GEODESIC CURRENTS AND TEICHMÜLLER SPACE

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Abstract. Consider a hyperbolic surface X of infinite area. The Liouville map $\mathcal{L}$ assigns to any quasiconformal deformation of X a measure on the space $G(\tilde{X})$ of geodesics of the universal covering $\tilde{X}$ of X. We show that the Liouville map $\mathcal{L}$ is a homeomorphism from the Teichmüller space $\mathcal{T}(X)$ onto its image, and that the image $\mathcal{L}(\mathcal{T}(X))$ is closed and unbounded. The set of asymptotic rays to $\mathcal{L}(\mathcal{T}(X))$ consists of all bounded measured laminations on X. Hence, the set of projective bounded measured laminations is a natural boundary for $\mathcal{T}(X)$. The action of the quasiconformal mapping class group on $\mathcal{T}(X)$ continuously extends to this boundary for $\mathcal{T}(X)$.

1. Introduction

Teichmüller spaces are studied extensively. At this point, we have a good understanding of the geometry of Teichmüller spaces of finite area hyperbolic surfaces as well as of infinite area hyperbolic surfaces. However, little is known about the geometry at infinity of Teichmüller spaces of infinite surfaces.

The geometry at infinity of Teichmüller spaces of finite area hyperbolic surfaces is well-developed. Thurston [15], [7] introduced a natural boundary to Teichmüller spaces of finite surfaces. In the case of a finite surface, Thurston's closure of the Teichmüller space is a compact space homeomorphic to the closed unit ball in Euclidean space of an appropriate dimension. The action of the mapping class group on the Teichmüller space extends continuously to an action by homeomorphisms on Thurston's compactification.

In this paper we study the asymptotic geometry at infinity of Teichmüller spaces of infinite area hyperbolic surfaces. We introduce a Thurston-type boundary to Teichmüller spaces of infinite surfaces. At present such boundary was known to exist only for finite surfaces. The action of the mapping class group continuously extends to Thurston-type closure of the Teichmüller space of an infinite surface, but the closure is not compact.

The crucial ingredients in proofs of the above results for finite surfaces were lengths of simple closed geodesics and intersection numbers. These techniques cannot be used in Teichmüller spaces of infinite surfaces. The notion of Liouville currents is the essential tool that allowed us to introduce a Thurston-type boundary for Teichmüller spaces of infinite surfaces.

Liouville currents for finite surfaces were introduced by Bonahon [2]. He used Liouville currents to give a different description of Thurston's boundary for the Teichmüller space of a finite surface. Liouville currents are also defined for infinite surfaces. However, this is where the similarity between the finite and the infinite

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case stops. The infinite case requires more saddle topology on the space of Liouville currents than the finite case. Our considerations, in a related paper [13], has led us to such topology. Equipped with the new topology and using new techniques adapted for the infinite case we prove the existence of a Thurston-type boundary for infinite surfaces.

From now on, we assume that X is an infinite surface. In other words, X is a Riemann surface which has an infinite area hyperbolic metric compatible with the complex structure. The Teichmüller space $\mathcal{T}(X)$ is infinite-dimensional as opposed to the Teichmüller space of a finite surface which is finite-dimensional. Consequently $\mathcal{T}(X)$ is not locally compact and any closure of $\mathcal{T}(X)$ by adding points at "infinity" cannot be compact.

Liouville currents on X are positive measures on the space of geodesics $G(\tilde{X})$ of the universal covering $\tilde{X}$ of X which are covering group invariant. By the definition, the Liouville map associates to any point of $\mathcal{T}(X)$ a Liouville current on X. The image of the Liouville map is contained in the space of bounded measures on $G(\tilde{X})$. Bonahon [2] used classical weak* topology on the space of measures on $G(\tilde{X})$ for the case of a finite surface. The fact that $\mathcal{T}(X)$ is locally non-compact suggests that we need a more delicate topology on the space measures on $G(\tilde{X})$ when the surface X is infinite.

In a related paper [13], we proved that the Liouville map is differentiable in the appropriate sense. To describe the derivative of the Liouville map, we introduced the space of Hölder distributions $\mathcal{H}(X)$ as follows. The space of geodesics $G(\tilde{X})$ is identified to $S^1 \times S^1 - diag$. The identification is not unique and any two identifications are Hölder equivalent. Thus, $S^1 \times S^1 - diag$ comes equipped with the family of Hölder equivalent metrics. The space $H(\tilde{X})$ of Hölder continuous functions with compact supports in $G(\tilde{X})$ is well-defined independently of a particular metric in the family. For a fixed metric in the family, we introduce the space $test(\nu)$ of ν-Hölder continuous test functions, $0 < \nu \leq 1$. The space $\mathcal{H}(\tilde{X})$ is the dual to $H(\tilde{X})$ such that $W \in \mathcal{H}(\tilde{X})$ if $\sup_{(\varphi,Q)\in test(\nu)} |W(\varphi)| < \infty$ for each Hölder exponent ν, $0 < \nu \leq 1$. The space $\mathcal{H}(X)$ of Hölder distributions for X consists of all $W \in \mathcal{H}(\tilde{X})$ that are covering group invariant. For a fixed ν, we introduce the ν-norm $\|W\|_\nu = \sup_{(\varphi,Q)\in test(\nu)} |W(\varphi)|$ on $\mathcal{H}(X)$. The family of ν-norms gives $\mathcal{H}(X)$ the structure of a complete metrizable vector space. The author [13] proved that the Liouville map

$$\mathcal{L} : \mathcal{T}(X) \to \mathcal{H}(X)$$

is differentiable for the topology on $\mathcal{H}(X)$ coming from the family of ν-norms.

The space of measures on $G(\tilde{X})$ embeds into $\mathcal{H}(\tilde{X})$ via integration. This suggested that the proper topology on the space of measures is the restriction of the topology on $\mathcal{H}(X)$.

We use the family of ν-norms in analyzing global properties of the Liouville map $\mathcal{L}$. As first result, we proved that $\mathcal{L} : \mathcal{T}(X) \to \mathcal{H}(X)$ is a topological embedding whose image is closed and unbounded.

Theorem 1. *The Liouville map $\mathcal{L} : \mathcal{T}(X) \to \mathcal{H}(X)$ is a homeomorphism onto its image. The image $\mathcal{L}(\mathcal{T}(X))$ of $\mathcal{T}(X)$ is closed and unbounded in $\mathcal{H}(X)$.*

The space $\mathcal{L}(\mathcal{T}(X))$ has no "new" natural boundary points in $\mathcal{H}(X)$ because it is closed. On the other hand, since $\mathcal{L}(\mathcal{T}(X))$ is unbounded we can define a natural boundary at "infinity" to be the set of equivalence classes of "controlled" paths that leave every bounded subset of $\mathcal{L}(\mathcal{T}(X))$. Since $\mathcal{H}(X)$ is a topological vector space, we instead define a boundary of $\mathcal{T}(X)$ to be the set of rays asymptotic to unbounded paths in $\mathcal{L}(\mathcal{T}(X))$. We say that a ray tW, $t>0$ and $W \in \mathcal{H}(X)$, is *asymptotic* to $\mathcal{L}(\mathcal{T}(X))$ if there exists a path $\alpha_t \in \mathcal{L}(\mathcal{T}(X))$ such that $\frac{1}{t}\alpha_t \to \beta$ as $t \to \infty$ in the topology of $\mathcal{H}(X)$. The path $\alpha_t \in \mathcal{L}(\mathcal{T}(X))$ asymptotically converges to the ray tW, $t>0$. The set of asymptotic rays to $\mathcal{L}(\mathcal{T}(X))$ in $\mathcal{H}(X)$ forms a natural boundary for $\mathcal{T}(X)$.

In the next theorem we identify asymptotic rays to $\mathcal{L}(\mathcal{T}(X))$ with the space of projective bounded measured lamination $\mathcal{PML}_b(X)$ of X. We define a Thurston-type boundary for the Teichmüller space $\mathcal{T}(X)$ of an infinite surface X to be equal to $\mathcal{PML}_b(X)$.

Theorem 2. *The Teichmüller space $\mathcal{T}(X)$ has a Thurston-type boundary which is equal to $\mathcal{PML}_b(X)$.*

More precisely, if $tW \in \mathcal{H}(X)$ is an asymptotic ray to $\mathcal{L}(\mathcal{T}(X))$ then $W=\beta$ for some $\beta \in \mathcal{ML}_b(X)-\{0\}$ and conversely, given $\beta \in \mathcal{ML}_b(X)-\{0\}$ there exists a path $\alpha_t \in \mathcal{L}(\mathcal{T}(X))$ such that $\frac{1}{t}\alpha_t$ converges to β as $t \to \infty$ in the topology of $\mathcal{H}(X)$.

We say that $\mathcal{L}(\mathcal{T}(X))$ is asymptotic to $\mathcal{ML}_b(X)$. In order to specify the topology on the closure $\mathcal{T}(X) \cup \mathcal{PML}_b(X)$ of the Teichmüller space $\mathcal{T}(X)$, we use the projection map π from $\mathcal{H}(X)-\{0\}$ onto the unit sphere S^1_ν for a fixed ν-norm. The space $\mathcal{L}(\mathcal{T}(X))$ is homeomorphic to its image under π on the unit sphere S^1_ν. To each element of $\mathcal{PML}_b(X)$, there is exactly one corresponding element on S^1_ν. The topology on $\mathcal{T}(X) \cup \mathcal{PML}_b(X)$ is by the definition the induced topology from S^1_ν. We prove that the topology of the closure is independent of the chosen sphere S^1_ν, $0<\nu\leq 1$.

For a finite surface, Thurston [7] showed that the action of the mapping class group extends continuously to the boundary of the Teichmüller space. We show that this is also true in the infinite case.

Theorem 3. *The action of the quasiconformal mapping class group $QMCG(X)$ extends continuously to the boundary of $\mathcal{T}(X)$ and each element of $QCMG(X)$ is a homeomorphism of the closure of $\mathcal{T}(X)$.*

Given a bounded measured lamination $\beta \in \mathcal{ML}_b(X)$ and $[f_0] \in \mathcal{T}(X)$ there exists an earthquake path in $\mathcal{T}(X)$ with the initial point $[f_0]$ and with the measure $t\beta$, $t>0$. It is natural to expect that the endpoint of the above earthquake path is $[\beta] \in \mathcal{PML}_b(X)$.

Theorem 4. *Let $[f_t] \in \mathcal{T}(X)$ be an earthquake path with the initial point $[f_0] \in \mathcal{T}(X)$ and with the measure $t\beta$, $t>0$. Then the earthquake path $[f_t]$ converges to $[\beta] \in \mathcal{PML}_b(X)$ as $t \to \infty$ in the topology of the closure of $\mathcal{T}(X)$.*

We make a connection between going to infinity in $\mathcal{T}(X)$ toward a boundary point $[\beta] \in \mathcal{PML}_b(X)$ and the lengths of simple closed geodesics on X that intersect β. Namely, in Theorem 4.5 we show that if $[f_t] \to [\beta]$ as $t \to \infty$ then the length of simple closed geodesics on $f_t(X)$ which intersect β goes to infinity. However, the converse is not true due to the fact that on an arbitrary X we do not have enough simple closed geodesics.

The paper is organized as follows. In Section 2 we introduce the space of Hölder distributions and remind the reader about measured laminations, earthquakes and Teichmüller spaces. In Section 3 we define the Liouville map, describe its basic properties and prove Theorem 1. In Section 4 we define asymptotic rays to the image of the Teichmüller space to be boundary points and give a topology for the closure. Then we proceed to prove Theorems 2,3 and 4. In Appendix we give basic lemmas on the distortion of the Liouville mass of a box under simple earthquakes. These lemmas are heavily used in the proof of Corollary 4.1 which gives the second part of Theorem 2.

2. **Preliminaries**

The unit disk, the complex plane and the Riemann sphere are only simply connected Riemann surfaces, up to conformal equivalence. The Uniformization Theorem states that any Riemann surface X has exactly one of the three as a holomorphic universal covering space. We consider only Riemann surfaces which have the unit disk Δ for the universal covering. The unit disk admits a canonical hyperbolic metric and this metric projects to the unique hyperbolic metric on X which is compatible with the complex structure. Conversely, given a hyperbolic metric on X there exists a complex structure on X unique up to conformal equivalence which is compatible with the metric. A Riemann surface is *finite* if the unique hyperbolic metric compatible with the complex structure has finite area. A Riemann surface is *infinite* if the hyperbolic metric has infinite area.

2.1. **Angle Metrics.**

A hyperbolic metric on a surface X lifts to a unique hyperbolic metric on the universal covering $\tilde{X}$ of X. Given a hyperbolic metric on $\tilde{X}$ we define the boundary $\partial_\infty\tilde{X}$ of $\tilde{X}$ as follows. Fix a point $\tilde{x}\in\tilde{X}$. The boundary $\partial_\infty\tilde{X}$ is the set of geodesic rays from $\tilde{x}$. It can be shown that this definition does not depend on the choice of $\tilde{x}$ (see [6]). The boundary $\partial_\infty\tilde{X}$ is homeomorphic to S^1 and the homeomorphism can be obtained by continuously extending an isometry between $\tilde{X}$ and Δ to their boundaries.

A geodesic for a hyperbolic metric on $\tilde{X}$ has two distinct endpoints on $\partial_\infty\tilde{X}$. Conversely, given two distinct points on $\partial_\infty\tilde{X}$ there exists a unique geodesic in $\tilde{X}$ whose endpoints are equal to them. Thus, *the space* $G(\tilde{X})$ *of oriented geodesics in* $\tilde{X}$ is identified with $\partial_\infty\tilde{X}\times\partial_\infty\tilde{X}-diag$, where $diag$ denotes the diagonal of $\partial_\infty\tilde{X}\times\partial_\infty\tilde{X}$.

Definition 2.1. Let M be any set. Let d_1 and d_2 be two metrics on M. Then metric d_1 is *Hölder equivalent* to metric d_2 if there exists $C>0$ and ν, $0<\nu\leq 1$, such that
$$\frac{1}{C}d_2(x,y)^{-\nu}\leq d_1(x,y)\leq Cd_2(x,y)^{\nu}$$
for all $x,y\in M$. If $\nu=1$ in the above inequality then d_1 is *Lipschitz equivalent* to d_2.

We define an *angle metric* on $\partial_\infty\tilde{X}$. Fix $\tilde{x}\in\tilde{X}$. The distance between $\tilde{a}$ and $\tilde{b}$ on $\partial_\infty\tilde{X}$ is given by the angle at $\tilde{x}$ between the geodesic rays with the initial point $\tilde{x}$ and with the terminal points $\tilde{a}$ and $\tilde{b}$. This metric depends on the choice of $\tilde{x}\in\tilde{X}$. The metrics that arise from two different choices are Lipschitz equivalent.

However, there are more ambiguities in the definition of an angle metric to consider. Let X_1 be a Riemann surface and let $f:X_1\to X$ be a quasiconformal map. A lift $\tilde{f} : \tilde{X}_1 \to \tilde{X}$ of f is a quasiconformal mapping and hence a quasi-isometry for the hyperbolic metrics on $\tilde{X}_1$ and $\tilde{X}$ (see [5]). Then $\tilde{f}$ maps geodesics in $\tilde{X}_1$ onto quasi-geodesics in $\tilde{X}$. The mapping $\tilde{f}$ extends to a Hölder bi-continuous map between $\partial_\infty \tilde{X}_1$ and $\partial_\infty \tilde{X}$ for the angle metrics on $\tilde{X}_1$ and on $\tilde{X}$ (see [1] or [9]).

An angle metric on $\partial_\infty \tilde{X}$ gives the *product metric* on $G(\tilde{X}) \cong \partial_\infty \tilde{X} \times \partial_\infty \tilde{X} - diag$. The lift $\tilde{f}$ maps $G(\tilde{X}_1)$ onto $G(\tilde{X})$ and it is Hölder bi-continuous for the product metric. Thus we can identify $G(\tilde{X}_1)$ with $G(\tilde{X})$ if we consider the class of Hölder equivalent product metrics to a fixed product metric on $G(\tilde{X})$.

From now on we fix identification $\tilde{X} \cong \Delta$. Then $\partial_\infty \tilde{X} \cong S^1$ and $G(\tilde{X}) \cong S^1 \times S^1 - diag$. The unit circle S^1 has the *standard angle metric*. The distance between $x, y \in S^1$ in the standard angle metric is the angle at the origin between the radius which ends at x and the radius which ends at y. We denote by d the *standard product metric* induced on $S^1 \times S^1 - diag$ by the standard angle metric. Then we consider $G(\tilde{X}) \equiv S^1 \times S^1 - diag$ with the class of product metrics d_1 which are Hölder equivalent to d.

2.2. **Hölder Distributions.** If a function $\varphi : G(\tilde{X}) \to \mathbb{R}$ is Hölder continuous with respect to one metric in the above class then it is Hölder continuous with respect to any other metric. Therefore, the Hölder continuity of functions from $G(\tilde{X})$ to $\mathbb{R}$ is independent of the specific metric. The space $H(\tilde{X})$ consists of all Hölder continuous functions $\varphi : G(\tilde{X}) \to \mathbb{R}$ with compact support. For a ν-Hölder continuous function φ in $H(\tilde{X})$, in the standard product metric d, we define its ν-norm by

$$\|\varphi\|_\nu = \max\{\max|\varphi(x,y)|, \sup|\varphi(x,y) - \varphi(x_1,y_1)|d((x,y),(x_1,y_1))^{-\nu}\}$$

where the maximum inside the brackets is over all (x, y) in $G(\tilde{X})$ and where the supremum is over all distinct $(x, y), (x_1, y_1) \in G(\tilde{X})$. The space $H^\nu(\tilde{X})$ consists of all ν-Hölder continuous functions $\varphi : G(\tilde{X}) \to \mathbb{R}$ in the metric d with compact support. Then $H(\tilde{X}) = \cup_{0<\nu\leq 1} H^\nu(\tilde{X})$.

Definition 2.2. The *cross-ratio* of a quadruple (a, b, c, d) is given by $cr(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$.

For our purposes it will be convenient to consider a subset of the set of Hölder continuous functions with compact support. We consider ν-Hölder continuous functions in the metric d whose support is in a *box* $Q := [a, b] \times [c, d] \subset G(\tilde{X})$ with $cr(a, b, c, d) = 2$. Let Θ_Q be the Möbius transformation which maps $-i$, 1, i and -1 onto a, b, c and d, respectively. Such Θ_Q exists because $cr(a, b, c, d) = 2$. We introduce the set of test functions $test(\nu)$ to be the set of all (φ, Q), where $\varphi : G(\tilde{X}) \to \mathbb{R}$ is a ν-Hölder continuous functions on $G(\tilde{X})$ whose support is in $Q = [a, b] \times [c, d]$ with $cr(a, b, c, d) = 2$ and such that $\|\varphi \circ \Theta_Q\|_\nu \leq 1$.

We introduce the space $\mathcal{H}(\tilde{X})$ of Hölder distribution on $\tilde{X}$ using $test(\nu)$. The space $\mathcal{H}(\tilde{X})$ consists of all real linear functionals W on $H(\tilde{X})$ such that

$$\sup_{(\varphi,Q)\in test(\nu)} |W(\varphi)| < \infty$$

for all $0 < \nu \leq 1$. The supremum depends on the Hölder exponent ν in general. For a fixed ν, we define ν-norm of $W \in \mathcal{H}(\tilde{X})$ by

$$\|W\|_\nu = \sup_{(\varphi,Q)\in test(\nu)} |W(\varphi)|.$$

The family of ν-norms makes $\mathcal{H}(\tilde{X})$ into a topological vector space.

The space $\mathcal{H}^\nu(\tilde{X})$ consists of all real linear functionals W on $H^\nu(\tilde{X})$ such that $\|W\|_\nu = \sup_{(\varphi,Q)\in test(\nu)} |W(\varphi)| < \infty$. The ν-norm makes $\mathcal{H}^\nu(\tilde{X})$ into a Banach space. By restricting the elements $W \in \mathcal{H}(\tilde{X})$ to the space $H^\nu(\tilde{X})$ we obtain an inclusion of $\mathcal{H}(\tilde{X})$ into $\mathcal{H}^\nu(\tilde{X})$. Further $\mathcal{H}(\tilde{X}) = \cap_{0<\nu\leq 1}\mathcal{H}^\nu(\tilde{X})$ and the space $\mathcal{H}(\tilde{X})$ is a Frechét space. The idea of introducing above spaces comes from a related paper [13]. For the proofs of properties of these spaces see the above paper.

The action $\pi_1(X)$ on $\tilde{X}$ is by isometry. Since we fixed isometry $\tilde{X} \cong \Delta$ we identify $\pi_1(X)$ with a Fuchsian group Γ such that $X \cong \Delta/\Gamma$. The space $\mathcal{H}(X)$ of Hölder distributions on X consists of all $W \in \mathcal{H}(\tilde{X})$ such that $W(\varphi \circ \gamma) = W(\varphi)$ for all $\gamma \in \Gamma$ and for all $\varphi \in H(\tilde{X})$. The space $\mathcal{H}^\nu(X)$ consists of all $W \in \mathcal{H}^\nu(\tilde{X})$ such that $W(\varphi\circ\gamma) = W(\varphi)$ for all $\gamma \in \Gamma$ and for all $\varphi \in H^\nu(\tilde{X})$. The space $\mathcal{H}^\nu(X)$ is a Banach subspace of $\mathcal{H}^\nu(\tilde{X})$ and the space $\mathcal{H}(X)$ is a Fréchet subspace of $\mathcal{H}(\tilde{X})$.

We define the Liouville measure L on $G(\tilde{X})$. Let

$$L([a,b]\times[c,d]) := \log\frac{(a-c)(b-d)}{(a-d)(b-c)} = \log cr(a,b,c,d)$$

for a box $Q = [a,b]\times[c,d] \subset G(\tilde{X}) \equiv S^1 \times S^1 - diag$ and let

$$L(\{a\}\times[c,d]) = L([a,b]\times\{c\}) = 0.$$

It is possible to extend the quantity L to a positive Radon measure on $G(\tilde{X})$. Bonahon [2] showed that L is extendible to a smooth measure with the density $\frac{d\alpha d\beta}{|e^{i\alpha}-e^{i\beta}|^2}$. Note that the definition of L is independent of an identification $\tilde{X} \equiv \Delta$ because the cross-ratio is invariant under Möbius transformations.

In the light of the definition of the Liouville measure we can define $test(\nu)$ to consists of all (φ, Q) where $\varphi : G(\tilde{X}) \to \mathbb{R}$ is ν-Hölder continuous function with support in a box $Q = [a,b]\times[c,d]$ whose mass is $\log 2$ with respect to the Liouville measure and such that $\|\varphi \circ \Theta_Q\|_\nu \leq 1$.

Consider a positive Radon measure α on $G(\tilde{X})$. The measure α is *bounded* if

$$\sup \alpha(Q) < \infty$$

where the supremum is over all boxes $Q = [a,b]\times[c,d] \subset G(\tilde{X})$ with $L(Q) = \log 2$. We introduce the *norm* of a bounded measure α as

$$\|\alpha\| = \sup \alpha(Q)$$

where the supremum is over all boxes $Q = [a,b]\times[c,d]$ with $L(Q) = \log 2$. For $\varphi \in H(\tilde{X})$ we define

$$\alpha(\varphi) = \int_{G(\tilde{X})} \varphi d\alpha.$$

It is easy to see that if α is a bounded measure then $\alpha \in \mathcal{H}(\tilde{X})$. Thus the space of positive bounded measures α naturally embeds into the space of Hölder distributions $\mathcal{H}(\tilde{X})$. If α is Γ-invariant then $\alpha \in \mathcal{H}(X)$.

2.3. **Measured Laminations.** If the support of a positive measure β on $G(\tilde{X})$ consists of non-intersecting geodesics then β is called a *measured lamination*. For technical reasons we require that measured laminations are invariant under the self map of $G(\tilde{X})$ which changes the direction of each geodesic. The support of β is a *geodesic lamination*, namely a closed set which can be written as a union of non-intersecting geodesics. The space of all bounded measured laminations on $\tilde{X}$ is denoted by $\mathcal{ML}_b(\tilde{X})$. The space $\mathcal{ML}_b(X)$ consists of all Γ-invariant elements of $\mathcal{ML}_b(\tilde{X})$. In $\mathcal{ML}_b(\tilde{X}) - \{0\}$ we define the projective equivalence relation by $\lambda_1 \sim \lambda_2$ if there exists $t > 0$ such that $\lambda_1 = t\lambda_2$. The projective class of λ is denoted by $[\lambda]$. The set of all projective bounded measured laminations is denoted by $\mathcal{PML}_b(\tilde{X})$. The set of all projective bounded measured laminations invariant under Γ is denoted by $\mathcal{PML}_b(X)$.

2.4. **Teichmüller Space.** Using the identification $\tilde{X} \cong \Delta$ it is customary to define the Teichmüller space $\mathcal{T}(X)$ as a space of all quasiconformal maps $f : \Delta \to \Delta$ such that $f \circ \gamma \circ f^{-1}$ is a Möbius transformation for all $\gamma \in \Gamma$ modulo an equivalence relation. Two such quasiconformal maps f_1 and f_2 are equivalent if there exists a Möbius mapping Θ such that $\Theta \circ f_1|_{S^1} = f_2|_{S^1}$. We write $[f]$ for the equivalence class of f.

Equivalently, we can define $\mathcal{T}(X)$ to be the space of all quasisymmetric maps h of S^1 fixing $-i$, 1 and i, and such that $h\gamma h^{-1}$ is a Möbius map for all $\gamma \in \Gamma$.

2.5. **Earthquakes.** Thurston [14] introduced left (and right) earthquakes. We consider only left earthquakes. The (left) *earthquake map* E is a mapping from $\mathcal{ML}_b(X) \times \mathcal{T}(X)$ onto $\mathcal{T}(X)$ (see [14], [8] or [12]). The restriction of the earthquake map to $\{\beta\} \times \mathcal{T}(X)$ is called a left *earthquake* with the measure $\beta \in \mathcal{ML}_b(X)$. For a fixed $[f] \in \mathcal{T}(X)$ and for a fixed $\beta \in \mathcal{ML}_b(X)$ the image of $\{(t\beta, [f]) \in \mathcal{ML}_b(X) \times \mathcal{T}(X);\ t \geq 0\}$ under the earthquake map is called an *earthquake path* with the measure $t\beta$ and with the parameter $t > 0$. The above earthquake path has the initial point $[f]$.

An earthquake for a finite measured lamination is called a *finite earthquake*. A *simple earthquake* is an earthquake for a measured lamination whose support consists of only one geodesic.

We define $E(\beta, [id]) : \Delta \to \Delta$, where $[id] \in \mathcal{T}(\tilde{X})$ is the base point. By taking the restriction of $E(\beta, [id])$ to S^1 we obtain a point in $\mathcal{T}(\tilde{X})$. Assume first that the support of β has finitely many geodesics $\{g_1, g_2, \ldots, g_n\}$. The connected components of the complement of the support of β are called *gaps* of β. Finite measured lamination β has finitely many gaps $\{G_1, G_2, \ldots, G_k\}$. We fix one gap, say G_1. Finite earthquake $E_\beta := E(\beta, [id])$ is the identity on this gap. For any other gap G_j we connect it to G_1 by a geodesic arc s. The arc s intersects finitely many geodesics g_{j_i}, $i = 1, 2, \ldots r$ of the support of β given in order from G_1 to G_j. We orient g_{j_i} to the left as seen from G_1. Denote by A_{j_i} the hyperbolic translation with the oriented axis g_{j_i} and with the translation length $\beta(g_{j_i})$. Earthquake E_β on G_{j_i} is given by $E_\beta|G_{j_i} = A_{j_1} \circ A_{j_2} \circ \cdots \circ A_{j_r}$. Note that finite earthquake E_β is not continuously extendible to g_j. To make earthquake defined on the whole Δ we set E_β on g_j to be equal to $E_\beta|G_j$, where G_j is the gap adjacent to g_j from the left as seen from G_1. Then $E_\beta : \Delta \to \Delta$ is onto and one to one but it is not continuous. The extension of E_β to S^1 is continuous and moreover it is a quasisymmetric map.

We normalize E_β by postcomposing it with a Möbius transformation such that it fixes $-i$, 1 and i, and we obtain an element of $\mathcal{T}(\tilde{X})$.

The earthquake $E_\beta := E(\beta, [id])$ for non-finite bounded measured lamination β is defined as the limit of finite earthquakes with finite measures approximating non-finite measure β. By taking the Teichmüller class of a Γ-invariant extension of $E_\beta|S^1$ we get a point in $\mathcal{T}(X)$.

If $[f] \in \mathcal{T}(X)$ is not the base point $[id]$ then we define $E(\beta, [f]) = f \circ E(\beta, [id])$. It is easy to see that $E(\beta, [f]) = E(f^*(\beta), [id]) \circ f$ where $f^*(\beta)$ is the push-forward of β to $f(\tilde{X}) \cong \Delta$.

Definition 2.3. We say that some quantity b_n is *of the order* a_n if there exists $C \geq 1$ such that

$$\frac{1}{C} a_n \leq b_n \leq C a_n.$$

3. **Embedding of $\mathcal{T}(X)$**

In this section we define the Liouville map $\mathcal{L} : \mathcal{T}(X) \to \mathcal{H}(X)$ and investigate its global properties.

Given $[f] \in \mathcal{T}(X)$ we construct a measure α on $G(\tilde{X})$. Denote by h the extension of f to S^1 and note that h is a bi-Hölder continuous map of S^1 onto itself. Thus h maps Borel sets in $G(\tilde{X}) = S^1 \times S^1 - diag$ onto Borel sets in $G(\tilde{X})$. Define

$$\alpha(B) = L(h(B)) \tag{1}$$

for all Borel sets $B \subset G(\tilde{X})$. The measure α is called a *Liouville current.* By the definition, the *Liouville map* $\mathcal{L} : \mathcal{T}(X) \to \mathcal{H}(X)$ is given by $\mathcal{L}([f]) = \alpha$.

Given two different $[f]$ and $[f_1]$ in $\mathcal{T}(X)$ it is easy to see that we get different measures on $G(\tilde{X})$. Consequently, the Liouville map is injective.

Bonahon [2] proved that any Liouville current $\alpha = \mathcal{L}([f])$, $[f] \in \mathcal{T}(X)$, satisfies

$$e^{-\alpha([a,b]\times[c,d])} + e^{-\alpha([b,c]\times[d,a])} = 1 \tag{2}$$

for all boxes $Q = [a, b] \times [c, d] \subset G(\tilde{X})$.

Conversely, if α is a measure which satisfies (2) then α is obtained as in (1) from some homeomorphism h of S^1 (see [2]). The homeomorphism h is unique up to post-composition with a Möbius map. If α is a bounded measure which satisfies (2) then h is obviously a quasisymmetric map. Consequently h extends to a quasiconformal map f of Δ onto itself and α is a Liouville current.

Remark 3.1. One might expect to have a bound from below to the set $\{\alpha(Q); L(Q) = \log 2\}$ in the definition of a bounded measure α in order to be able to claim that h is quasisymmetric. But, for α obtained from some homeomorphism h of S^1 as in (1), if $\sup \alpha(Q) = M < \infty$ then $\inf \alpha(Q) = M_1 > 0$ where the supremum and the infimum are over all boxes Q with $L(Q) = \log 2$. This fact is easily proved by considering $Q = [a, b] \times [c, d]$ and $\tilde{Q} = [b, c] \times [d, a]$, simultaneously.

We show that $f : \Delta \to \Delta$ can be chosen to be Γ-invariant if α is Γ-invariant. By the existence of Barycentric extension [4] it is enough to show that $h\gamma h^{-1}$ is a Möbius map, for all $\gamma \in \Gamma$. Since α is Γ-invariant we obtain that $\alpha(\gamma(Q)) = \alpha(Q)$ for all boxes $Q = [a, b] \times [c, d] \subset G(\tilde{X})$ and for all $\gamma \in \Gamma$. By the definition of α and by the above invariance we get

$$L(h\gamma(Q)) = L(h(Q))$$

for all boxes $Q \subset G(\tilde{X})$.

The above inequality is equivalent to

$$L(h\gamma h^{-1}(Q_1)) = L(Q_1)$$

for all boxes $Q_1 \subset G(\tilde{X})$. Thus the map $h\gamma h^{-1}$ preserves the cross-ratios hence it is a Möbius map. Consequently $[f] \in \mathcal{T}(X)$.

We gather above results into a theorem.

Theorem 3.1. *The Liouville map $\mathcal{L} : \mathcal{T}(X) \to \mathcal{H}(X)$ is well-defined and one to one. The image of $\mathcal{T}(X)$ consists of all bounded Γ-invariant measures $\alpha \in \mathcal{H}(X)$ which satisfy* (2) *for all boxes $[a, b] \times [c, d] \subset G(\tilde{X})$.* □

We need the following technical lemma:

Lemma 3.1. *Let $[f_0] \in \mathcal{T}(X)$, $k > 1$ and $\epsilon > 0$ be given and let $\mathcal{L}([f_0]) = \alpha_0$. Then there exists a neighborhood $N([f_0]; k, \epsilon)$ of $[f_0]$ in $\mathcal{T}(X)$ such that*

$$|\alpha(Q) - \alpha_0(Q)| < \epsilon$$

for all $\alpha = \mathcal{L}([f])$ with $[f] \in N([f_0]; k, \epsilon)$ and for all $Q = [a, b] \times [c, d]$ with $\log(1 + \frac{1}{k}) \leq L(Q) \leq \log k$.

Proof. By definition, a K-quasiconformal map f satisfies

$$\frac{1}{K} m(a, b, c, d) \leq m(f(a, b, c, d)) \leq K m(a, b, c, d) \tag{3}$$

where $m(a, b, c, d)$ is the module of the quadrilateral whose sides lie on S^1 and whose vertices are elements of (a, b, c, d), and where $f(a, b, c, d) = (f(a), f(b), f(c), f(d))$ (see [1] or [9]).

The module $m(a, b, c, d)$ is a continuous function of the cross-ratio $cr(a, b, c, d)$ and vice versa. In particular, the cross-ratio $cr(a, b, c, d)$ is a uniform function of $m(a, b, c, d)$, for $m(a, b, c, d)$ in a compact set. Consequently, if $1+\frac{1}{k_1} \leq cr(a, b, c, d) \leq k_1$ then

$$\frac{1}{C(k_1)} \leq m(a, b, c, d) \leq C(k_1) \tag{4}$$

where $C(k_1)$ is a constant depending on k_1. Also $m(a, b, c, d) \to 0$ if and only if $cr(a, b, c, d) \to 1$, and $m(a, b, c, d) = 1$ if and only if $cr(a, b, c, d) = 2$.

By (3) and by (4) we can choose k_1 big enough such that

$$\{Q : \log(1 + \frac{1}{k}) \leq L(Q) \leq \log k\} \subseteq \{Q : \log(1 + \frac{1}{k_1}) \leq L(f_0^{-1}Q) \leq \log k_1\}. \tag{5}$$

Let g be a $(1+\delta)$-quasiconformal map. By (3), for δ small enough, the difference $m(g(a, b, c, d)) - m(a, b, c, d)$ is small for all (a, b, c, d) which satisfy (4). Thus, for δ small enough, by the uniform continuity of $cr(a, b, c, d)$ in $m(a, b, c, d)$ and by (4) we get

$$|cr(g(a, b, c, d)) - cr(a, b, c, d)| < \frac{\epsilon}{2} \tag{6}$$

for all (a, b, c, d) which satisfy $1 + \frac{1}{k_1} \leq cr(a, b, c, d) \leq k_1$. The constant $\delta > 0$ depends on k and ϵ.

We choose a neighborhood $N([f_0]; k, \epsilon)$ of $[f_0]$ in $\mathcal{T}(X)$ such that $[f] \in N([f_0]; k, \epsilon)$ if $f \circ f_0^{-1}$ is a $(1 + \delta)$-quasiconformal. From (6) we get that

$$\left| cr(f \circ f_0^{-1}(a, b, c, d)) - cr(a, b, c, d) \right| < \frac{\epsilon}{2} \tag{7}$$

for all a, b, c and d satisfying (4). We divide (7) with $cr(a,b,c,d)$ and get

$$\frac{1}{1+\epsilon} \leq \frac{cr(f \circ f_0^{-1}(a,b,c,d))}{cr(a,b,c,d)} \leq 1+\epsilon$$

for all (a,b,c,d) satisfying $1+\frac{1}{k_1} \leq cr(a,b,c,d) \leq k_1$ and for all $[f] \in N([f_0]; k, \epsilon)$. From the above inequality we get

$$\frac{1}{1+\epsilon} \leq \frac{cr(f(\tilde{a},\tilde{b},\tilde{c},\tilde{d}))}{cr(f_0(\tilde{a},\tilde{b},\tilde{c},\tilde{d}))} \leq 1+\epsilon \tag{8}$$

for all $(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = f_0^{-1}(a,b,c,d)$ with $1+\frac{1}{k_1} \leq cr(a,b,c,d) \leq k_1$ and for all $f \in N([f_0]; k, \epsilon)$.

By taking the logarithm in (8) and noting (5) we get the conclusion. □

We proceed to prove the continuity of the Liouville map.

Theorem 3.2. *The Liouville map $\mathcal{L} : \mathcal{T}(X) \to \mathcal{H}(X)$ is continuous.*

Proof. Let $[f_0] \in \mathcal{T}(X)$ and let $\mathcal{L} : [f_0] \mapsto \alpha_0$. Fix $0 < \nu \leq 1$ and fix $\epsilon > 0$. Let $(\varphi, Q) \in test(\nu)$ where $Q = [a,b] \times [c,d]$ with $L(Q) = \log 2$. Then $\Theta_Q : S^1 \to S^1$ is the Möbius mapping which maps $-i, 1, i$ and -1 onto a, b, c and d, respectively. We define β_0 to be the pull-back of the measure α_0 by Θ_Q, i.e. $\beta_0 = (\Theta_Q)_* \alpha_0 = \alpha_0 \circ \Theta_Q$.

Let $N([f_0]; \frac{1}{e^{\frac{C}{n^2}}-1}, \frac{1}{n^3})$ be a neighborhood of $[f_0]$ in $\mathcal{T}(X)$ as in Lemma 3.1. In particular, $[f] \in N([f_0]; \frac{1}{e^{\frac{C}{n^2}}-1}, \frac{1}{n^3})$ implies that

$$|\alpha_0(\tilde{Q}) - \alpha(\tilde{Q})| < \frac{1}{n^3} \tag{9}$$

for all boxes $\tilde{Q}$ which satisfy $\frac{C}{n^2} \leq L(\tilde{Q}) \leq \log 2$ where constant $C > 0$ is to be given later, and where $\mathcal{L} : [f] \mapsto \alpha$. Let $\beta = (\Theta_Q)_* \alpha$.

Divide arc $[-i, 1] \subset S^1$ into n equal subarcs $[a_{i-1}, a_i]$, for $i = 1, 2, \ldots, n$ with $a_0 = -i$ and $a_n = 1$. Divide arc $[i, -1] \subset S^1$ into n equal subarcs $[c_{j-1}, c_j]$, for $j = 1, 2, \ldots, n$ with $c_0 = i$ and $c_n = -1$. We form boxes $E_{ij} = [a_{i-1}, a_i] \times [c_{j-1}, c_j]$ for $i, j = 1, 2, \ldots n$. Note that $\cup_{i,j=1}^n E_{ij} = [-i, 1] \times [i, -1]$ and each pairwise intersection of two E_{ij} either is empty or is $\{a_i\} \times [c_{j-1}, c_j]$ or is $[a_{i-1}, a_i] \times \{c_j\}$. Thus, the intersection of two distinct $E_{i,j}$ has zero mass for the measures β_0 and β.

The diameter of each $[a_{i-1}, a_i]$ and of each $[c_{j-1}, c_j]$ in the standard angle metric on S^1 is $\frac{\pi/2}{n}$. Since the arcs $[a_{i-1}, a_i]$ lie in the fixed interval $[-i, 1]$ and the arcs $[c_{j-1}, c_j]$ lie in the fixed interval $[i, -1]$ the Liouville measure of $E_{ij} = [a_{i-1}, a_i] \times [c_{j-1}, c_j]$ is comparable to $\frac{1}{n^2}$. By (9) we get

$$|\beta(E_{ij}) - \beta^{'}(E_{ij})| < \frac{1}{n^3}. \tag{10}$$

Note that

$$\begin{aligned} \textstyle\int \varphi d\alpha_0 = \int \varphi \circ \Theta_Q d\beta_0 \\ \textstyle\int \varphi d\alpha = \int \varphi \circ \Theta_Q d\beta. \end{aligned} \tag{11}$$

We define a step function approximation $\varphi_n \circ \Theta_Q = \sum_{i,j=1}^n p_{ij} \chi_{E_{ij}}$ to $\varphi \circ \Theta_Q$, where $p_{ij} = \varphi \circ \Theta_Q(g_{ij})$ for a fixed geodesic $g_{ij} \in E_{ij}$. By the Hölder continuity of $\varphi \circ \Theta_Q$ we obtain

$$|\varphi \circ \Theta_Q - \sum p_{ij} \chi_{E_{ij}}| \leq \frac{(\pi/2)^\nu}{n^\nu}. \tag{12}$$

Integrating the expression $\varphi\circ\Theta_Q-\sum p_{ij}\chi_{E_{ij}}$ and using inequality (12) we get

$$\begin{aligned}&\Big|\int\varphi\circ\Theta_Q d\beta_0-\int\varphi_n\circ\Theta_Q d\beta_0\Big|\leq\frac{(\pi/2)^\nu}{n^\nu}\beta_0([-i,1]\times[i,-1])\\&\Big|\int\varphi\circ\Theta_Q d\beta-\int\varphi_n\circ\Theta_Q d\beta\Big|\leq\frac{(\pi/2)^\nu}{n^\nu}\beta([-i,1]\times[i,-1]).\end{aligned}\tag{13}$$

We estimate $\Big|\int\varphi_n d\alpha_0-\int\varphi_n d\alpha\Big|$. Using the definition of φ_n we get

$$\Big|\int\varphi_n d\alpha_0-\int\varphi_n d\alpha\Big|\leq\sum|p_{ij}|\cdot|\beta_0(E_{ij})-\beta(E_{ij})|.\tag{14}$$

Further by (10) we get

$$\sum|p_{ij}|\cdot|\beta_0(E_{ij})-\beta(E_{ij})|<\sum_{i,j=1}^{n}\frac{1}{n^3}=\frac{1}{n}.\tag{15}$$

From (14) and (15) we obtain

$$\Big|\int\varphi_n d\alpha_0-\int\varphi_n d\alpha\Big|<\frac{1}{n}.\tag{16}$$

Using the triangle inequality

$$\begin{aligned}\Big|\int\varphi d\alpha_0-\int\varphi d\alpha\Big|\leq\Big|\int\varphi d\alpha_0-\int\varphi_n d\alpha_0\Big|+\\\Big|\int\varphi_n d\alpha_0-\int\varphi_n d\alpha\Big|+\Big|\int\varphi_n d\alpha-\int\varphi d\alpha\Big|\end{aligned}\tag{17}$$

and inequalities (13) and (16) we obtain

$$\Big|\int\varphi d\alpha_0-\int\varphi d\alpha\Big|\leq\frac{(\pi/2)^\nu}{n^\nu}[\alpha_0(Q)+\alpha(Q)]+\frac{1}{n}.\tag{18}$$

Since $[f_0]$ is fixed, there exists a constant $C_2>0$ such that $\alpha_0(Q)\leq C_2$ and for any α as above $\alpha(Q)\leq C_2+1$ by inequality (9). Thus the right side of (18) is smaller than ϵ for n big enough. Hence

$$\Big|\int\varphi d\alpha_0-\int\varphi d\alpha\Big|<\epsilon$$

for all α in the image under $\mathcal{L}$ of a sufficiently small neighborhood of $[f_0]$ and for all $(\varphi,Q)\in test(\nu)$. This means that α is close to α_0 in the topology of $\mathcal{H}(\tilde{X})$. Thus $\mathcal{L}$ is continuous. □

It remains to prove the continuity of $\mathcal{L}^{-1}$.

Theorem 3.3. *The map $\mathcal{L}^{-1}:\mathcal{L}(\mathcal{T}(X))\to\mathcal{T}(X)$ is continuous.*

Proof. Fix $\alpha_0\in\mathcal{L}(\mathcal{T}(X))$. We show the continuity of $\mathcal{L}^{-1}$ at α_0. By the definition of the topology on $\mathcal{H}(\tilde{X})$ it is enough to show the continuity of $\mathcal{L}^{-1}$ for one ν-norm.

Fix $0<\nu\leq 1$. Let $\mathcal{L}:[f_0]\mapsto\alpha_0$ and fix $\epsilon(\nu)>0$. Let $U_{\epsilon(\nu)}$ be the $\epsilon(\nu)$-neighborhood of α_0 in $\mathcal{L}(\mathcal{T}(X)))$ for the ν-norm, namely $\alpha\in U_{\epsilon(\nu)}$ if

$$\sup\Big|\int\varphi d\alpha_0-\int\varphi d\alpha\Big|<\epsilon(\nu)$$

where the supremum is over all $(\varphi,Q)\in test(\nu)$.

First, we show that $\mathcal{L}^{-1}(U_{\epsilon(\nu)})$ is bounded in $\mathcal{T}(X)$. To see this it is enough to show that there exists a constant $M>0$ such that

$$\alpha(Q)\leq M$$

for any $\alpha \in U_{\epsilon(\nu)}$ and for all boxes $Q = [a,b] \times [c,d]$ with $L(Q) = \log 2$. Assume on the contrary that there exists a sequence of measures $\alpha_n \in U_{\epsilon(\nu)}$ and a sequence of boxes $Q_n = [a_n, b_n] \times [c_n, d_n] \subset G(\tilde{X})$ with $L(Q_n) = \log 2$ such that $\alpha_n(Q_n) \to \infty$ as $n \to \infty$. Then we can subdivide $[a_n, b_n] \times [c_n, d_n]$ such that we get a new sequence of boxes $Q_n^{'} = [a_n^{'}, b_n^{'}] \times [c_n^{'}, d_n^{'}] \subset Q_n$ for which $L(Q_n^{'}) \to 0$ and $\alpha_n(Q_n^{'}) \to \infty$ as $n \to \infty$. Let Θ_n be a Möbius transformation such that Θ_n^{-1} maps $[a_n^{'}, b_n^{'}] \times [c_n^{'}, d_n^{'}]$ close to the geodesic $(e^{i\frac{7\pi}{4}}, e^{i\frac{3\pi}{4}}) \in G(\tilde{X})$ in the sense that $\Theta_n^{-1}(a_n^{'})$ and $\Theta_n^{-1}(b_n^{'})$ converge to $e^{i\frac{7\pi}{4}}$, and $\Theta_n^{-1}(c_n^{'})$ and $\Theta_n^{-1}(d_n^{'})$ converge to $e^{i\frac{3\pi}{4}}$ as $n \to \infty$ for the standard angle metric. It is obvious that for n large we can find a sequence of functions φ_n such that the support of $\varphi_n \circ \Theta_n$ is in $[-i, 1] \times [i, -1]$, $\varphi_n \geq 0$, $\varphi_n = 1$ on $\Theta_n^{-1}(Q_n^{'})$ and $\|\varphi_n \circ \Theta_n\|_\nu \leq 2$. Thus $(\frac{\varphi_n}{2}, \Theta_n([-i,1] \times [i,-1])) \in test(\nu)$. Consequently $\int \frac{\varphi_n}{2} d\alpha_n \geq \frac{1}{2}\alpha_n(Q_n^{'}) \to \infty$ as $n \to \infty$. But this is a contradiction with $\alpha_n \in U_{\epsilon(\nu)}$. Thus $\mathcal{L}^{-1}(U_{\epsilon(\nu)})$ is bounded in $\mathcal{T}(X)$.

Let $D_\delta = [-ie^{i\delta}, 1e^{-i\delta}] \times [ie^{i\delta}, -1e^{-i\delta}]$ and $B_\delta = [-i, 1] \times [i, -1] - D_\delta$ for $\delta > 0$ small. Let $Q = [a,b] \times [c,d]$ be any box such that $L(Q) = \log 2$. As in Section 2, Θ_Q is the Möbius transformation which maps $-i$, 1, i and -1 onto a, b, c and d, respectively. We define function φ as follows: $\varphi \circ \Theta_Q(e^{ix}, e^{iy}) = 1$ for $(e^{ix}, e^{iy}) \in D_\delta$; $\varphi \circ \Theta_Q(e^{ix}, e^{iy}) = 0$, outside $[-i, 1] \times [i, -1]$; and $\varphi \circ \Theta_Q(e^{ix}, e^{iy}) = \frac{\min\{|x - \frac{3\pi}{2}|, |x - 2\pi|, |y - \frac{\pi}{2}|, |y - \pi|\}}{\delta}$ for $(e^{ix}, e^{iy}) \in B_\delta$.

It is clear that $\|\varphi \circ \Theta_Q\|_\nu = \delta^{-\nu}$. We define $\varphi_{\delta,\nu} \circ \Theta_Q = \delta^\nu \varphi \circ \Theta_Q$. Then $\|\varphi_{\delta,\nu} \circ \Theta_Q\|_\nu = 1$ and consequently $(\varphi_{\delta,\nu}, Q) \in test(\nu)$.

Let $\mathcal{L} : [f] \mapsto \alpha$ and let $\beta = (\Theta_Q)_* \alpha$ for $\alpha \in U_{\epsilon(\nu)}$. We proved above that $[f]$ lies in a bounded subset of $\mathcal{T}(X)$, i.e. there exists a bound on the constant of quasiconformality for all such f. Consequently we can choose δ small enough such that

$$\beta_0(B_\delta) + \beta(B_\delta) < \frac{\epsilon}{4}. \tag{19}$$

Since $\alpha_0, \alpha \in U_{\epsilon(\nu)}$ and $(\varphi_{\delta,\nu}, Q) \in test(\nu)$ we get

$$\Big| \int \varphi_{\delta,\nu} d\alpha_0 - \int \varphi_{\delta,\nu} d\alpha \Big| < \epsilon(\nu). \tag{20}$$

By the definition of $\varphi_{\delta,\nu}$

$$\begin{aligned} \Big| \int \varphi_{\delta,\nu} d\alpha_0 - \int \varphi_{\delta,\nu} d\alpha \Big| \geq \Big| \Big| \int_{B_\delta} \varphi_{\delta,\nu} \circ \Theta_Q d\beta_0 - \int_{B_\delta} \varphi_{\delta,\nu} \circ \Theta_Q d\beta \Big| - \\ \delta^\nu \cdot |\beta_0(D_\delta) - \beta(D_\delta)| \Big|. \end{aligned} \tag{21}$$

The right side of the above inequality is greater than or equal to

$$\delta^\nu \cdot |\beta_0(D_\delta) - \beta(D_\delta)| - \delta^\nu \cdot (\beta_0(B_\delta) + \beta(B_\delta)). \tag{22}$$

Combining (20), (21) and (22) we obtain

$$|\beta_0([-i,1] \times [i,-1]) - \beta([-i,1] \times [i,-1])| \leq \frac{\epsilon(\nu)}{\delta^\nu} + 2[\beta_0(B_\delta) + \beta(B_\delta)].$$

By the above inequality and by (19) we get

$$|\alpha_0(Q) - \alpha(Q)| \leq \epsilon$$

for $\epsilon(\nu) = \frac{\epsilon\delta^\nu}{2}$. This implies that $[f]$ is close to $[f_0]$ for ϵ small enough. Thus $\mathcal{L}^{-1}$ is continuous. □

Theorem 3.4. *The image $\mathcal{L}(\mathcal{T}(X))$ of $\mathcal{L}$ is closed and unbounded.*

Proof. In the proof of Theorem 3.3 we showed that $\mathcal{L}^{-1}(U)$ is bounded for U bounded subset of $\mathcal{L}(\mathcal{T}(X))$. Hence $\mathcal{L}(\mathcal{T}(X))$ is unbounded because $\mathcal{T}(X)$ is unbounded.

We show that $\mathcal{L}(\mathcal{T}(X))$ is closed. By the same method as in the proof of Theorem 4.1 one can show that an element β in the closure of $\mathcal{L}(\mathcal{T}(X))$ is a positive measure. The measure β is bounded because it is an element of $\mathcal{H}(\tilde{X})$ and it is Γ-invariant because it is the limit of a Γ-invariant measures α_t. Because $\alpha_t \to \beta$ as $t \to \infty$, there exists $t_0 > 0$ such that $\{\alpha_t; t \geq t_0\}$ is a bounded set in $\mathcal{H}(\tilde{X})$.

Let $[f_t] \in \mathcal{T}(X)$ such that $\mathcal{L}([f_t]) = \alpha_t$. Then $\{f_t; t > t_0\}$ have a bounded constant of quasiconformality. We choose representatives f_t of $[f_t]$ such that f_t fixes $-i$, 1 and i. Then a subsequence of f_t converges uniformly on compact subsets to a quasiconformal map g which fixes $-i$, 1 and i.

Let $\beta_1 = \mathcal{L}([g])$ and let $Q = [a,b] \times [c,d]$ be any box. Then $\alpha_t(Q) \to \beta_1(Q)$ as $t \to \infty$ by the pointwise convergence of f_t to g. Consequently, $\alpha_t(\varphi) \to \beta_1(\varphi)$ as $t \to \infty$ for all $\varphi \in H^\nu(\tilde{X})$. By the uniqueness part of Riesz Representation Theorem, we get $\beta = \beta_1$. Thus $\beta \in \mathcal{L}(\mathcal{T}(X))$ and $\mathcal{L}(\mathcal{T}(X))$ is closed. □

Theorem 1 follows directly from Theorems 3.1, 3.2, 3.3 and 3.4.

4. **Closure of $\mathcal{T}(X)$**

In this section we use the embedding of $\mathcal{T}(X)$ into $\mathcal{H}(X)$ to define a natural boundary to the Teichmüller space. By Theorem 3.4, the image $\mathcal{L}(\mathcal{T}(X))$ is closed and unbounded. The idea is to use asymptotic rays to $\mathcal{L}(\mathcal{T}(X))$ in $\mathcal{H}(X)$ to introduce a boundary at infinity for $\mathcal{L}(\mathcal{T}(X))$. By the definition, this boundary will be a boundary for $\mathcal{T}(X)$.

A ray tW, for $W \in \mathcal{H}(X)$ and for $t > 0$, is *asymptotic* to $\mathcal{L}(\mathcal{T}(X))$ if there exists a path $\alpha_t \in \mathcal{L}(\mathcal{T}(X))$ such that $\frac{1}{t}\alpha_t$ converges to W as $t \to \infty$ in the topology of $\mathcal{H}(X)$. A different parametrization of the path α_t might give a path α_t' such that $\frac{1}{t}\alpha_t'$ does not converge in $\mathcal{H}(X)$.

To avoid ambiguities which arise from reparametrizations and to give a topology on the closure of $\mathcal{T}(X)$, we introduce the projectivization of $\mathcal{H}(X)$. Namely, the space of projective Hölder distributions $\mathcal{PH}(X)$ consists of equivalence classes of elements in $\mathcal{H}(X) - \{0\}$ where $W_1 \sim W_2$ if there exists $\lambda > 0$ such that $W_1 = \lambda W_2$. Let $\pi : \mathcal{H}(X) - \{0\} \to \mathcal{PH}(X)$ be the natural projection map given by $\pi(W) = W/\sim$. The space $\mathcal{PH}(X)$ has the quotient topology. There is a one to one correspondence between $\mathcal{PH}(X)$ and the unit sphere S^1_ν in $\mathcal{H}(X)$ for a fixed ν-norm, $0 < \nu \leq 1$, given by $I(W/\sim) = \frac{W}{\|W\|_\nu}$. The unit sphere S^1_ν has the subspace topology inherited from $\mathcal{H}(X)$.

Proposition 4.1. *The map I from the projective Hölder distributions $\mathcal{PH}(X)$ to the unit sphere S^1_ν in $\mathcal{H}(X)$ for a fixed ν-norm, $0 < \nu \leq 1$, is a homeomorphism.*

Proof. The map I is one to one and onto. It remains to show that I is continuous and open.

To show that I is continuous, it suffices to show that $I \circ \pi : \mathcal{H}(X) - \{0\} \to S^1_\nu$ is continuous. The map $I \circ \pi : W \mapsto \frac{W}{\|W\|_\nu}$ is continuous if it is continuous for a sequence of ν_n-norms where $\nu_n \to 0$ as $n \to \infty$. In a related paper [13], we prove that

$$\|\varphi\|_\mu \leq \left(\frac{\pi}{2}\right)^{\nu-\mu} \|\varphi\|_\nu \tag{23}$$

for all $(\varphi, Q) \in test(\nu)$ and for all $\mu < \nu$. By (23), if $(\varphi, Q) \in test(\nu)$ then $((\frac{\pi}{2})^{\mu-\nu}\varphi, Q) \in test(\mu)$ for $\mu < \nu$. We fix $W \in \mathcal{H}(X) - \{0\}$ and $\nu_n < \nu$. A neighborhood of W consists of all W_1 such that $\|W - W_1\|_{\nu_n} < \epsilon$. By the remark following (23), $\|W - W_1\|_\nu < (\frac{\pi}{2})^{\nu-\nu_n}\epsilon$. This implies that $\|W\|_\nu$ and $\|W_1\|_\nu$ are close depending on ϵ. By the triangle inequality

$$\Big\|\frac{W}{\|W\|_\nu} - \frac{W_1}{\|W_1\|_\nu}\Big\|_{\nu_n} \leq \|W\|_{\nu_n}(\frac{1}{\|W\|_\nu} - \frac{1}{\|W_1\|_\nu}) + \frac{1}{\|W_1\|_\nu}\|W - W_1\|_{\nu_n}.$$

The right hand side of the above inequality is as small as we want for ν_n fixed and for $\epsilon > 0$ small enough. This proves the continuity of $I \circ \pi$. Consequently, I is continuous.

We show that I is open. Let $U/\sim$ be an open subset of $\mathcal{PH}(X)$ where $U = \pi^{-1}(U/\sim)$ is an open subset of $\mathcal{H}(X)$. Let $W/\sim$ be any point in $U/\sim$. Let $W \in U$ be one point in $\pi^{-1}(W/\sim)$. Since U is open, there exists ν_1, $0 < \nu_1 \leq 1$, and $\epsilon > 0$ such that $N_W = \{W_1; \|W - W_1\|_{\nu_1} < \epsilon\} \subset U$. Then the set $\frac{1}{\|W\|_\nu}N_W = \{\frac{W_1}{\|W\|_\nu}; \|W - W_1\|_{\nu_1} < \epsilon\}$ is an open neighborhood of $\frac{W}{\|W\|_\nu}$ in $\mathcal{H}(X)$. Thus, $(\frac{1}{\|W\|_\nu}N_W) \cap S^1_\nu$ is an open neighborhood of $\frac{W}{\|W\|_\nu}$ in S^1_ν and it is contained in the image of $U/\sim$ under I. The map I is open. □

The map $I \circ \pi : \mathcal{H}(X) - \{0\} \to S^1_\nu$ when restricted to $\mathcal{L}(\mathcal{T}(X))$ is one to one. To see this note that any $\alpha \in \mathcal{L}(\mathcal{T}(X))$ satisfies equation (2). Then $\lambda\alpha$, for $\lambda \neq 1$ cannot satisfy (2). We show that the restriction of $I \circ \pi$ to $\mathcal{L}(\mathcal{T}(X))$ is a homeomorphism onto its image.

Proposition 4.2. *The map $I \circ \pi$ when restricted to $\mathcal{L}(\mathcal{T}(X))$ is a homeomorphism onto its image.*

Proof. Since $I \circ \pi$ is one to one, it is enough to show that $I \circ \pi$ is continuous and open.

By Proposition 4.1, $I \circ \pi : \mathcal{H}(X) - \{0\} \to S^1_\nu$ is continuous and consequently its restriction is continuous. The arguments in the proof of Proposition 4.1 show that $I \circ \pi$ is open. Thus, the restriction of $I \circ \pi$ is open in the relative topology of the restricted domain. Consequently, the map $I \circ \pi$ is a homeomorphism onto its image. □

We introduce a boundary for $\mathcal{T}(X)$ using the image of $\mathcal{L}(\mathcal{T}(X))$ on the unit sphere S^1_ν. Namely, a *boundary point* for $\mathcal{T}(X)$ is by the definition a boundary point of $(I \circ \pi)(\mathcal{L}(\mathcal{T}(X)))$ on S^1_ν. Since S^1_ν is identified with $\mathcal{PH}(X)$ the boundary is a subset of projective Hölder distributions $\mathcal{PH}(X)$. Because $\mathcal{L}(\mathcal{T}(X))$ is closed, each boundary point corresponds to an asymptotic ray to $\mathcal{L}(\mathcal{T}(X))$.

In the definition of the boundary there is a choice of a Hölder exponent ν, $0 < \nu \leq 1$. We show that the boundary is well-defined, namely independent of ν. Assume that ν and ν_1 are two different Hölder exponents. Proposition 4.1 gives two homeomorphisms $I_\nu : \mathcal{PH}(X) \to S^1_\nu$ and $I_{\nu_1} : \mathcal{PH}(X) \to S^1_{\nu_1}$. Then $I_{\nu_1} \circ (I_\nu)^{-1} : S^1_\nu \to S^1_{\nu_1}$ is a homeomorphism under which we identify images of $\mathcal{L}(\mathcal{T}(X))$ in S^1_ν and $S^1_{\nu_1}$. Then the boundary of $(I_\nu \circ \pi)(\mathcal{L}(\mathcal{T}(X)))$ in S^1_ν is homeomorphically identified with the boundary of $(I_{\nu_1} \circ \pi)(\mathcal{L}(\mathcal{T}(X)))$ in $S^1_{\nu_1}$.

The *closure* of $\mathcal{T}(X)$ equals the closure of $(I \circ \pi)(\mathcal{L}(\mathcal{T}(X)))$ in S^1_ν where $\mathcal{T}(X)$ is homeomorphically identified with its image $(I \circ \pi)(\mathcal{L}(\mathcal{T}(X)))$ in S^1_ν. We just observed that two closures obtained by taking different Hölder exponents are homeomorphic.

An interesting property of the topology on $\mathcal{H}(X)$ is that when restricted to positive measures it can be described by using only one ν-norm. More precisely,

Proposition 4.3. *Let $\|\cdot\|_\nu$ be a fixed norm on $\mathcal{H}(X)$. When restricted to subspace of positive measures, the topology induced by the ν-norm is the same as the induced topology from $\mathcal{H}(X)$.*

Proof. Let α be a fixed positive measure and let $\nu_1 < \nu$. In the proof of Theorem 3.2, we showed that we can approximate in the supremum norm any $(\varphi, Q) \in test(\nu_1)$ by step functions φ_n with supports again in Q. Each φ_n can be approximated in the supremum norm by differentiable function ψ_n which has derivative bounded in terms of n and $\sup|\varphi_n - \psi_n|$. Thus we established a "uniform" density of $H^\nu(\tilde{X})$ in $H^{\nu_1}(\tilde{X})$. More precisely, each $(\varphi, Q) \in test(\nu_1)$ can be approximated by $\psi_n \in H^\nu(\tilde{X})$ where $\|\psi_n\|_\nu$ is bounded in terms of n. By again the proof of Theorem 3.2, this is enough to claim that any neighborhood of α in ν_1-norm contains a neighborhood of α in ν-norm. $\square$

In the following theorem we show that the boundary of $\mathcal{T}(X)$ is a subset of $\mathcal{PML}_b(X)$.

Theorem 4.1. *The boundary for $\mathcal{T}(X)$ is contained in the space of projective bounded measured laminations $\mathcal{PML}_b(X)$.*

Proof. In what follows, it will be convenient to consider asymptotic rays to $\mathcal{L}(\mathcal{T}(X))$ as boundary points. Let ν be a fixed Hölder exponent. We assume that tW, $t > 0$ and $W \in \mathcal{H}(X)$, is an asymptotic ray to $\mathcal{L}(\mathcal{T}(X))$. There is no loss of generality if we assume that $\|W\|_\nu = 1$. Then there exists a path $[f_t] \in \mathcal{T}(X)$ with the following properties. The path $\alpha_t = \mathcal{L}([f_t])$ satisfies $\frac{1}{\|\alpha_t\|_\nu}\alpha_t \to W$ as $t \to \infty$ in the ν-norm and $\|\alpha_t\|_\nu \to \infty$ as $t \to \infty$. The fact that $\mathcal{L}(\mathcal{T}(X))$ is closed forces $\|\alpha_t\|_\nu \to \infty$ as $t \to \infty$. Otherwise, $\|\alpha_t\|_\nu$ being bounded imply that a positive multiple of W is in $\mathcal{L}(\mathcal{T}(X))$. For convenience of notation we assume that $\|\alpha_t\|_\nu = t$.

Then $\frac{1}{t}\alpha_t \to W$ as $t \to \infty$ in the ν-norm. Namely, for any $\epsilon > 0$ there exists t_0 such that

$$\Big|\frac{1}{t}\int \varphi d\alpha_t - W(\varphi)\Big| < \epsilon \tag{24}$$

for $t > t_0$ and for all $(\varphi, Q) \in test(\nu)$. The inequality (24) holds for any $\varphi \in H^\nu(\tilde{X})$ but the constant $t_0 = t_0(\varphi)$ depends on the function φ, if φ is not a test function. This follows from the fact that any $\varphi \in H^\nu(\tilde{X})$ can be written as a linear combination of finitely many elements of $test(\nu)$.

By the definition of $\mathcal{H}(X)$, the set $\{W(\varphi); (\varphi, Q) \in test(\nu)\}$ is bounded. We show that there exists $M > 0$ such that

$$\frac{1}{t}\alpha_t(Q) < M \tag{25}$$

for all boxes $Q = [a, b] \times [c, d]$ with $L(Q) = \log 2$ and for $t > 1$. Assume not. Then we can find sequences $\{Q_n = [a_n, b_n] \times [c_n, d_n]\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ such that $\frac{1}{t_n}\alpha_{t_n}(Q_n) \to \infty$, $L(Q_n) \to 0$ and $t_n \to \infty$ as $n \to \infty$. Thus, for n big enough we can find $(\varphi_n, Q^n) \in test(\nu)$ with $\varphi_n = \frac{1}{2}$ on $Q_n \subset Q^n$ and $\varphi_n \geq 0$. Then, by (24), for $t_n > t_0$ we have

$$W(\varphi_n) \geq \frac{1}{t_n}\int \varphi_n d\alpha_{t_n} - \epsilon \geq \frac{1}{2t_n}\alpha_{t_n}(Q_n) - \epsilon$$

and consequently $W(\varphi_n) \to \infty$ as $n \to \infty$. This gives contradiction with $W \in \mathcal{H}(X)$. Thus (25) holds.

We show that W can be extended to act on real continuous functions with compact support in $G(\tilde{X})$. Let ψ be a continuous function on $G(\tilde{X})$ with compact support. We use a sequence of convolutions with "bump" functions with supports around the "origin" $(1,1)$ in $S^1 \times S^1$ shrinking to $(1,1)$ to get a sequence of smooth approximations φ_n to ψ in the topology of $L^\infty(G(\tilde{X}))$. This could be made precise by choosing identification $S^1 \times S^1 \equiv \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ and bump functions with supports around $(0,0)$. We note that the support of each φ_n is compact subset of $G(\tilde{X})$ and all supports of the sequence $\{\varphi_n\}$ are contained in a fixed compact set K if we choose the supports of φ_n small enough. Denote by $supp(\varphi)$ the *support* of a function φ. Thus $\|\psi - \varphi_n\|_\infty \to 0$ as $n \to \infty$ where $\varphi_n \in H^\nu(\tilde{X})$ with $supp(\varphi_n) \subset K$. Taking φ_n in inequality (24) we get

$$\Big|\frac{1}{t}\int \varphi_n d\alpha_t - W(\varphi_n)\Big| < \epsilon \tag{26}$$

for $t > t_0(n)$. Given n and m, we define $t_{n,m} = \max\{t_0(n), t_0(m)\}$. Then

$$\Big|W(\varphi_n) - W(\varphi_m)\Big| \leq \Big|W(\varphi_n) - \frac{1}{t_{n,m}}\int \varphi_n d\alpha_{t_{n,m}}\Big| + \\ \Big|\frac{1}{t_{n,m}}\int \varphi_n d\alpha_{t_{n,m}} - \frac{1}{t_{n,m}}\int \varphi_m d\alpha_{t_{n,m}}\Big| + \Big|\frac{1}{t_{n,m}}\int \varphi_m d\alpha_{t_{n,m}} - W(\varphi_m)\Big|.$$

The fixed compact set K can be covered by finitely many boxes $Q_i, i = 1, 2, \ldots, r$ with $L(Q_i) = \log 2$ for each i. Then by (26) and by (25) the right hand side of the above inequality is less than $2\epsilon + \frac{M\cdot r\cdot L(K)}{t_{n,m}} \cdot \|\varphi_n - \varphi_m\|_\infty$. Thus $W(\varphi_n)$ is a Cauchy sequence. We define $\tilde{W}(\psi) = \lim_{n\to\infty} W(\varphi_n)$. The extension $\tilde{W}$ of W is a linear functional on the set of continuous functions with compact support. The functional $\tilde{W}$ is positive on all $\varphi \geq 0$, $\varphi \in H^\nu(\tilde{X})$ because it is the limit of $\frac{1}{t}\int \varphi d\alpha_t \geq 0$. Further, $\tilde{W}$ is positive on all $\psi \geq 0$, ψ continuous with compact support, because $\tilde{W}(\psi)$ is the limit of $\tilde{W}(\varphi_n) \geq 0$ as $n \to \infty$ with $\varphi_n \in H^\nu(\tilde{X})$ and $\tilde{W}(\varphi_n) \geq 0$. Thus $\tilde{W}$ is a positive linear functional on the set of continuous functions with compact support. By the Riesz Representation Theorem (see [10]) there exists a unique positive Radon measure β on $G(\tilde{X})$ which represents W.

The measure β is bounded. To see this we take an arbitrary $Q = [a,b] \times [c,d]$ with $L(Q) = \log 2$. There exists $\varphi \in H^\nu(\tilde{X})$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on Q and $L(supp(\varphi)) \leq 1$. The support of φ can be covered by two boxes whose Liouville mass is $\log 2$. By (24) and by (25) and by the above, there exists $t_0(\varphi)$ such that for $t > t_0(\varphi)$

$$W(\varphi) = \int \varphi d\beta \leq \frac{1}{t}\int \varphi d\alpha_t + \epsilon \leq 2M + \epsilon.$$

Since $\beta(Q) \leq \int \varphi d\beta \leq 2M + \epsilon$ the measure β is bounded.

It remains to show that the support of β consists of a geodesic lamination. Assume on the contrary that geodesics g_1 and g_2 in the support of β intersect. We find $Q = [a,b] \times [c,d]$ such that Q contains g_1 in its interior and such that $Q_1 = [b,c] \times [d,a]$ contains g_2 in its interior. Let $\varphi \geq 0$, $\varphi \in H^\nu(\tilde{X})$ is nonzero on g_1. Then $\int \varphi d\beta$ is nonzero. Consequently $\alpha_t(Q) \to \infty$ as $t \to \infty$ otherwise $\int \varphi d\beta = \lim_{t\to\infty} \frac{1}{t}\int \varphi d\alpha_t = 0$. Similarly $\alpha_t(Q_1) \to \infty$ as $t \to \infty$. Since α_t is in the image of $\mathcal{T}(X)$, by Theorem 3.1 we have

$$e^{-\alpha_t(Q)} + e^{-\alpha_t(Q_1)} = 1.$$

The above equality together with $\alpha_t(Q) \to \infty$ and $\alpha_t(Q_1) \to \infty$ as $t \to \infty$ gives contradiction. Thus the support of β consists of non-intersecting geodesics. □

We proceed to prove that every $[\beta] \in \mathcal{PML}_b(X)$ is in the boundary of $\mathcal{T}(X)$. Let $[f_t]$ be the earthquake path in $\mathcal{T}(X)$ starting at the identity with the earthquake measure $t\beta$. Denote by α_t the image of $[f_t]$ in $\mathcal{H}(X)$.

We prove several lemmas which are needed for the proof of the above.

For the box $[-i, 1] \times [i, -1]$ of geodesics we consider the first coordinate in the product to be the *horizontal direction* and the second coordinate to be the *vertical direction.*

Lemma 4.1. *Let β be a geodesic lamination with $\|\beta\| = 1$. There exists a subset E_n of $[-i, 1] \times [i, -1]$ with the following properties:*
1. $\beta(E_n)$ is of the order $\frac{1}{n}$
2. E_n is a union of $n-1$ vertical and $n-1$ horizontal strips of the widths of order $\frac{1}{n^2}$
3. Each vertical and each horizontal strip intersect in a small box; we take centers of all such boxes, and the centers of the intersections of strips with the sides of $[-i, 1] \times [i, -1]$, and the vertices of $[-i, 1] \times [i, -1]$ to form the boxes $A_{i,j}$. We obtain n^2 boxes $A_{i,j}$ and $L(A_{i,j})$ is of the order $\frac{1}{n^2}$ for each (i, j).
Proof. We use the upper half plane model $\mathbb{H}^2$. We replace $[-i, 1] \times [i, -1]$ by $[-2, -1] \times [1, 2]$. If we prove the Lemma for $[-2, -1] \times [1, 2]$ it will follow for $[-i, 1] \times [i, -1]$ because the standard angle metric on S^1 is Lipschitz equivalent to the Euclidean metric on a compact set of $\widehat{\mathbb{R}}$. We divide segments $[-2, -1]$ and $[1, 2]$ into n segments of the same length using the division points $x_0 = -2$, $x_n = -1$, $x_i = -2 + \frac{i}{n}$, $i = 1, 2, \ldots, n-1$ for $[-2, -1]$ and the division points $y_0 = 1$, $y_n = 2$, $y_j = 1 + \frac{j}{n}$, $j = 1, 2, \ldots, n-1$ for $[1, 2]$. Further, we divide each of the segments $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ into n segments of the same size with the division points $x_i^0 = x_{i-1}$, $x_i^n = x_i$, $x_i^k = x_{i-1} + \frac{k}{n^2}$, $k = 1, 2, \ldots, n-1$ for $[x_{i-1}, x_i]$ and the division points $y_j^0 = y_{j-1}$, $y_j^n = y_j$, $y_j^h = y_{j-1} + \frac{h}{n^2}$, $h = 1, 2, \ldots, n-1$ for $[y_{j-1}, y_j]$. We form n vertical strips $V_k = \cup_{i=1}^n [x_i^{k-1}, x_i^k] \times [1, 2]$ for $k = 1, 2, \ldots, n$ and n horizontal strips $H_h = \cup_{j=1}^n [-2, -1] \times [y_j^{h-1}, y_j^h]$ for $h = 1, 2, \ldots, n$. The union of V_k covers $[-2, -1] \times [1, 2]$ and each point is covered at most twice. Hence $\sum_{k=1}^n \beta(V_k) \leq 2$. There exists at least one k such that $\beta(V_k) \leq \frac{2}{n}$. Fix such k. The same holds for some h, i.e. $\beta(H_h) \leq \frac{2}{n}$. We define E_n as the union of H_k and V_h. Then $\beta(E_n) \leq \frac{4}{n}$. Let $a_i = \frac{x_i^{k-1} + x_i^k}{2}$ be the midpoint of $[x_i^{k-1}, x_i^k]$ for $i = 1, 2, \ldots, n-1$; and let $a_0 = -2$ and $a_n = -1$. Let $c_j = \frac{y_j^{h-1} + y_j^h}{2}$ be the midpoint of $[y_j^{h-1}, y_j^h]$ for $j = 1, 2, \ldots, n-1$; and let $c_0 = 1$ and $c_n = 2$.

The points (a_i, c_j), (a_i, c_{j+1}), (a_{i+1}, c_j) and (a_{i+1}, c_{j+1}) are vertices of the rectangles $A_{i,j}$ for $i, j = 0, 1, 2, \ldots, n-1$. The difference between the x-coordinates of the vertices of $A_{i,j}$ is of the order $\frac{1}{n}$. The same holds for y-coordinates of the vertices of $A_{i,j}$. Thus we get that $L(A_{i,j})$ is of the order $\frac{1}{n^2}$. □

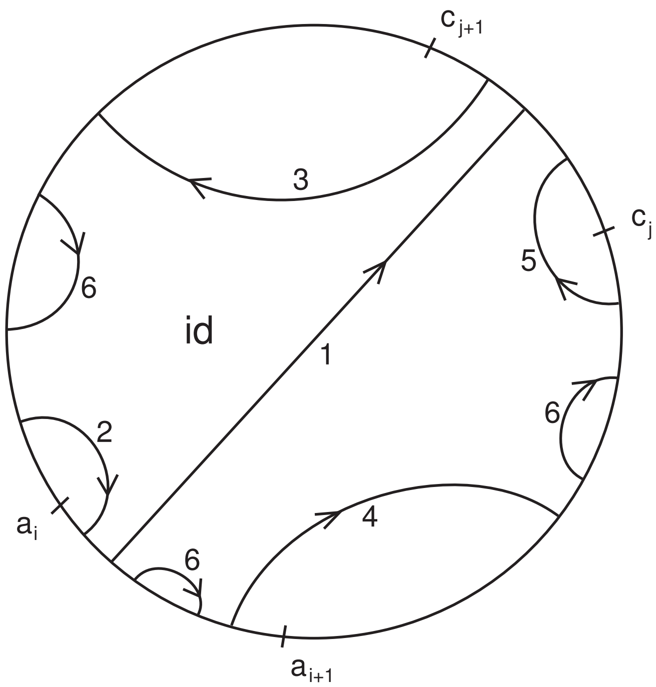

FIGURE 1. Case $\beta(A_{i,j})\neq 0$

We keep the notation of the previous lemma and compare $\frac{1}{t}\alpha_t(A_{i,j})$ to $\beta(A_{i,j})$ for $i,j=1,2,\dots,n-2$. Note that we consider only on the "inside" boxes $A_{i,j}$ of the above division of $[-i,1]\times[i,-1]$. The next two lemmas give essential estimates.

Lemma 4.2. *Let E_n and $A_{i,j}$ be as above. Then*

$$\frac{1}{t}\alpha_t(A_{i,j})-\beta(A_{i,j})\leq \frac{1}{t}L([a_i,y_j^{h-1}]\times[c_j,x_i^{k-1}])+\beta([x_i^{k-1},a_i]\times[c_j,c_{j+1}]) \\ +\beta([a_i,a_{i+1}]\times[y_j^{h-1},c_j])+\beta([x_i^{k-1},a_i]\times[y_j^{h-1},c_j]).$$

for all $t>0$ and for all $i,j=1,2,\dots,n-2$.

Proof. We use the upper half plane model $\mathbb{H}^2$ and replace $[-i,1]\times[i-1]$ with $[-2,-1]\times[1,2]$. Let f_t denotes the earthquake path for the measure $t\beta$ and let $\alpha_t=\mathcal{L}([f_t])$. We divide our reasoning into several cases.

Case 1: $\beta(A_{i,j})\neq 0$.

By our assumption the support of β does not contain geodesics with one endpoint in the interval (a_{i+1},c_j) and the other endpoint in the interval (c_{j+1},a_i). The intervals are taken with respect to the orientation of $\widehat{\mathbb{R}}$ as the boundary of the upper half plane $\mathbb{H}^2$. We divide the support of β into six groups (see figure 1):

1. The geodesics which belong to $A_{i,j}=[a_i,a_{i+1}]\times[c_j,c_{j+1}]$.
2. The geodesics which belong to $(c_{j+1},a_i)\times[a_i,a_{i+1}]$.
3. The geodesics which belong to $[c_j,c_{j+1}]\times(c_{j+1},a_i)$.
4. The geodesics which belong to $[a_i,a_{i+1}]\times(a_{i+1},c_j)$.
5. The geodesics which belong to $(a_{i+1},c_j)\times[c_j,c_{j+1}]$.
6. The geodesics which does not belong to any of the above five groups.

We normalize the earthquake path f_t to be the identity on the stratum I which separates the geodesics of group 1 from the geodesics of groups 2 and 3. If groups 2 and 3 are empty then we normalize the earthquake path to be the identity on the strata which separates group 1 from (c_{j+1}, a_i). Note that boxes in our division of the support of β are written such that the first coordinate is the repelling fixed point and the second coordinate is the attracting fixed point of the hyperbolic translation along the geodesic for the given normalization in the definition of earthquake path f_t.

We analyze the effect of earthquake path f_t on the Liouville measure of $A_{i,j}$. We divide the measure β into six measures β_i, $i = 1, 2, \dots 6$; by taking its restriction to the above six groups. Each newly obtained measured lamination β_i has stratum I_i which contains I. Define six earthquake paths f_t^i, $i = 1, 2, \dots, 6$; for the measures β_i such that they are all identity on stratum I_i which contains I. Then $f_t = f_t^1 \circ \cdots \circ f_t^6$. Note that f_t^5 ,f_t^4, f_t^3 and f_t^2 commute with each other.

Earthquake f_t^6 fixes $A_{i,j}$ and we can disregard it.

By Lemma A.1, the Liouville mass of $A_{i,j}$ is an increasing function of the distance between the geodesics with endpoints a_i and c_{j+1}, and the geodesic with the endpoints a_{i+1} and c_j.

Because of our normalization, earthquake path f_t^2 moves a_i closer to a_{i+1} and fixes a_{i+1}, c_j and c_{j+1}. Earthquake path f_t^5 moves c_j closer to c_{j+1} and fixes c_{j+1}, a_i and a_{i+1}. Thus the Liouville mass of $A_{i,j}$ is decreasing under earthquake paths f_t^2 and f_t^5. Since we are interested in the upper bound we can disregard earthquake paths f_t^2 and f_t^5.

Consider earthquake path f_t^3. We divide the measure β_3 into two measures. The first measure β_3^1 equals the restriction of β_3 to $[c_j, c_{j+1}] \times (c_{j+1}, x_i^{k-1})$. Earthquake path for the measure β_3^1 moves c_{j+1} at most to x_i^{k-1} and leaves a_i, a_{i+1} and c_j fixed. The second measure β_3^2 equals the restriction of β_3 to $[c_j, c_{j+1}] \times [x_i^{k-1}, a_i)$. By Lemma A.2, if we replace earthquake path for the measure β_3^2 by the hyperbolic translation with the repelling fixed point c_j, with the attracting fixed point a_i and with the translation length $t\beta([c_j, c_{j+1}] \times [x_i^{k-1}, a_i))$ then the Liouville mass of the image of $A_{i,j}$ increases. By Lemma A.1, we increase the Liouville mass of the image of $A_{i,j}$ by not more than $t\beta([c_j, c_{j+1}] \times [x_i^{k-1}, a_i))$. Let us denote by $\alpha_t^{'}$ the image in $\mathcal{H}(X)$ of earthquake path f_t^3. From above we get that

$$\alpha_t^{'}(A_{i,j}) \leq L([a_i, a_{i+1}] \times [c_j, x_i^{k-1}]) + t\beta([c_j, c_{j+1}] \times [x_i^{k-1}, a_i)).$$

Similar conclusions can be made for the Liouville mass of the image of $f_t^3(A_{i,j})$ under earthquake path f_t^4.

We compose earthquake paths f_t^3 and f_t^4. Denote by $\alpha_t^{''}$ the image in $\mathcal{H}(X)$ of $[f_t^3 \circ f_t^4]$. Then

$$\begin{aligned} &\alpha_t^{''}(A_{i,j}) \leq L([a_i, y_j^{h-1}] \times [c_j, x_i^{k-1}]) + \\ &t\Big\{\beta([c_j, c_{j+1}] \times [x_i^{k-1}, a_i)) + \beta([a_i, a_{i+1}] \times [y_j^{h-1}, c_j))\Big\}. \end{aligned}$$

We are left to consider the image of $f_t^3 \circ f_t^4(A_{i,j})$ under earthquake path f_t^1. By Lemmas A.4 and A.5, we increase the Liouville measure of the image of $f_t^3 \circ f_t^4(A_{i,j})$ if we replace earthquake path f_t^1 by the hyperbolic translation with the repelling fixed point a_i, with the attracting fixed point c_j and with the translation length $t\beta(A_{i,j})$. We combine all earthquake paths to obtain original earthquake path f_t.

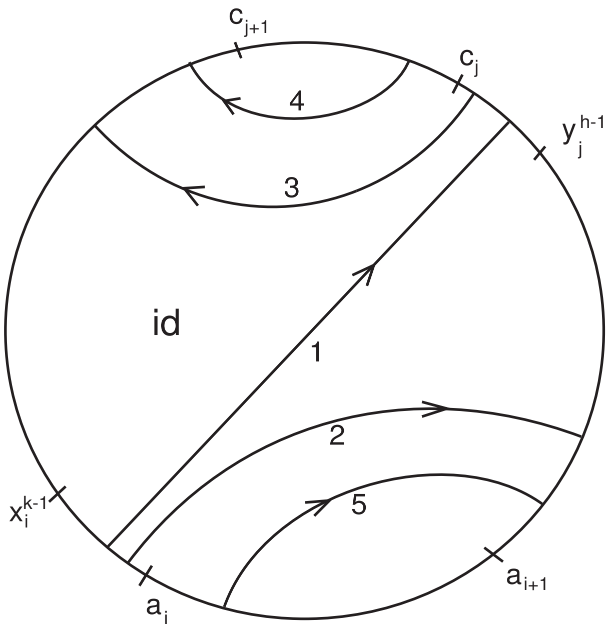

FIGURE 2. Case $\beta(A_{i,j})=0$, $\beta([x_i^{k-1},a_i]\times[y_j^{h-1},c_j])\neq 0$

By Lemma A.1 and by the above discussion we get

$$\begin{aligned}\alpha_t(A_{i,j}) &\leq L([a_i,y_j^{h-1}]\times[c_j,x_i^{k-1}])+t\beta(A_{i,j})+\\ &t\Big\{\beta([c_j,c_{j+1}]\times[x_i^{k-1},a_i))+\beta([a_i,a_{i+1}]\times[y_j^{h-1},c_j))\Big\}.\end{aligned}\tag{27}$$

We divide inequality (27) by t which concludes the proof in Case 1.

Case 2: $\beta(A_{i,j})=0$ and $\beta([a_{i+1},c_j]\times[c_{j+1},a_j])=0$.

By a similar argument as in Case 1 we get inequality (27) without the second term on the right.

Case 3: $\beta(A_{i,j})=0$, $\beta([a_{i+1},c_j]\times[c_{j+1},a_i])\neq 0$ and $\beta([x_i^{k-1},a_i]\times[y_j^{h-1},c_j])\neq 0$.

As a consequence of the above conditions we get $\beta([c_{j+1},x_i^{k-1})\times[a_{i+1},y_j^{h-1}))=0$. In this case we divide the support of β into the following six groups (see figure 2):

1. The geodesics which belong to $[x_i^{k-1},a_i]\times[y_j^{h-1},c_j]$.
2. The geodesics which belong to $[x_i^{k-1},a_i]\times[a_{i+1},y_j^{h-1})$.
3. The geodesics which belong to $[y_j^{h-1},c_j]\times[c_{j+1},x_i^{k-1})$.
4. The geodesics which belong to $(c_j,c_{j+1})\times(c_{j+1},a_i]\cup\{c_j\}\times(c_{j+1},x_i^{k-1})$.
5. The geodesics which belong to $(a_i,a_{i+1})\times(a_{i+1},c_j]\cup\{a_i\}\times(a_{i+1},y_j^{h-1})$.
6. All other geodesics.

We normalize earthquake path f_t to be the identity on the strata I which separates group 1 from group 3. If group 3 is empty then we normalize f_t to be the identity on the stratum I which separates group 1 and group 4. If groups 3 and 4 are empty then we use the stratum I which separates group 1 from c_{j+1}. We change measured lamination β to new measured lamination $\beta^{'}$ which gives new earthquake path $f_t^{'}$ such that the Liouville mass of $f_t(A_{i,j})$ is smaller than the Liouville mass of $f_t^{'}(A_{i,j})$.

The geodesics in group 1 are replaced by the geodesic with endpoints a_i and c_j, and with the weight $t\beta([x_i^{k-1}, a_i] \times [y_j^{h-1}, c_j])$. Any geodesic in group 2 with endpoints $a \in [x_i^{k-1}, a_i]$ and $b \in (a_{i+1}, y_j^{h-1})$ is replaced by the geodesic with endpoints a_i and b. The measure on the new group 2 is given by the push-forward of the measure β on the old group 2. Similarly the geodesics in group 3 are replaced by the geodesics with one endpoint c_j and the measure is the push-forward of the measure β on the old group 3.

The earthquake along the geodesics in group 6 either does not move $A_{i,j}$ or it moves it to the set with smaller Liouville measure. We disregard group 6 and obtain new measured lamination $\beta^{'}$.

Measured lamination $\beta^{'}$ gives earthquake path $f_t^{'}$. Let $\alpha_t^{'}$ denotes the image of $[f_t^{'}]$ in $\mathcal{H}(X)$. By Lemma A.3, we get $\alpha_t(A_{i,j}) \leq \alpha_t^{'}(A_{i,j})$.

The measure $\beta^{'}$ satisfies $\beta^{'}(A_{i,j}) = \beta([x_i^{k-1}, a_i] \times [y_j^{h-1}, c_j]) \neq 0$. Thus we are in the Case 1. We obtain

$$\frac{1}{t}\alpha_t^{'}(A_{i,j}) - \beta^{'}(A_{i,j}) \leq \frac{1}{t}L([a_i, y_j^{h-1}] \times [c_j, x_i^{k-1}]) + \beta^{'}([c_j, c_{j+1}] \times [x_i^{k-1}, a_i)) + \beta^{'}([a_i, a_{i+1}] \times [y_j^{h-1}, c_j)).$$

By the definition $\beta^{'}([c_j, c_{j+1}] \times [x_i^{k-1}, a_i)) = \beta([c_j, c_{j+1}] \times [x_i^{k-1}, a_i))$ and $\beta^{'}([a_i, a_{i+1}) \times [y_j^{h-1}, c_j)) = \beta([a_i, a_{i+1}] \times [y_j^{h-1}, c_j))$. Since $\beta(A_{i,j}) = 0$ and by the above we obtain

$$\begin{aligned}\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j}) \leq L([a_i, y_j^{h-1}] \times [c_j, x_i^{k-1}]) + \beta([x_i^{k-1}, a_i] \times [y_j^{h-1}, c_j]) \\ + \beta([c_j, c_{j+1}] \times [x_i^{k-1}, a_i)) + \beta([a_i, a_{i+1}] \times [y_j^{h-1}, c_j))\end{aligned} \tag{28}$$

Case 4: $\beta(A_{i,j}) = 0$, $\beta([a_{i+1}, c_j] \times [c_{j+1}, a_j] \neq 0$ and $\beta([x_i^{k-1}, a_i] \times [y_j^{h-1}, c_j]) = 0$.

In this case we also construct new measured lamination $\beta^{'}$ which gives earthquake path $f_t^{'}$ (see figure 3). Let $\alpha_t^{'}$ denote the image of $[f_t^{'}]$ in $\mathcal{H}(X)$. For this path we also get that $\alpha_t(A_{i,j}) \leq \alpha_t^{'}(A_{i,j})$.

We start by replacing the earthquake along the geodesics in $[y_j^{h-1}, c_j] \times [c_{j+1}, x_i^{k-1}]$ by the hyperbolic translation with the repelling fixed point c_j, with the attracting fixed point x_i^{k-1} and with the translation length $t\beta([y_j^{h-1}, c_j] \times [c_{j+1}, x_i^{k-1}])$. Further, the earthquake along the geodesics in $[x_i^{k-1}, a_i] \times [a_{i+1}, y_j^{h-1}]$ is replaced by the hyperbolic translation with the repelling fixed point a_i, with the attracting fixed point y_j^{h-1} and with the translation length $t\beta([x_i^{k-1}, a_i] \times [a_{i+1}, y_j^{h-1}])$. The earthquake along the geodesics in $[a_{i+1}, y_j^{h-1}] \times [c_{j+1}, x_i^{k-1}]$ is replaced by the hyperbolic translation with the repelling fixed point x_i^{k-1}, with the attracting fixed point y_j^{h-1} and with the translation length $t\beta([a_{i+1}, y_j^{h-1}] \times [c_{j+1}, x_i^{k-1}])$.

We get the new measured lamination $\beta^{'}$ and the corresponding path $\alpha_t^{'}$. Then by Lemma A.3, we get $\alpha_t(A_{i,j}) \leq \alpha_t^{'}(A_{i,j})$. The set $f_t^{'}(A_{i,j})$ is a subset of $[a_i, y_j^{h-1}] \times [c_j, x_i^{k-1}]$. Since $\beta(A_{i,j}) = 0$ we get

$$\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j}) \leq \frac{1}{t}\alpha_t^{'}(A_{i,j}).$$

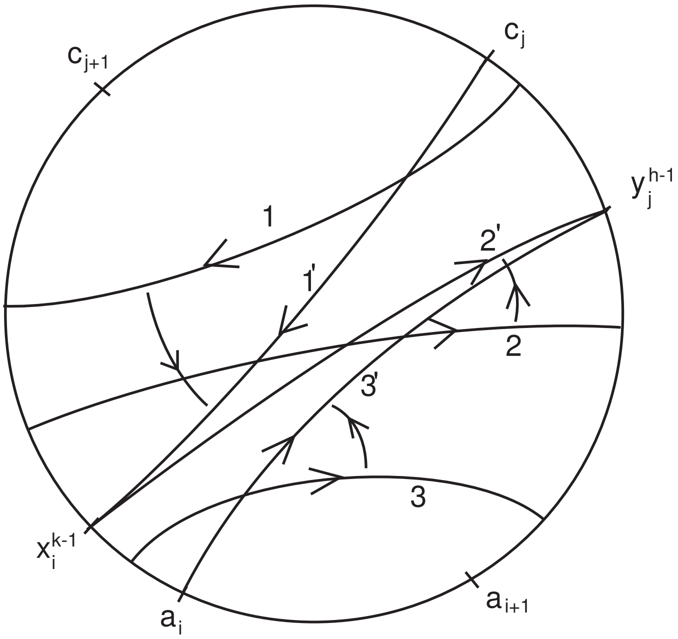

FIGURE 3. Replacing β with $\beta^{'}$

From above

$$\frac{1}{t}\alpha^{'}_t(A_{i,j}) \leq \frac{1}{t}L([a_i, y_j^{h-1}] \times [c_j, x_i^{k-1}]).$$

Then

$$\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j}) \leq \frac{1}{t}L([a_i, y_j^{h-1}] \times [c_j, x_i^{k-1}]). \tag{29}$$

□

Lemma 4.3. *Let E_n and $A_{i,j}$ be as above. Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j}) \geq \frac{-C_1 \log n}{t} - \frac{C_2}{t} - \beta([x_{i+1}^{k-1}, a_{i+1}] \times [y_{j+1}^{h-1}, c_{j+1}])$$

for all $t > 0$ and for all $i, j = 1, 2, \ldots, n-2$.

Proof. Either $\beta(A_{i,j}) \neq 0$ or $\beta(A_{i,j}) = 0$. We divide our proof in several cases.

Case 1: $\beta(A_{i,j}) = 0$

Then

$$\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j}) = \frac{1}{t}\alpha_t(A_{i,j}) \geq 0.$$

Thus the lower bound in this case is 0.

Case 2: $\beta(A_{i,j}) \neq 0$ and $\beta([a_i, x_{i+1}^{k-1}] \times [c_j, y_{j+1}^{h-1}]) \neq 0$.

Consequently $\beta((x_{i+1}^{k-1}, a_{i+1}] \times (y_{j+1}^{h-1}, c_{j+1}]) = 0$. We consider the group of all geodesics of the support of β which lie in set $[a_i, x_{i+1}^{k-1}] \times [c_j, y_{j+1}^{k-1}]$. Let d be the endpoint in the interval $[c_j, c_{j+1}]$ of a geodesic in the above group which is closest to c_{j+1}. Let b be the endpoint of a geodesic in the above group in the interval $[a_i, x_{i+1}^{k-1}]$ which is closest to a_{i+1}. Let a be the endpoint of a geodesic in the above

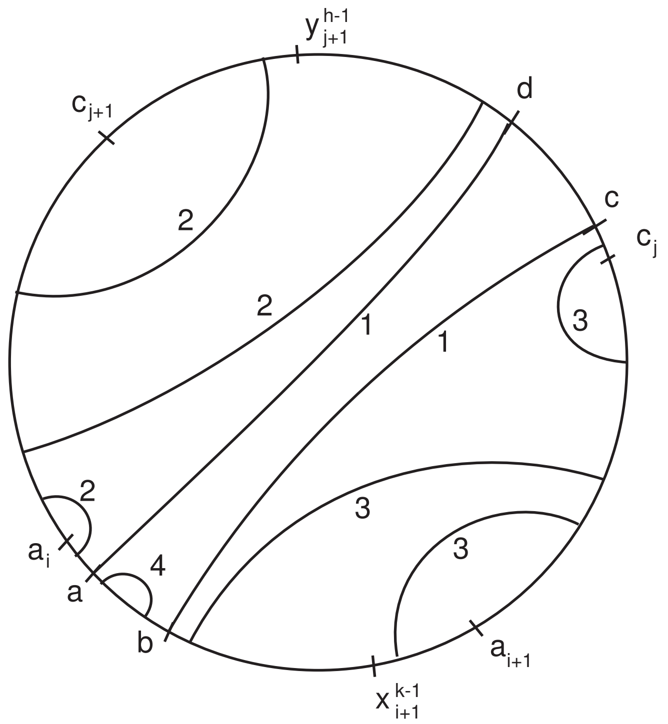

FIGURE 4. Case $\beta(A_{i,j}) \neq 0$, $\beta([a_i, x^{k-1}_{i+1}] \times [c_j, y^{h-1}_{j+1}]) \neq 0$

group in the interval $[a_i, x^{k-1}_{i+1}]$ which is closest to a_i. Let c be the endpoint of a geodesic in the above group in the interval $[c_j, y^{h-1}_{j+1}]$ which is closest to c_j. We divide the geodesics in the support of β in the following four groups (see figure 4):
1. The geodesics which belong to $[a_i, x^{k-1}_{i+1}] \times [c_j, y^{k-1}_{j+1}]$.
2. The geodesics whose both endpoints lie in the interval $[d, a]$ except the geodesic with endpoints a and d.
3. The geodesics whose both endpoints lie in the interval $[b, c]$ except the geodesic with endpoints b and c.
4. All other geodesics.

We are interested in the lower bound for $\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j})$. Thus we can replace $\frac{1}{t}\alpha_t(A_{i,j})$ with $\frac{1}{t}\alpha_t(A^{'}_{i,j})$ where $A^{'}_{i,j}$ is a subset of $A_{i,j}$. Define $A^{'}_{i,j} = [a, a_{i+1}] \times [c, c_{j+1}]$. By definition, $A^{'}_{i,j} \supset [x^{k-1}_{i+1}, a_{i+1}] \times [y^{h-1}_{j+1}, c_{j+1}]$. Let f_t be earthquake path for the measure β. We normalize earthquake path f_t to be the identity on the strata I which separates the geodesics of group 1 from the geodesics of group 2.

As in previous lemma, we divide β into four measured laminations β_i, $i = 1, 2, 3, 4$, such that β_i is the restriction of β to the geodesics of group i. Each measured lamination β_i has stratum I_i that contains I. We define f^i_t to be earthquake path for the measure β_i normalized such that $f^i_t|_{I_i} = id$. Earthquake f^4_t fixes $A^{'}_{i,j}$. Thus we can disregard it. Earthquake f^2_t either fixes $A^{'}_{i,j}$ or it fixes $[a, a_{i+1}]$

and maps $[c, c_{j+1}]$ onto interval $[c, e]$ with $[c, c_{j+1}] \subset [c, e]$. We can disregard earthquake f_t^2 because we are interested in the lower bound for the Liouville measure of the image of $A'_{i,j}$. Similarly we can disregard earthquake f_t^3.

Let A be the hyperbolic translation with the repelling fixed point x_{i+1}^{k-1}, with the attracting fixed point y_{j+1}^{h-1} and with the translation length $t\beta([a_i, x_{i+1}^{k-1}]\times[c_j, y_{j+1}^{h-1}])$. Then by Lemmas A.4 and A.5, $\alpha_t(A'_{i,j})$ is less than or equal to the Liouville measure of $[x_{i+1}^{k-1}, A(a_{i+1})] \times [y_{j+1}^{h-1}, c_{j+1}]$. By Lemma A.1, we get that $L([x_{i+1}^{k-1}, A(a_{i+1})] \times [y_{j+1}^{h-1}, c_{j+1}])$ is greater than or equal to $t\beta([a_i, x_{i+1}^{k-1}] \times [c_j, y_{j+1}^{h-1}]) + \log\frac{l^2}{4}$, where l is the hyperbolic distance between the geodesics with the endpoints x_{i+1}^{k-1} and c_{j+1}, and the geodesic with the endpoints a_{i+1} and y_{j+1}^{h-1}. The angle distance between x_{i+1}^{k-1} and a_{i+1} is of the order $\frac{1}{n^2}$. Also the angle distance between y_{j+1}^{h-1} and c_{j+1} is of the order $\frac{1}{n^2}$. Consequently, the distance between the geodesic with endpoints x_{i+1}^{k-1} and c_{j+1} and the geodesic with the endpoints a_{i+1} and y_{j+1}^{h-1} is of the order $\frac{1}{n^4}$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j}) \geq \frac{-C_1 \log n}{t} - \frac{C_2}{t}.$$

Case 3: $\beta(A_{i,j}) \neq 0$ and $\beta([a_i, x_{i+1}^{k-1}] \times [c_j, y_{j+1}^{h-1}]) = 0$.

Then $\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j}) \geq -\beta(A_{i,j}) = -\beta([x_{i+1}^{k-1}, a_{i+1}] \times [y_{j+1}^{h-1}, c_{j+1}])$. □

We gather the above estimates in the following useful form.

Lemma 4.4. *Let E_n and $A_{i,j}$ be as above. There exists a constant $C(n) > 0$ such that*

$$\sum_{i,j=1}^{n-2} \Big|\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j})\Big| \leq \frac{C(n)}{t} + \frac{6}{n}.$$

Proof. By Lemmas 4.2 and 4.3 we get that

$$\begin{aligned}
&\Big|\frac{1}{t}\alpha_t(A_{i,j}) - \beta(A_{i,j})\Big| \leq \\
&\max\Big\{\frac{1}{t}L([a_i, y_j^{h-1}] \times [c_j, x_i^{k-1}]) + \beta([x_i^{k-1}, a_i] \times [c_j, c_{j+1}]) + \\
&\beta([a_i, a_{i+1}] \times [y_j^{h-1}, c_j]) + \beta([x_i^{k-1}, a_i] \times [y_j^{h-1}, c_j]), \\
&\frac{C_1 \log n}{t} + \frac{C_2}{t} + \beta([x_{i+1}^{k-1}, a_{i+1}] \times [y_{j+1}^{h-1}, c_{j+1}])\Big\}
\end{aligned} \tag{30}$$

Since

$$\{\cup_{i,j=1}^{n-2}[a_i, a_{i+1}] \times [y_j^{h-1}, c_j]\} \cup \{\cup_{i,j=1}^{n-2}[x_i^{k-1}, a_i] \times [y_j^{h-1}, c_j]\} \cup \{\cup_{i,j=1}^{n-2}[x_i^{k-1}, a_i] \times [c_j, c_{j+1}]\} \subset E_n$$

and each point in E_n is covered at most six times by the sets on the left we get

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{n-2}\Big\{\beta([x_i^{k-1}, a_i] \times [c_j, c_{j+1}]) + \beta([a_i, a_{i+1}] \times [y_j^{h-1}, c_j]) + \\
&\beta([x_i^{k-1}, a_i] \times [y_j^{h-1}, c_j])\Big\} \leq 6\beta(E_n) \leq \tfrac{6}{n}.
\end{aligned} \tag{31}$$

Since $\cup_{i,j=1}^{n-2}[x_{i+1}^{k-1}, a_{i+1}] \times [y_{j+1}^{h-1}, c_{j+1}] \subset E_n$ we get

$$\sum_{i,j=1}^{n-2} \beta([x_{i+1}^{k-1}, a_{i+1}] \times [y_{j+1}^{h-1}, c_{j+1}]) \leq \frac{1}{n}. \tag{32}$$

From (30), (31) and (32) we get

$$\sum_{i,j=1}^{n-2}\Big|\frac{1}{t}\alpha_t(A_{i,j})-\beta(A_{i,j})\Big|\leq$$
$$\max\Big\{\frac{1}{t}\sum_{i,j=1}^{n-2}L([a_i,y_j^{h-1}]\times[c_j,x_i^{k-1}])+\frac{6}{n},\frac{1}{t}\sum_{i,j=1}^{n-2}(C_1\log n+C_2)+\frac{1}{n}\Big\}.$$

By choosing

$$C(n)=\max\Big\{\sum_{i,j=1}^{n-2}L([a_i,y_j^{h-1}]\times[c_j,x_i^{k-1}]),(C_1\log n+C_2)(n-2)^2\Big\}$$

we obtain the desired inequality. □

We use above lemma to show that $\frac{1}{t}\alpha_t$ is bounded.

Lemma 4.5. *Let α_t and β be as above. Then $\frac{1}{t}\alpha_t(Q)$ is bounded for $t\geq 1$ and for all boxes $Q=[a,b]\times[c,d]$ such that $L(Q)=\log 2$.*

Proof. By Lemma 4.4 we get

$$\sum_{i,j=1}^{n-2}\frac{1}{t}\alpha_t(A_{i,j})\leq\beta(Q)+\frac{C_1(n)}{t}+\frac{6}{n}$$

where $A_{i,j}$ are as above and n is fixed. Note that $\cup_{i,j=1}^{n-2}A_{i,j}=[a^{'},b^{'}]\times[c^{'},d^{'}]$ is a proper subset of $Q=[a,b]\times[c,d]$. But Q can be covered by finitely many such $[a^{'},b^{'}]\times[c^{'},d^{'}]$ and the conclusion follows. □

Now we prove that $\frac{1}{t}\alpha_t$ converges to β in the ν-norm.

Theorem 4.2. *If $\alpha_t\in\mathcal{H}(X)$ denotes the image of an earthquake path $[f_t]$ with the measure $\beta\in\mathcal{ML}_b(X)$, where $\|\beta\|=1$, then $\frac{1}{t}\alpha_t$ converges to β as $t\to\infty$ in the ν-norm.*

Proof. Let $(\varphi,Q)\in test(\nu)$ where $Q=[a,b]\times[c,d]$. We keep the notation of the previous lemmas.

We define a sequence of step function φ_n to approximate φ. Fix $g_{i,j}\in A_{i,j}$ for each $i,j=0,1,\dots,n-1$. Define $\varphi_n\circ\Theta_{abcd}(g)=\varphi\circ\Theta_{abcd}(g_{i,j})$, for $g\in A_{i,j}$ where $i,j=1,2,\dots,n-2$; and $\varphi_n\circ\Theta_{abcd}(g)=0$ for $g\in A_{i,j}$ where $i=0$ or $i=n-1$ or $j=0$ or $j=n-1$; and $\varphi_n\circ\Theta_{abcd}(g)=0$ for g not in $[-i,1]\times[1,-i]$. Note that φ_n is not defined on the set of measure zero which is not important for the integration.

By the definition of φ_n, by the Hölder continuity of φ and because $A_{i,j}$ has diameter of the order $\frac{1}{n}$ there exists $C>0$ such that $|\varphi(g)-\varphi_n(g)|\leq\frac{C^\nu}{n^\nu}$. Then we obtain the following inequalities

$$\Big|\frac{1}{t}\int\varphi d\alpha_t-\frac{1}{t}\int\varphi_n d\alpha_t\Big|\leq\frac{C^\nu}{n^\nu}\frac{1}{t}\alpha_t(Q)$$
$$\Big|\frac{1}{t}\int\varphi_n d\alpha_t-\int\varphi_n d\beta\Big|\leq\sum_{i,j=1}^{n-2}\Big|\frac{1}{t}\alpha_t(A_{i,j})-\beta(A_{i,j})\Big|$$
$$\Big|\int\varphi_n d\beta-\int\varphi d\beta\Big|\leq\frac{C^\nu}{n^\nu}\beta(Q).$$

By Lemma 4.4, $\sum_{i,j=1}^{n-2}|\frac{1}{t}\alpha_t(A_{i,j})-\beta(A_{i,j})|$ can be made small for n and t large independently of the box Q. By Lemma 4.5, $\frac{1}{n^\nu}\frac{1}{t}\alpha_t([a,b]\times[c,d])$ is small for n large. Above inequalities combined imply the convergence. □

Corollary 4.1. *Let $\beta\in\mathcal{ML}_b(X)$ such that $\|\beta\|_\nu=1$. Let $[f_t]\in\mathcal{T}(X)$ be an earthquake path with the measure β and let $\alpha_t=\mathcal{L}([f_t])$. Then $\frac{1}{\|\alpha_t\|}\alpha_t\to\beta$ as $t\to\infty$ in the ν-norm. In other words, any $[\beta]\in\mathcal{PML}_b(X)$ is a boundary point for $\mathcal{T}(X)$.*

Proof. By Theorem 4.2, $\frac{1}{t}\alpha_t\to\beta$ as $t\to\infty$ in the ν-norm. Consequently $\|\frac{1}{t}\alpha_t\|_\nu\to 1$ as $t\to\infty$. Then $\frac{\|\alpha_t\|_\nu}{t}\to 1$ as $t\to\infty$ and

$$\Big\|\frac{1}{\|\alpha_t\|}\alpha_t-\frac{1}{t}\alpha_t\Big\|_\nu\to\infty$$

as $t\to\infty$. The Corollary follows by the triangle inequality. □

Theorem 2 of the Introduction is a direct consequence of Theorem 4.1 and Corollary 4.1.

In the proof of Theorem 4.2, we showed that $\frac{1}{t}\alpha_t\to\beta$ as $t\to\infty$, where $\alpha_t=\mathcal{L}([f_t])$ for an earthquake path $[f_t]\in\mathcal{T}(X)$ starting at $[id]$ with the measure β. In the following theorem we prove that this is still true even if an earthquake path does not start from the basepoint $[id]$ of $\mathcal{T}(X)$.

Let $[f_t]$ be an earthquake path with the initial point $[f]\in\mathcal{T}(X)$ and with the measure $t\beta$. Then $[f_t]=[g_t\circ f]$ where $[g_t]$ is an earthquake path starting at $[id]$ with the measure $f^*\beta$.

Theorem 4.3. *Suppose $f:\Delta\to\Delta$ is a Γ-invariant quasiconformal map. Let $\beta\in\mathcal{ML}_b(X)$ and let $[f_t]$ be an earthquake path with the initial point $[f]\in\mathcal{T}(X)$ and with the measure $t\beta$ such that $\|\beta\|=1$. Denote by α_t the image of $[f_t]$ in $\mathcal{H}(X)$. Then $\frac{1}{t}\alpha_t$ converges to β as $t\to\infty$ in $\mathcal{H}(X)$.*

Proof. The map f extends to a quasisymmetric map of S^1 and we denote it by f, again. Earthquake path f_t extends to a path of quasisymmetric maps of S^1 and we denote this path by f_t, again. If we normalize them properly we get $f_0=f$. Let g_t be earthquake path with the measure $f^*\beta$ and $g_0=id$. Let α_t' stands for the image of $[g_t]$ in $\mathcal{H}(X')$ where $X'=f(X)$. Then $f_t=g_t\circ f$ and consequently $\alpha_t'=f^*\alpha_t$.

As in the proof of Theorem 4.2, it is enough to show that $|\frac{1}{t}\int\varphi_n d\alpha_t-\int\varphi_n d\beta|$ is small for large enough fixed n and for $t>t_0(n)$ independently of the choice of the step function φ_n. We note that $\int\varphi_n d\alpha_t=\sum_{i,j=1}^{n-2}p_{i,j}\alpha_t(A_{i,j})=\sum_{i,j=1}^{n-2}p_{i,j}\alpha_t'(A_{i,j}')$ where $A_{i,j}'=f(A_{i,j})$ and $p_{i,j}$ is the value of φ at one point in $A_{i,j}$. Then

$$\Big|\frac{1}{t}\int\varphi_n d\alpha_t-\int\varphi_n d\beta\Big|\leq\sum_{i,j=1}^{n-2}\Big|\frac{1}{t}\alpha_t'(A_{i,j}')-f^*\beta(A_{i,j}')\Big|$$

where by the definition $\alpha_t'(A_{i,j}')=\alpha_t(A_{i,j})$ and $\beta(A_{i,j})=f^*\beta(A_{i,j}')$. By Lemma 4.5, $\sum_{i,j=1}^{n-2}\Big|\frac{1}{t}\alpha_t'(A_{i,j}')-f^*\beta(A_{i,j}')\Big|\leq\frac{C(n)}{t}+\frac{3}{n}$ because $f(A_{i,j})=A_{i,j}'$ and $f(E_n)=E_n'$ have similar properties as $A_{i,j}$ and E_n by the Hölder continuity of f. The theorem follows. □

Corollary 4.2. *Let α_t be as in Theorem 4.3 and let $\|\beta\|_\nu=1$. Then $\frac{1}{\|\alpha_t\|_\nu}\alpha_t\to\beta$ as $t\to\infty$ in the ν-norm.*

Proof. Similar to the proof of Corollary 4.1. □

Theorem 4 follows directly from Corollary 4.2.

We consider the action of the quasiconformal mapping class group $QMCG(X)$ on $\mathcal{T}(X)$. The group $QMCG(X)$ consists of all quasiconformal maps of Δ onto itself which conjugate Γ onto itself up to an equivalence relation. Two such maps are equivalent if their extensions to S^1 are equal after postcomposing one of them by an element of Γ. For $g \in QMCG(X)$ the action on $\mathcal{T}(X)$ is given by $[f] \mapsto [f \circ g^{-1}]$ for $[f] \in \mathcal{T}(X)$. We keep the same notation g for its extension to the boundary S^1. We define the action of $g \in QMCG(X)$ on $\mathcal{H}(X)$ by

$$g^* W(\varphi) = W(\varphi \circ g),$$

for all $W \in \mathcal{H}(X)$ and for all $\varphi \in H(X)$. Then, for a measure α in $\mathcal{H}(X)$ we get $g^*\alpha(\varphi) = \int \varphi \circ g(x) d\alpha(x)$. By substituting $x = g^{-1}(y)$ we get $g^*\alpha(\varphi) = \int \varphi(y) d\alpha(g^{-1}(y))$ and it implies that $g^*\alpha(\varphi) = \int \varphi d(g^*\alpha)$. Thus the action of g on $\mathcal{H}(X)$ restricts to the usual action of g on $\mathcal{T}(X) \cup \mathcal{PML}_b(X)$.

We show that the action of g is continuous on $\mathcal{H}(X)$.

Theorem 4.4. *The action of $QMCG(X)$ on $\mathcal{H}(X)$ is continuous.*

Proof. Let $|W_1(\varphi) - W_2(\varphi)| < \epsilon$ for all $(\varphi, Q) \in test(\nu)$. Function $\varphi \circ g$ has support in $g^{-1}(Q)$. Let $\Theta_{g^{-1}(Q)}$ be the Möbius transformation which maps $-k$, -1, 1 and k onto $g^{-1}(a)$, $g^{-1}(b)$, $g^{-1}(c)$ and $g^{-1}(d)$, for a unique constant $k > 1$. We obtain

$$\|\varphi \circ g \circ \Theta_{g^{-1}(Q)}\|_{\nu\nu_1} \leq \|\varphi \circ \Theta_Q\|_{\nu} \|\Theta_Q^{-1} \circ g \circ \Theta_{g^{-1}(Q)}\|_{\nu_1}^{\nu}$$

where $0 < \nu_1 \leq 1$ is equal to the Hölder exponent of g (see [13]). The Liouville mass of the support of $\varphi \circ g$ is possibly greater than $\log 2$. We can write $\varphi \circ g$ as a sum of finitely many $\nu\nu_1$-Hölder continuous functions each having support with Liouville mass less than or equal to $\log 2$. The number of such functions depends on the Liouville measure of $g^{-1}(Q)$ which in turn depends on the constant of quasiconformality of g and it is independent of Q. Consequently, there exists a constant $C > 0$ such that

$$|g^* W_1(\varphi) - g^* W_2(\varphi)| = |W_1(\varphi \circ g) - W_2(\varphi \circ g)| < C\epsilon.$$

Thus the action is continuous. □

The inverse of the action of $g \in QMCG(X)$ on $\mathcal{H}(X)$ is the action of g^{-1}. The action of $QMCG(X)$ on $\mathcal{H}(X)$ restricts to the classical action on $\mathcal{T}(X) \cup \mathcal{PML}_b(X)$. Thus Theorem 4.4 implies Theorem 3 of the Introduction.

Our description of a Thurston-type boundary for $\mathcal{T}(X)$ when X is infinite surface differs from the original Thurston's description for finite surfaces. We make a "weak" connection between the two.

Let γ be simple closed geodesic on X and $[f] \in \mathcal{T}(X)$. We define $l_{[f]}(\gamma)$ to be the length of the unique geodesic in the homotopy class of $f(\gamma)$ on $f(X)$.

Theorem 4.5. *Let $[f_t] \in \mathcal{T}(X)$ such that $[f_t] \to [\beta]$ as $t \to \infty$, where $[\beta] \in \mathcal{PML}_b(X)$. Let γ be a simple closed geodesic on X such that $\beta \cap \gamma \neq \emptyset$. Then*

$$l_{[f_t]}(\gamma) \to \infty$$

as $t \to \infty$.

Proof. Let Γ_t be the covering group of $X_t = f_t(X)$. The length of the geodesic γ_t in the homotopy class of $f_t(\gamma)$ is equal to the translation length of the corresponding hyperbolic isometry $A_t \in \Gamma_t$. Let r_t and a_t be the repelling and the attracting fixed point of A_t. Similarly, let $A \in \Gamma$ be a hyperbolic element which corresponds to γ and, let r and a be the repelling and the attracting fixed point of A.

An elementary calculation shows that $L([r_t, x_t] \times [A_t x_t, a_t]) = \log \frac{\lambda_t}{\lambda_t - 1}$ where λ_t is the multiplier of A_t and $x_t = f_t(x)$ for arbitrary $x \in (r, a)$. Note that $\log \lambda_t$ is the translation length of A_t, and $\log \lambda_t \to \infty$ if and only if $\lambda_t \to \infty$.

To prove the theorem, it is enough to show that $L([r_t, x_t] \times [A_t x_t, a_t]) \to 0$ as $t \to \infty$. Note that $L([r_t, x_t] \times [A_t x_t, a_t]) = \alpha_t([r, x] \times [Ax, a])$. We choose $x \in (r, a)$ such that $\beta((x, Ax) \times (a, r)) > 0$. Then $\alpha_t([x, Ax] \times [a, r]) \to \infty$ as $t \to \infty$. Since each α_t satisfies equation (2), we get that $\alpha_t([r, x] \times [Ax, a]) \to 0$ as $t \to \infty$. □

The above connection is just in one direction. Namely, for a fixed $[\beta] \in \mathcal{PML}_b(X)$ the condition $l_{[f_t]}(\gamma) \to \infty$ as $t \to \infty$ for each simple closed geodesic γ on X such that $\gamma \cap \beta \neq \emptyset$ does not imply that $[f_t]$ converges to $[\beta]$. The main problem is that we do not have enough simple closed geodesics on an arbitrary infinite surface X.

Appendix

We give several lemmas on the Liouville measure of a box under the action of a hyperbolic isometry. Lemma A.2 and Lemma A.3 are similar to lemmas given in [8] and [12]. Lemmas given here will be used in section 5.

In the next lemma we estimate the change in the Liouville measure of a box under a simple left earthquake whose support is the geodesic located in the lower left corner of the box.

Lemma A.1. *Let a, b, c and d be points on S^1 in the counter-clockwise order. Denote by l the distance between the geodesic with endpoints a and d and the geodesic with endpoints b and c. Let A be the hyperbolic translation with the repelling fixed point a, with the attracting fixed point c and with the translation length βt. Then*

$$L([a, A(b)] \times [c, d]) = \beta t + \log \Big(\frac{\cosh l - 1}{2} + e^{-\beta t} \Big)$$

and consequently,

$$\beta t + \log \frac{l^2}{4} \leq L([a, A(b)] \times [c, d]) \leq \beta t + L([a, b] \times [c, d]).$$

Proof. For the simplicity of the computations we use the upper half plane $\mathbb{H}^2$. Because L is invariant under the Möbius maps we can assume that $a = 0$, $b > 0$, $c = \infty$ and $d = -1$. Then $L([a, b] \times [c, d]) = \log(b + 1)$ and $b = \frac{\cosh l - 1}{2}$, where l is the distance between the geodesic with the endpoints a and d and the geodesic with the endpoints b and c. By the same formula $L([a, A(b)] \times [c, d]) = \log(e^{\beta t} b + 1) = \beta t + \log \Big(\frac{\cosh l - 1}{2} + e^{-\beta t} \Big)$. □

In the following four lemmas we compare the Liouville measures of the images of a box under the action of two simple left earthquakes.

In the next lemma we compare the Liouville measures of the images of a box $Q = [a, b] \times [c, d]$ under two simple earthquakes E_1 and E_2 for measures β_1 and β_2 with their supports l_1 and l_2 in $[b, c) \times (c, d]$ where $\beta_1(l_1) = \beta_2(l_2)$. If one component of the complement of l_2 contains l_1 and c then the mass of $E_1(Q)$ for the Liouville measure is greater than or equal to the mass of $E_2(Q)$. Similarly, let E_3 and E_4 be two simple earthquakes with supports $l_3, l_4 \in [a,b) \times (b,c]$, respectively. If $\beta_3(l_3) = \beta_4(l_4)$ and if one component of the complement of l_3 contains l_4 and b then the mass of $E_3(Q)$ for the Liouville measure is greater or equal to the mass of $E_4(Q)$.

Lemma A.2. *Let a, g, $g^{'}$, b, $e^{'}$, e, c, f, $f^{'}$ and d be points on S^1 given in the counter-clockwise order. Let A be the hyperbolic translation with the repelling fixed point e, with the attracting fixed point f and with the translation length $\lambda > 0$. Let $A^{'}$ be the hyperbolic translation with the repelling fixed point $e^{'}$, with the attracting fixed point $f^{'}$ and with the translation length $\lambda > 0$. Let A_1 be the hyperbolic translation with the repelling fixed point g, with the attracting fixed point e and with the translation length $\lambda_1 > 0$. Let $A_1^{'}$ be a hyperbolic translation with the repelling fixed point $g^{'}$, the attracting fixed point $e^{'}$ and the translation length λ_1. Then*

$$L([a,b] \times [A(c), d]) \geq L([a,b] \times [A^{'}(c), d])$$

and

$$L([a, A_1(b)] \times [c,d]) \geq L([a, A_1^{'}(b)] \times [c,d]).$$

Proof. By Lemma A.1, the Liouville measure of $[a,b] \times [c,d]$ is an increasing function of the hyperbolic distance between geodesic with the endpoints a and d, and the geodesic with the endpoints b and c. Thus it is enough to show that $A(c)$ lies between c and $A^{'}(c)$, and that $A_1^{'}(b)$ lies between b and $A_1(b)$ for the given counter-clockwise orientation of S^1. But this is trivial. □

Given $Q = [a,b] \times [c,d]$, let E_1 and E_2 be two simple earthquakes for measures β_1 and β_2 with support geodesics l_1 and l_2 in $[b,c] \times [d,a]$, respectively. Assume that $\beta_1(l_1) = \beta_2(l_2)$ and that the endpoints in $[d,a]$ of l_1 and l_2 are equal. Assume that one component of the complement of l_1 contains l_2 and (c,d). We show that $E_1(Q)$ has smaller or equal mass to $E_2(Q)$ for the Liouville measure. Let E_3 and E_4 be two simple left earthquakes for measures β_3 and β_4 with supports l_3 and l_4 in $[b,c] \times [d,a]$, respectively. Assume that $\beta_3(l_3) = \beta_4(l_4)$ and that the endpoints in $[b,c]$ of l_3 and l_4 are equal. We show that if one component of the complement of l_3 contains l_4 and (c,d) then the mass of $E_3(Q)$ is greater than or equal to the mass of $E_4(Q)$ for the Liouville measure.

Lemma A.3. *Let a, b, x, y, c, d and e be points on S^1 given in the counter-clockwise order. Let A_x be the hyperbolic translation with the repelling fixed point x, with the attracting fixed point e and with the translation length l. Let A_y be the hyperbolic translation with the repelling fixed point y, with the attracting fixed point e and with the translation length l. Let A^x be the hyperbolic translation with the repelling fixed point e, with the attracting fixed point x and with the translation length l. Let A^y be the hyperbolic translation with the repelling fixed point e, with the attracting fixed point y and with the translation length l. Then*

$$L([a,b] \times [A_x(c), A_x(d)]) \leq L([a,b] \times [A_y(c), A_y(d)])$$

and

$$L([A^x(a), A^x(b)] \times [c,d]) \geq L([A^y(a), A^y(b)] \times [c,d]).$$

Proof. We use the upper half plane model model $\mathbb{H}^2$. By the invariance of L under the Möbius maps we can assume that $e=\infty$. Let $\lambda=e^l$. Then $L([a,b]\times[A_x(c),A_x(d)])=\log\frac{[\lambda(c-x)+x-a][\lambda(d-x)+x-b]}{[\lambda(d-x)+x-a][\lambda(c-x)+x-b]}$. The derivative of the above expression with respect to x is

$$\lambda(\lambda-1)(d-c)\Big\{\frac{1}{[\lambda(d-x)+x-b][\lambda(c-x)+x-b]}-\frac{1}{[\lambda(d-x)+x-a][\lambda(c-x)+x-a]}\Big\}.$$

It is positive for $b<x<c$ and for $\lambda>1$. Consequently, the expression $L([a,b]\times[A_x(c),A_x(d)])$ is increasing in x. Hence the first inequality is proved. The second inequality follows by applying A^x to the box $[a,b]\times[A_x(c),A_x(d)]$ and by applying A^y to the box $[a,b]\times[A_y(c),A_y(d)]$ and noting that the Liouville measure is Möbius invariant. □

Let $Q=[a,b]\times[c,d]$. Let E_1 be a simple earthquake for measure β_1 whose support geodesic l_1 has endpoints a and $e\in[c,d]$. Let E_2 be a simple earthquake for measure β_2 whose support geodesic l_2 has endpoints $a^{'}$ and $e\in[c,d]$. We assume that $\beta_1(l_1)=\beta_2(l_2)$. We show that the Liouville mass of $E_1(Q)$ is greater than the Liouville mass of $E_2(Q)$.

Lemma A.4. *Let a, $a^{'}$, b, c, e and d be points on S^1 given in the counter-clockwise order. Let A be the hyperbolic translation with the repelling fixed point a, with the attracting fixed point e and with the translation length $l>0$. Let $A^{'}$ be the hyperbolic translation with the repelling fixed point $a^{'}$, with the attracting fixed point e and with the translation length $l>0$. Then*

$$L([a,A(b)]\times[A(c),d])\geq L([a,A^{'}(b)]\times[A^{'}(c),d]).$$

Proof. We use $\mathbb{H}^2$ model and we can assume that $e=\infty$. Then computations give $L([a,A(b)]\times[A(c),d])=\log\left(\frac{c-a}{c-b}\frac{a-d+\lambda(b-a)}{a-d}\right)$ and $L([a,A^{'}(b)]\times[A^{'}(c),d])=\log\left[\frac{c-a^{'}+\frac{1}{\lambda}(a^{'}-a)}{c-b}\frac{a^{'}-d+\lambda(b-a^{'})}{a-d}\right]$. But $a^{'}-d+\lambda(b-a^{'})\leq a-d+\lambda(b-a)$ and $c-a^{'}+\frac{1}{\lambda}(a^{'}-a)\leq c-a$ for $a^{'}\geq a$ and the conclusion follows. □

Let $Q=[a,b]\times[c,d]$ and let E_1 be a simple earthquake for measure β_1 with support geodesic l_1 whose endpoints are a and c. Let E_2 be a simple earthquake for measure β_2 with support geodesic l_2 whose endpoints are a and $c^{'}\in(c,d)$. We assume that $\beta_1(l_1)=\beta_2(l_2)$. We show that the Liouville mass of $E_1(Q)$ is greater than or equal to $E_2(Q)$.

Lemma A.5. *Let a, b, c, $c^{'}$ and d be points on S^1 given in the counter-clockwise order. Let A be the hyperbolic translation with the repelling fixed point a, with the attracting fixed point c and with the translation length $l>0$. Let $A^{'}$ be the hyperbolic translation with the repelling fixed point a, with the attracting fixed point $c^{'}$ and with the translation length $l>0$. Then*

$$L([a,A(b)]\times[c,d])\geq L([a,A^{'}(b)]\times[A^{'}(c),d]).$$

Proof. We use $\mathbb{H}^2$ model and we can assume that $a = \infty$. Let $\lambda = e^l$. The multiplier for A and $A^{'}$ is $\frac{1}{\lambda}$. Then computations give $L([a, A(b)] \times [c, d]) = \log\left[\frac{d-c+\frac{1}{\lambda}(c-b)}{\frac{1}{\lambda}(c-b)}\right]$ and $L([a, A^{'}(b)] \times [A^{'}(c), d]) = \log\left[\frac{d-c^{'}+\frac{1}{\lambda}(c^{'}-b)}{\frac{1}{\lambda}(c-b)}\right]$. But $d - c + \frac{1}{\lambda}(c-b) \geq d - c^{'} + \frac{1}{\lambda}(c^{'} - b)$ for $c^{'} \geq c$ and the conclusion follows. □

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