent map $T_{[id]}L$ is linear, because it is the limit of linear maps $T_{[id]}L_n$.

Again, by the same theorem from calculus as in Lemma 5.1, to show (25) it is enough to note that the series (16) in Lemma 5.1 when divided by $\|\lambda\|_\infty$ converges uniformly in λ and t for $\|\lambda\|_\infty$ bounded and t in a small neighborhood of 0 independently of $(\varphi, Q) \in test(\nu)$. That is easily seen from the inequality (24). □

This concludes the proof that L is differentiable at the point $[id]$.

General Case:

Let $\alpha_t = L([f^{\mu+t\lambda}])$, where $[f^{\mu+t\lambda}]$ is a variation at $[f^{\mu}]$. Following Ahlfors [2], we write

$$f^{\mu+t\lambda} = f^{\rho(t)} \circ f^{\mu}$$

where $\rho(t) = \{\frac{t\lambda}{1-\overline{\mu}(\mu+t\lambda)}(\frac{f^{\mu}_z}{|f^{\mu}_z|})^2\} \circ (f^{\mu})^{-1}$. Let $\alpha_t' = L([f^{\rho(t)}]) = L_{[f^{\rho(t)}]}$. Then $(f^{\mu})_*\alpha_t' = \alpha_t$ where $(f^{\mu})_*\alpha_t'(A) = \alpha_t'(f^{\mu}(A))$ for any Borel set $A \subset G(\tilde{X})$. It follows that

$$\int \varphi d\alpha_t = \int \varphi \circ (f^{\mu})^{-1} d\alpha_t'$$

for $(\varphi, Q) \in test(\nu)$. Let $Q = [a,b]\times[c,d]$ with $cr(a,b,c,d) = 2$. The quasiconformal map f^{μ} maps any four points on $\widehat{\mathbb{R}}$ with the cross ratio 2 onto four points with the cross ratio k_1 bounded from above by constant $k > 2$ and bounded from below by $1 + \frac{1}{k}$. The constant k depends only on the quasiconformal constant of f^{μ} (see [12]). Then the function $\varphi \circ (f^{\mu})^{-1}$ has its support in $f^{\mu}(Q) = [f^{\mu}(a), f^{\mu}(b)] \times [f^{\mu}(c), f^{\mu}(d)]$ such that

$$1 + \frac{1}{k} \leq cr(f^{\mu}(a), f^{\mu}(b), f^{\mu}(c), f^{\mu}(d)) \leq k.$$

Denote by Θ^{μ}_Q the Möbius mapping which maps $[-k_1, -1] \times [1, k_1]$ onto $f^{\mu}(Q)$ where $1 < k_1 \leq k$. Then $(\Theta_Q)^{-1} \circ (f^{\mu})^{-1} \circ \Theta^{\mu}_Q$ maps $-k_1$, -1, 1 and k_1 onto -2, -1, 1 and 2, respectively. As we vary over all $Q = [a,b] \times [c,d]$ in $G(\tilde{X})$ with $cr(a,b,c,d) = 2$, we obtain a family of quasiconformal maps $(\Theta_Q)^{-1} \circ (f^{\mu})^{-1} \circ \Theta^{\mu}_Q$. The constant of quasiconformality is bounded for the whole family because it is equal to the constant of quasiconformality for f^{μ}. The family is normalized to map -1, 1 and k_1 onto -1, 1 and 2, respectively, where $k_1 > 1$ is bounded away from 1 and ∞. The boundedness of k_1 follows from the condition $cr(a,b,c,d) = 2$ and the fact that f^μ is quasiconformal. The family $\{(\Theta_Q)^{-1} \circ (f^\mu)^{-1} \circ \Theta_Q^\mu ; Q = [a,b] \times [c,d]$ with $cr(a,b,c,d) = 2\}$ is uniformly Hölder continuous with the Hölder exponent ν_1 (see [12]). Therefore,

$$\|\varphi \circ (f^\mu)^{-1} \circ \Theta_Q^\mu\|_{\nu\nu_1} \leq \|\varphi \circ \Theta_Q\|_\nu \|(\Theta_Q)^{-1} \circ (f^\mu)^{-1} \circ \Theta_Q^\mu\|_{\nu_1}^\nu .$$

By the above inequality $\varphi \circ (f^\mu)^{-1} \circ \Theta_Q^\mu$ is $\nu\nu_1$-Hölder continuous with

$$\|\varphi \circ (f^\mu)^{-1} \circ \Theta_Q^\mu\|_{\nu\nu_1} \leq C$$

where $C = \max_Q \|(\Theta_Q)^{-1} \circ (f^\mu)^{-1} \circ \Theta_Q^\mu\|_{\nu_1}^\nu$.

We define $test^\mu(\nu)$ to be the set of all $(\varphi \circ (f^\mu)^{-1}, f^\mu(Q))$ with $(\varphi, Q) \in test(\nu)$. Note that the arguments in the Special Case $[f] = [id]$ are still valid if we replace $test(\nu)$ by $test^\mu(\nu)$. It follows that L is differentiable at any $[f^\mu]$ if it is differentiable at the base point $[id]$ of $\mathcal{T}(X)$. Thus general case is proved.

Remark 5.1. We point out one surprising feature of our result. A quotient space of Zygmund bounded functions is the tangent to the Teichmüller space. The space of Zygmund bounded functions is a subset of the space of Hölder continuous functions. In general, the dual of a subset contains the dual of a set. Thus, it would be expected that in order to describe the tangent space to the Teichmüller space we need the space of Zygmund distributions which contains the space of Hölder distributions. However, we showed that it is enough to consider Hölder distributions.

We can use estimates and techniques of Lemmas 5.1 and 5.2 to obtain an explicit expression for the tangent map $T_{[id]}L$ at the base point $[id]$ of $\mathcal{T}(X)$.

Lemma 5.3. *The tangent map $T_*L : T_*(\mathcal{T}(X)) \to \mathcal{H}^\nu(X)$ at the base point $[id]$ is given by the formula*

$$\begin{aligned} T_{[id]}L(\lambda/\sim)(\varphi) = \\ -\frac{2}{\pi} Re \int_{\mathbb{H}^2} \lambda(\zeta) \Big[\int_{G(\mathbb{H}^2)} \frac{\varphi(x,y)}{(\zeta - x)^2(\zeta - y)^2} dxdy \Big] d\xi d\eta \end{aligned} \tag{26}$$

where λ is a Beltrami differential representing a tangent vector at $[id]$, where $(\varphi, Q) \in test(\nu)$, where $\zeta = \xi + i\eta \in \mathbb{H}^2$ and where $(x,y) \in G(\mathbb{H}^2) = \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} - \Delta$.

Proof. Let $E_{i,j} = [a_{i-1}, a_i] \times [c_{j-1}, c_j]$ be a subset of $[-2,-1] \times [1,2]$ as in Lemma 5.1. Note that

$$\int_{E_{i,j}} \frac{dxdy}{(\zeta - x)^2(\zeta - y)^2} = \frac{(a_{i-1} - a_i)(c_{j-1} - c_j)}{(\zeta - a_{i-1})(\zeta - a_i)(\zeta - c_{j-1})(\zeta - c_j)}. \tag{27}$$

Then the expression under the integral in (20), for $t = 0$, can be written as

$$\lambda'(\zeta) \int_{[-2,-1]\times[1,2]} \frac{\varphi_n \circ \Theta_Q(x,y)}{(\zeta - x)^2(\zeta - y)^2} dxdy =: g_n(\zeta). \tag{28}$$

Further, the series

$$\int_{\mathbb{H}^2} |g_1(\zeta)| d\xi d\eta + \sum_{n=1}^{\infty} \int_{\mathbb{H}^2} |g_{n+1}(\zeta) - g_n(\zeta)| d\xi d\eta$$

converges by the proof of Lemma 5.1. In more details, the right hand side of (21) is greater than $\int_{\mathbb{H}^2}|g_{n+1}(\zeta)-g_n(\zeta)|d\xi d\eta$. By the sequence of inequalities that lead to (24) we get that the right hand side of (21) is less than the right hand side of (24). Thus the convergence of the above series follows.

The convergence of the above series implies the convergence of the series $|g_1(\zeta)|+\sum_{n=1}^{\infty}|g_{n+1}(\zeta)-g_n(\zeta)|$ in a.e. sense. The sum $\tilde{g}(\zeta)=|g_1(\zeta)|+\sum_{n=1}^{\infty}|g_{n+1}(\zeta)-g_n(\zeta)|$ is an integrable function and $|g_n(\zeta)|\leq\tilde{g}(\zeta)$ for all n and almost all ζ. Note that $g_n(\zeta)$ converges to

$$\lambda^{'}(\zeta)\int_{[-2,-1]\times[1,2]}\frac{\varphi\circ\Theta_Q(x,y)}{(\zeta-x)^2(\zeta-y)^2}dxdy$$

in a.e. sense. By the Lebesgue's dominated convergence theorem we get that $\int_{\mathbb{H}^2}g_n(\zeta)d\xi d\eta\to\int_{\mathbb{H}^2}g(\zeta)d\xi d\eta$ as $n\to\infty$.

We define

$$K(\zeta;a_{i-1},a_i,c_{j-1},c_j)=\frac{(a_{i-1}-a_i)(c_{j-1}-c_j)}{(\zeta-a_{i-1})(\zeta-a_i)(\zeta-c_{j-1})(\zeta-c_j)}.$$

Wolpert [17] noticed that

$$\begin{aligned}K(\zeta;a_{i-1},a_i,c_{j-1},c_j)=K(\Theta(\zeta);\Theta(a_{i-1}),\\ \Theta(a_i),\Theta(c_{j-1}),\Theta(c_j))\Big(\Theta^{'}(\zeta)\Big)^2\end{aligned}\tag{29}$$

for any Möbius transformation Θ. Then by (29) and by the definition of $\lambda^{'}$, we get

$$\begin{aligned}g_n(\zeta)=&\Big[\int_{[-2,-1]\times[1,2]}\frac{\varphi_n(\Theta_Q(x,y))}{[\Theta_Q(\zeta)-\Theta_Q(x)][\Theta_Q(\zeta)-\Theta_Q(y)]}dxdy\Big]\times\\ &\times\lambda(\Theta_Q(\zeta))|\Theta_Q^{'}(\zeta)|^2.\end{aligned}\tag{30}$$

Note that the integral in (30) does not change if we integrate over $G(\tilde{X})$ because $\varphi_n\circ\Theta_Q$ has its support in $[-2,-1]\times[1,2]$. By a substitution of variable $(x^{'},y^{'})=\Theta_Q(x,y)$ in the integral in (30) and above we get

$$\begin{aligned}g_n(\zeta)=&\Big[\int_{G(\tilde{X})}\frac{\varphi_n(x^{'},y^{'})}{[\Theta_Q(\zeta)-x^{'}][\Theta_Q(\zeta)-y^{'}]}dx^{'}dy^{'}\Big]\times\\ &\times\lambda(\Theta_Q(\zeta))|\Theta_Q^{'}(\zeta)|^2.\end{aligned}\tag{31}$$

To obtain(26), by (31) it is enough to do a change of variable $\zeta^{'}=\Theta_Q(\zeta)$ in the integral $-\frac{2}{\pi}Re\int_{\mathbb{H}^2}g_n(\zeta)d\xi d\eta$ and let $n\to\infty$. □

Remark 5.2. In the integral formula (26) the change of the order of integration is not allowed. The double integral would diverge. We also note that the proof of the convergence of (26) strongly depends on the fact that φ is Hölder continuous. The double integral in (26) does not converge if we replace φ by an arbitrary continuous function with compact support. Thus the image of the tangent map $T_{[id]}L$ does not contain measures on $G(\tilde{X})$.

Remark 5.3. Gardiner [10] defines a map from the Beltrami differentials to the space of distributions on $G(\tilde{X})$. The image of the map consists of absolutely continuous measures to the Euclidean measure $dxdy$ on $G(\tilde{X})\cong\mathbb{R}\times\mathbb{R}-diag$. The formulas (20) and (21) in [10] are similar to the formula in Lemma 5.3, but the space of test functions in [10] consists of only smooth functions. In the same paper, it is described the process of inverting the map using the Bers's reproducing formula [5]. We note that similar process would give the inverse of $T_{[id]}L$.

Using the change of the base point of $\mathcal{T}(X)$ as in the discussion of the differentiability in the General Case, we give the tangent map at any point $[f^\mu]\in\mathcal{T}(X)$. Let $[f^{\mu+t\lambda}]$ be a variation at $[f^\mu]$. Then $L^\mu\lambda$ is a Beltrami differential on $f^\mu(X)$.

Proposition 5.1. *The tangent map $T_*L : T_*\mathcal{T}(X)\to\mathcal{H}^\nu(X)$ at the point $[f^\mu]$ is given by the formula*

$$T_{[f^\mu]}L(L^\mu\lambda/\sim)(\varphi)=$$
$$-\frac{2}{\pi}Re\int_{\mathbb{H}^2}L^\mu\lambda(\zeta)\Big[\int_{G(\tilde{X})}\frac{\varphi\circ(f^\mu)^{-1}(x,y)}{(\zeta-x)^2(\zeta-y)^2}dxdy\Big]d\xi d\eta.\Box$$

We establish the continuity of $T_{[f^\mu]}L$ on the Teichmüller space $\mathcal{T}(X)$.

Lemma 5.4. *The tangent map $T_*L : T_*\mathcal{T}(X)\to\mathcal{H}^\nu(X)$ varies continuously with $[f]\in\mathcal{T}(X)$.*

Proof. By the discussion in the General Case, it is enough to show that $T_{[f]}L$ varies continuously at $[id]$.

Let λ be a harmonic Beltrami differential. We consider variations $[f^{t\lambda}]$ and $[f^{\mu+t\lambda}]$ at $[id]$ and at $[f^\mu]$, respectively. The tangent vectors to these variations are $\lambda/\sim$ and $L^\mu\lambda/\sim$ at $[id]$ and at $[f^\mu]$, respectively. The tangent vector $L^\mu\lambda/\sim$ is the parallel translation of $\lambda/\sim$ on $T_*\mathcal{T}(X)$. To prove that $T_{[f]}L$ varies continuously at $[id]$ it is enough to prove that $\|T_{[id]}L(\lambda/\sim)-T_{[f^\mu]}L(L^\mu\lambda/\sim)\|_\nu$ is small for $[f^\mu]$ close to $[id]$ independently of $\lambda/\sim$ as long as $\|\lambda\|_\infty$ is bounded.

Let $(\varphi,Q)\in test(\nu)$. Given n, define $(\varphi_n,Q)\in test(\nu)$ to be the step function approximation to (φ,Q) as in the proof of Lemma 5.1. Let $\alpha_t=L([f^{t\lambda}])$ and let $\alpha_t^\mu=L([f^{\mu+t\lambda}])$. Define $\beta_t=(\Theta_Q)_*(\alpha_t)$ and $\beta_t^\mu=(\Theta_Q)_*(\alpha_t^\mu)$. Then $\int\varphi d\alpha_t=\int\varphi\circ\Theta_Q d\beta_t$ and $\int\varphi d\alpha_t^\mu=\int\varphi\circ\Theta_Q d\beta_t^\mu$. In the proof of Lemma 5.1 we showed that $\frac{d}{dt}\int\varphi_n d\alpha_t\to\frac{d}{dt}\int\varphi d\alpha_t$ as $n\to\infty$ uniformly in t and independently of $(\varphi,Q)\in test(\nu)$. By the discussion in the General Case it is true that $\frac{d}{dt}\int\varphi_n d\alpha_t^\mu\to\frac{d}{dt}\int\varphi d\alpha_t^\mu$ as $n\to\infty$ uniformly in t and independently of $(\varphi,Q)\in test(\nu)$.

In Lemma 5.2 we defined $L_n([f])(\varphi)=\int\varphi_n d\alpha$ where $\alpha=L([f])$. By the above convergence,

$$T_{[id]}L_n(\lambda/\sim)(\varphi)\to T_{[id]}L(\lambda/\sim)(\varphi)$$

and

$$T_{[f^\mu]}L_n(L^\mu\lambda/\sim)(\varphi)\to T_{[f^\mu]}L(L^\mu\lambda/\sim)(\varphi)$$

as $n \to \infty$ independently of $(\varphi, Q) \in test(\nu)$. By above and by the triangle inequality we get

$$\begin{aligned}\Big|T_{[id]}L(\lambda/\sim)(\varphi) - T_{[f^\mu]}L(L^\mu\lambda/\sim)(\varphi)\Big| \leq \\ \Big|T_{[id]}L(\lambda/\sim)(\varphi) - T_{[id]}L_n(\lambda/\sim)(\varphi)\Big| + \\ \Big|T_{[id]}L_n(\lambda/\sim)(\varphi) - T_{[f^\mu]}L_n(L^\mu\lambda/\sim)(\varphi)\Big| \\ + \Big|T_{[f^\mu]}L_n(L^\mu\lambda/\sim)(\varphi) - T_{[f^\mu]}L(L^\mu\lambda/\sim)(\varphi)\Big|.\end{aligned} \tag{32}$$

To prove the lemma it is enough to show that for any n there exists a neighborhood of $[id]$ in $\mathcal{T}(X)$ which depends on n such that for $[f^\mu]$ in this neighborhood

$$\Big|T_{[id]}L_n(\lambda/\sim)(\varphi) - T_{[f^\mu]}L_n(L^\mu\lambda/\sim)(\varphi)\Big| \tag{33}$$

is small. To show that (33) can be made arbitrary small we estimate

$$\Big|\frac{d}{dt}\beta_t(E_{i,j})|_{t=0} - \frac{d}{dt}\beta_t^\mu(E_{i,j})|_{t=0}\Big| \tag{34}$$

where $E_{i,j} = [a_{i-1}, a_i] \times [c_{j-1}, c_j]$ are defined in Lemma 5.1. Note that

$$\frac{d}{dt}\beta_t(E_{i,j})|_{t=0} = \frac{\dot{f}[\lambda](a_{i-1}) - \dot{f}[\lambda](c_{j-1})}{a_{i-1} - c_{j-1}} - \frac{\dot{f}[\lambda](a_{i-1}) - \dot{f}[\lambda](c_j)}{a_{i-1} - c_j} + \frac{\dot{f}[\lambda](a_i) - \dot{f}[\lambda](c_j)}{a_i - c_j} - \frac{\dot{f}[\lambda](a_i) - \dot{f}[\lambda](c_{j-1})}{a_i - c_{j-1}}$$

and

$$\frac{d}{dt}\beta_t^\mu(E_{i,j})|_{t=0} = \frac{\dot{f}^\mu[\lambda](a_{i-1}) - \dot{f}^\mu[\lambda](c_{j-1})}{f^\mu(a_{i-1}) - f^\mu(c_{j-1})} - \frac{\dot{f}^\mu[\lambda](a_{i-1}) - \dot{f}^\mu[\lambda](c_j)}{f^\mu(a_{i-1}) - f^\mu(c_j)} + \frac{\dot{f}^\mu[\lambda](a_i) - \dot{f}^\mu[\lambda](c_j)}{f^\mu(a_i) - f^\mu(c_j)} - \frac{\dot{f}^\mu[\lambda](a_i) - \dot{f}^\mu[\lambda](c_{j-1})}{f^\mu(a_i) - f^\mu(c_{j-1})}.$$

To estimate (34) we estimate

$$\Big|\frac{\dot{f}[\lambda](a_{i-1}) - \dot{f}[\lambda](c_{j-1})}{a_{i-1} - c_{j-1}} - \frac{\dot{f}^\mu[\lambda](a_{i-1}) - \dot{f}^\mu[\lambda](c_{j-1})}{f^\mu(a_{i-1}) - f^\mu(c_{j-1})}\Big| \tag{35}$$

and other three corresponding differences in the expressions for $\frac{d}{dt}\beta_t(E_{i,j})|_{t=0}$ and $\frac{d}{dt}\beta_t^\mu(E_{i,j})|_{t=0}$. Since $f^{\mu+t\lambda}$ is an analytic function of μ and $t\lambda$ it follows that $|\dot{f}[\lambda](a_{i-1}) - \dot{f}^\mu[\lambda](a_{j-1})|$ is small for $\|\mu\|_\infty$ small. For the same reason $|a_{i-1} - f^\mu(a_{i-1})|$ is small for $\|\mu\|_\infty$ small. Thus (35) can be made arbitrary small for $\|\mu\|_\infty$ small independently of a_i, $i = 1, 2, \ldots, 2^n$ and of c_j, $j = 1, 2, \ldots, 2^n$. Then we can make (34) arbitrary small for $\|\mu\|_\infty$ small. Also, we can make (33) small for $[f^\mu]$ close enough to $[id]$ depending on n.

Finally, to make (32) small we choose n big enough and small enough neighborhood of $[id]$ depending on n. The lemma follows. □

Theorem 1 and Theorem 2 follow directly from Lemmas 5.1, 5.2, 5.3 and 5.4.

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