REAL AND COMPLEX EARTHQUAKES

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ABSTRACT. We consider earthquakes and bendings of the hyperbolic plane $\mathbb{H}^2$. We show that an earthquake restricted to the unit circle S^1 is a quasisymmetric map if and only if its earthquake measure is bounded. Multiplying an earthquake measure by a positive parameter we obtain an earthquake path. Consequently, earthquake paths with bounded measures are paths in the universal Teichmüller space. We extend the real parameter for bounded earthquakes into complex parameter with small imaginary part. Such obtained complex earthquakes (or bendings) are holomorphic in the parameter. Moreover, bendings with complex parameters of small imaginary parts when restricted to the unit circle S^1 are holomorphic motions of S^1. In particular, this says that real earthquake paths with bounded earthquake measures are analytic in their parameters.

1. INTRODUCTION

Earthquakes are maps which change hyperbolic metrics on a surface. Thurston [12] introduced earthquakes on closed surfaces as a certain completion of "continuous" left Dehn twists. He showed that any two hyperbolic metrics on a closed surface are related by a unique earthquake. This fact is referred to as the *earthquake theorem*. An earthquake is uniquely determined by a measured lamination called an earthquake measure. Multiplying the earthquake measure by a positive parameter we get an earthquake path in the Teichmüller space of the initial surface. Kerckhoff [13] showed that an earthquake path, for a closed surface, is analytic in its parameter.

In a subsequent paper [18], Thurston gave a more direct proof of the earthquake theorem. He defined earthquakes on the hyperbolic plane $\mathbb{H}^2$ and showed that any homeomorphism of the unit circle S^1 can be obtained by continuously extending a unique earthquake of $\mathbb{H}^2$ to S^1 (*geology is transitive*). The original earthquake theorem is obtained from *geology is transitive* as follows. Any two hyperbolic structures on a closed surface are related by a quasiconformal homeomorphism. Such homeomorphism, via universal covering of the surface, gives a Γ-invariant quasisymmetric map h of S^1, where Γ is the covering group. Conversely, any Γ-invariant quasisymmetric map h of S^1 gives a quasiconformal map of $\mathbb{H}^2/\Gamma$ onto another hyperbolic surface. An earthquake of $\mathbb{H}^2$ that realizes a Γ-invariant homeomorphism h of S^1 is also Γ-invariant. It descends to an earthquake of the closed surface which relates the two hyperbolic metrics establishing the earthquake theorem.

Thurston's work on earthquakes in the hyperbolic plane $\mathbb{H}^2$ opened the possibility of using earthquakes on infinite area hyperbolic surfaces. In particular, earthquakes

could be used in Teichmüller spaces of infinite surfaces. This paper investigates which earthquakes preserve Teichmüller spaces and how these earthquakes depend on the parameter.

The Teichmüller space of the hyperbolic plane $\mathbb{H}^2$ is identified with the space of quasisymmetric maps of S^1 modulo post composition with Möbius maps. Thus, our goal is to identify earthquakes whose continuous extensions to S^1 are quasisymmetric maps. An earthquake is uniquely determined, up to post composition with a Möbius map, by a measured lamination which is called an earthquake measure. If we multiply an earthquake measure λ with a positive parameter t, we obtain a family of earthquake measures $t\lambda$. The corresponding family of earthquakes is called an earthquake path with the parameter t. To talk about earthquake paths in the Teichmüller space, we need to show that if an earthquake with the earthquake measure λ extends to a quasisymmetric map of S^1 then the whole earthquake path with earthquake measures $t\lambda$, $t > 0$, consists of quasisymmetric maps of S^1. The following theorem characterize earthquake maps which give quasisymmetric maps in terms of their earthquake measures.

Theorem 1. *Let E^λ be an earthquake of $\mathbb{H}^2$ with earthquake measure λ. Then the continuous extension of E^λ to S^1 is quasisymmetric if and only if λ is bounded.*

An earthquake measure λ is *bounded* if $\sup \lambda(I) < \infty$ where the supremum is over all geodesic arcs of length 1 in $\mathbb{H}^2$. It is immediate that if λ is bounded then $t\lambda$ is bounded. Thus $t \mapsto [E^{t\lambda}|_{S^1}]$ is a path in the universal Teichmüller space $\mathcal{T}(\mathbb{H}^2)$. If $[E^{t\lambda}|_{S^1}]$ is the Teichmüller class of $E^{t\lambda}|_{S^1}$ in $\mathcal{T}(\mathbb{H}^2)$, it is enough to prove Theorem 1 for the universal Teichmüller space $\mathcal{T}(\mathbb{H}^2)$, because an earthquake preserves invariance with respect to the covering group. Consequently Theorem 1 holds for any hyperbolic surface.

Remark 1. One direction of the above theorem, was proved by Thurston [18]. Independently of the author, a version of Theorem 1 is proved by Gardiner, Hu and Lakic [9], and by Hu [10].

It is natural to extend the real parameter $t > 0$ to a complex parameter τ. The extended map $\tau \mapsto E^{\tau\lambda}$ is called bending. We embed $\mathbb{H}^2$ as a vertical upper half-plane above x-axis for the model $\mathbb{C} \times \{t|t > 0\}$ of hyperbolic three space $\mathbb{H}^3$. The bending map $E^{\tau\lambda} : \mathbb{H}^2 \to \mathbb{H}^3$ is given by bending $\mathbb{H}^2$ according to the complex measure $\tau\lambda$ (see Epstein and Marden [5]). The map $E^{\tau\lambda} : \mathbb{H}^2 \to \mathbb{H}^3$ trivially extends to the boundary of flat pieces (connected components of the complement of the geodesic lamination support of λ) and to the endpoints of the geodesics in the support of λ on $S^1 = \widehat{\mathbb{R}}$. However, we show that it extends to the whole boundary S^1 of $\mathbb{H}^2$. Namely, $E^{\tau\lambda} : \widehat{\mathbb{R}} \to \widehat{\mathbb{C}}$ is well defined for $\tau \in S_\lambda$, where S_λ is a neighborhood of $\mathbb{R}$ depending on λ. Without loss of generality we assume that $E^{\tau\lambda}(\infty) = \infty$.

Theorem 2. *Let λ be a bounded earthquake measure. Then the earthquake map $(x, t) \mapsto E^{t\lambda}(x)$, $x \in \mathbb{R}$ and $t > 0$, extends to a holomorphic motion $(x, \tau) \mapsto E^{\tau\lambda}(x)$, $x \in \mathbb{R}$ and $\tau \in S_\lambda$, of the real line.*

Remark 2. In the proof of Theorem 2 we analyze the bent plane $E^{\tau\lambda}(\mathbb{H}^2)$ in $\mathbb{H}^3$. It turns out that $E^{\tau\lambda}(\mathbb{H}^2)$ divides $\mathbb{H}^3$ into two sets one of them being convex. The convex set faces one of the two components of $\mathbb{C} - E^{\tau\lambda}(\mathbb{R})$ and the boundary of the set in $\mathbb{H}^3$ is $E^{\tau\lambda}(\mathbb{H}^2)$.

By a theorem of Slodkowski [17], a holomorphic motion of a closed set in $\mathbb{C}$ can be extended to a holomorphic motion of $\mathbb{C}$. Thus $(x, \tau) \mapsto E^{\tau\lambda}(x)$, $x \in \mathbb{R}$ and $\tau \in S_\lambda$, extends to $(z, \tau) \mapsto E^{\tau\lambda}(z)$, $z \in \mathbb{C}$ and $\tau \in S_\lambda$. By a theorem of Mañé, Sad and Sullivan [15], $E^{\tau\lambda} : \mathbb{C} \to \mathbb{C}$ is quasiconformal, for a fixed $\tau \in S_\lambda$. Let μ_τ be the Beltrami coefficient of $E^{\tau\lambda}$ in the upper half plane $\mathbb{H}^2$. Then $\tau \mapsto [\mu_\tau]$ is a holomorphic map from S_λ to $\mathcal{T}(\mathbb{H}^2)$, where $[\mu_\tau]$ is the Teichmüller class of μ_τ in $\mathcal{T}(\mathbb{H}^2)$. This map is called a complex earthquake and it agrees with McMullen's complex earthquake definition for finite surfaces with small imaginary part of the parameter.

Corollary 1. *Given a bounded measured lamination λ in $\mathbb{H}^2$ there is a well defined complex earthquake $\tau \mapsto [\mu_\tau]$ for all parameters τ in a neighborhood S_λ of the real line $\mathbb{R}$. The complex earthquake is holomorphic and it extends the earthquake path $t \mapsto [E^{t\lambda}|_{\widehat{\mathbb{R}}}]$. Consequently, the earthquake path is real analytic in the parameter.*

Because we work with the universal Teichmüller space $\mathcal{T}(\mathbb{H}^2)$ our results are more general than previous results [11],[13],[16]. They apply, in particular, to Teichmüller spaces of arbitrary Riemann surfaces. They show that if we bend a small amount the bent surface is convex, a result previously known for measured laminations of finite surfaces [11]. Our results extend Kerckhoff's result on analyticity of earthquake paths for finite surfaces and McMullen's result of holomorphicity of complex earthquakes for finite surfaces to arbitrary hyperbolic surfaces.

There is a significant difference in techniques used for finite surfaces case and for arbitrary surfaces case. Mainly, in the finite case the role of closed geodesics is distinguished and they are used to approximate general laminations. In the infinite case, we are forced to work in the hyperbolic plane were no closed geodesics exist.

The paper is organized as follows. In Section 2 we define earthquakes and earthquake measures, show that bounded measures correspond to earthquakes, and give one direction of the proof of Theorem 1. In Section 3 we review the definition of bending. In Section 4 we recall some theorems on holomorphic motions. Using Theorem 2 and holomorphic motions we prove Corollary 1. Corollary 1 directly proves the second direction of Theorem 1. In Section 5 we give main geometric argument in this paper. It states that if the parameter has small imaginary part, then the image of a geodesic ray under the bending is contained in a cone of a fixed angle with vertex the initial point of the ray. In Section 6 we use the cone argument to prove Theorem 2.

2. Earthquakes and earthquake measures

We define earthquakes and investigate their properties. All our discussions are for the hyperbolic plane $\mathbb{H}^2$. The general theory for complete hyperbolic surfaces follows from the fact that $\mathbb{H}^2$ is the universal covering of any complete hyperbolic surface [18]. We begin with the definition of a geodesic lamination on $\mathbb{H}^2$. By a geodesic in $\mathbb{H}^2$ we mean a geodesic infinitely continued in both directions.

Definition 2.1. A *geodesic lamination* $\mathcal{L}$ on $\mathbb{H}^2$ is a closed subset of $\mathbb{H}^2$ that is written as a disjoint union of geodesics in $\mathbb{H}^2$. A *flat piece* of $\mathcal{L}$ is a connected component of the complement of $\mathcal{L}$ in $\mathbb{H}^2$. A *facet* of $\mathcal{L}$ is either a geodesic of $\mathcal{L}$ or a flat piece. The union of all facets of $\mathcal{L}$ covers $\mathbb{H}^2$.

The union of all simple geodesics on a finite area hyperbolic surface has Hausdorff dimension 1. In particular, the Lebesgue measure of a geodesic lamination on a finite surface is zero. Also, a geodesic lamination on a finite surface can be written in a unique way as a disjoint union of geodesics.

For $\mathbb{H}^2$ this is not true. In fact, a geodesic lamination $\mathcal{L}$ on $\mathbb{H}^2$ can fill the space. For this reason we require that the decomposition of $\mathcal{L}$ into a disjoint union of geodesics is a part of the above definition.

Definition 2.2. A *measured lamination* λ on $\mathbb{H}^2$ is given by the support geodesic lamination $|\lambda|$ and by the transverse measure to $|\lambda|$. In more details, we assign a finite Borel measure to each closed finite length geodesic arc in $\mathbb{H}^2$ whose support is on the transverse intersection of the arc with $|\lambda|$. The measure on arcs is required to be invariant under homotopies preserving geodesics of $|\lambda|$.

We allow for positive, real or complex measured laminations depending on our needs. Following Thurston [18], we give the definition of an earthquake.

Definition 2.3. An *earthquake* E is an injective map of $\mathbb{H}^2$ onto itself. The *support* of the earthquake is a geodesic lamination $\mathcal{L}$. On each facet of $\mathcal{L}$, the earthquake E is a Möbius map. For any two facets A and B, the map $E|_B \circ (E|_A)^{-1}$ is a hyperbolic translation with axis separating A and B. Further, $E|_B \circ (E|_A)^{-1}$ translates to the left as seen from A.

An earthquake is not necessarily a homeomorphism of $\mathbb{H}^2$. Thurston [18] showed that an earthquake continuously extends to an orientation preserving homeomorphism of the boundary $\partial_\infty \mathbb{H}^2 = S^1$ of $\mathbb{H}^2$. Conversely, any orientation preserving homeomorphism of the unit circle S^1 can be obtained by continuous extension of an earthquake of $\mathbb{H}^2$ (geology is transitive). We use the upper half-plane model for $\mathbb{H}^2$ and denote its boundary $\widehat{\mathbb{R}}$ by S^1, where $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

If the support of an earthquake E is a lamination $\mathcal{L}$ with finitely many geodesic then it is called a *finite* earthquake. Let l be a geodesic of $\mathcal{L}$, and let A and B be flat pieces adjacent to l. Then $E|_B \circ (E|_A)^{-1}$ has to be a hyperbolic translation with the axis l. We assign the translation length of $E|_B \circ (E|_A)^{-1}$ to l and obtain a positive measured lamination λ with support $\mathcal{L}$. Conversely, any positive finite measured lamination λ corresponds to a unique finite earthquake up to some ambiguities in the definition on isolated geodesics in $|\lambda|$ and more importantly up to post composition with a Möbius map [18].

Given a general earthquake E, there is associated positive measured lamination λ as follows. The support of λ equals to the support of E. Let I be a closed geodesic arc transversely intersecting the support of E with arbitrary orientation. For given n, choose facets $\mathcal{A}_n = \{A_1, A_2, \ldots, A_n\}$ of the support of E such that A_1 contains the left endpoint of I and A_n contains the right endpoint of I, and A_i intersect I in the given order. The sum of the translation lengths of $E|_{A_{i+1}} \circ (E|_{A_i})^{-1}$ for $i = 1, 2, \ldots, n-1$ is the approximate measure of I. If $n \to \infty$ and $\mathcal{A}_n$ are chosen such that the union of their elements is dense in I, then the limit of approximate measures for $\mathcal{A}_n$ is a well defined measure [18]. If the endpoints of I are not in the interior of flat pieces some additional care in the definition is needed. The measure of a point is the limit of measures of closed geodesic arcs transverse to the support of E which contain the point, and which descend to the point. If the left endpoint a of I is not on the flat piece of the support of E then we choose $\mathcal{A}_n$ such that A_1

does not contain a but it is close to a for n large. The mass of I is the limit of approximate measures for $\mathcal{A}_n$ plus half of the mass for a. Similarly for the right endpoint of I if it does not belong to a flat piece.

We call the associated positive measured lamination an *earthquake measure*. Thurston [18] showed that earthquakes are uniquelly determined up to post composition with a Möbius map by the associated earthquake measure.

More generally, we call a positive measured lamination an *earthquake measure*. An earthquake measure does not always give an earthquake, see examples in [19] and [9]. In general, for a given earthquake measure λ there is a cocycle map E^λ from $\mathbb{H}^2$ to $\mathbb{H}^2$ which satisfies all the properties of an earthquake except possibly being onto, see Epstein-Marden [5]. The cocyle for a given measure is constructed by taking the limit of finite earthquakes corresponding to finite measure approximation to the earthquake measure. More details on the construction of cocycles for (complex) measured laminations is given in Section 3. A natural question is which measures give an earthquake map. We give a sufficient condition which is not necessary.

Definition 2.4. An earthquake measure λ is *bounded* if $\sup \lambda(I)$ is finite, where the supremum is over all unit length closed geodesic arcs in $\mathbb{H}^2$.

We give a lower bound estimate in terms of an earthquake measure λ on the distance between the images under the corresponding cocycle map E^λ for two geodesics of $|\lambda|$, given their original distance.

Lemma 2.1. *Let λ be an earthquake measure on $\mathbb{H}^2$ and E^λ be the corresponding cocycle. Let l_1 and l_2 be two geodesics of $|\lambda|$ whose distance is d. The distance d_1 between $E^\lambda(l_1)$ and $E^\lambda(l_2)$ satisfies*

$$(1) \qquad\qquad d_1 \geq C(d)e^{-\lambda(I)}$$

where $C(d) = \frac{d}{\cosh d + 1}$ and $\lambda(I)$ is the measure of a closed arc I which connects l_1 and l_2.

Proof. The hyperbolic metric on $\mathbb{H}^2$ is invariant under Möbius maps. Thus without loss of generality, we can assume that l_1 has endpoints 0 and ∞, and that l_2 has endpoints 1 and c, $c > 1$. We normalize E^λ by post composing it with a Möbius map, if necessary, to be the identity on l_1. Then $E^\lambda(l_1) = l_1$.

The cocycle E^λ is obtained by taking the limit of finite compositions of hyperbolic translations with axis between l_1 and l_2 oriented to the left as seen from l_1. It is obvious that $E^\lambda(l_2)$ will be the closest to l_1 if all hyperbolic translations in the composition have the repelling fixed point 1 and the attracting fixed point ∞. In this case $E^\lambda(l_2)$ is the geodesic with endpoints 1 and Mc, where $M \leq e^{\lambda(I)}$.

A computation in the hyperbolic plane $\mathbb{H}^2$ gives $\cosh d = \frac{c+1}{c-1}$ and $\cosh d_1 = \frac{Mc+1}{Mc-1}$. Then

$$d_1 \geq \log \cosh d_1 = \log \frac{Mc+1}{Mc-1} \geq \frac{1}{Mc}$$

and

$$\frac{1}{c} = \frac{\cosh d - 1}{\cosh d + 1} \geq \frac{d}{\cosh d + 1} = C(d).$$

We combine above inequalities to obtain (1). $\square$

We show that a bounded earthquake measure gives a cocyle which is an earthquake. This condition is not necessary.

Proposition 2.1. *Let λ be a bounded earthquake measure. Then there exists an earthquake E^λ whose measure is λ.*

Proof. As remarked above, we only need to show that $E^\lambda : \mathbb{H}^2 \to \mathbb{H}^2$ is onto.

The union of facets of $|\lambda|$ covers $\mathbb{H}^2$. The images of facets of $|\lambda|$ under E^λ are mapped into $\mathbb{H}^2$ and they inherit separation properties from $|\lambda|$. Assume that $\{l_i\}$ is a sequence of geodesics in $|\lambda|$ and $\{P_i\}$ a sequence of half-planes with boundary l_i such that $P_{i+1} \subset P_i$ and the intersection of the closures of P_i in $\mathbb{H}^2 \cup S^1$, for all i, is a unique point on S^1. It is enough to show that for any such sequence $\{l_i\}$, the sequence $\{E^\lambda(l_i)\}$ satisfies the same property. There are two possibilities for $\{l_i\}$. Either all but finitely many geodesics l_i have a common endpoint or there is a subsequence, denoted by $\{l_i\}$ again, such that l_i and l_{i+1} are on the positive distance apart.

Assume that all l_i have a common endpoint $x \in \widehat{\mathbb{R}}$. Transverse measure λ can be considered as a measure on the space of geodesics of $\mathbb{H}^2$. We use this description simultaneously with the transverse description for the simplicity of notation. Since the distance between l_1 to l_i is zero and λ is bounded, we conclude that the mass in λ measure of all geodesics of $|\lambda|$ which have one endpoint x is finite. Without loss of generality, we can assume that l_i has endpoints ∞ and $x_i < 0$, and $x_i > x_{i+1}$, and $x_i \to -\infty$ as $i \to \infty$. Let E^λ be the identity on the geodesic with endpoints 0 and ∞. Then $E^\lambda|_{l_i}$ is the limit of finite compositions of hyperbolic translations with axis between l_{i-1} and l_i, and whose total translation length is $\lambda(I_i) = a_i$, where I_i is a closed geodesic arc connecting l_{i-1} and l_i. By the above $\sum_{i=1}^\infty a_i = a < \infty$. Then E^λ fixes ∞ and moves x_i toward 0. An easy observation shows that $E^\lambda(x_i) \le e^{-a} x_i \to -\infty$ as $i \to \infty$. Consequently, $E^\lambda(x_i) \to -\infty$ as $i \to \infty$ which proves the claim in this case.

Assume that we are in the second case. Namely, there exists a nested sequence $\{l_i\}$ of geodesics of $|\lambda|$ with the following properties. The hyperbolic plane $\mathbb{H}^2$ is divided into two hyperbolic half-planes by each l_i. Let P_i be the hyperbolic half-plane which contains l_{i+1}. The intersection of closures of P_i in $\mathbb{H}^2 \cup S^1$ equals a point $x \in S^1$. By choosing possibly different sequence $\{l_i\}$ of $|\lambda|$ we arrange that the distance between all consecutive l_i and l_{i+1} is in the interval $[1, 2]$ as follows. If two consecutive geodesics l_i and l_{i+1} are on the distance larger than 2, then we add to the sequence geodesics with zero mass in between l_i and l_{i+1}. If l_i and l_{i+1} are on the distance less than 1, then there exists l_j, $j > i + 1$, such that the distance between l_i and l_j is greater than or equal to 1. Then we replace l_{i+1} by l_j and proceed as above. It remains to show that for each l_i there exists l_j, $j > i$, on the distance greater than or equal to 1. If not, there are two possibilities. Either all l_j, $j > i$, are on the distance 0 from l_i or there is l_j, $j > i$, on the distance greater than 0 from l_i. The first possibility is in the contradiction with the fact that x is not on the boundary of P_i. If the second possibility holds then the set l_k, $k > j$, is compact because l_j separates l_i and l_k, the distance between l_j and l_i is greater than 0, and the distance between l_i and l_k is less than 1. Consequently, l_k do not accumulate on $x \in \widehat{\mathbb{R}}$ which gives contradiction.

It is enough to show that the distance between the images of l_1 and l_i under E^λ goes to infinity as $i \to \infty$. The distance between l_1 and l_i is greater than the sum of distances between l_j and l_{j+1} for $j = 1, 2, \ldots, i - 1$. Let I_j be a closed geodesic arc connecting l_j and l_{j+1}, and let d_j be the distance between l_j and l_{j+1}. By Lemma 2.1, the above sum is bounded below by $\sum_{j=1}^{i-1} e^{-\lambda(I_j)} C(d_j) \ge \overline{C} e^{-2\|\lambda\|}(i-1) \to \infty$

as $i \to \infty$. The constant $\overline{C} > 0$ exists because $C(d)$ is a positive continuous function for $d > 0$ and $d_j \in [1, 2]$. This establishes the proposition. $\square$

We prove that bounded earthquake measures give quasisymmetric maps and conversely, quasisymmetric maps determine earthquakes with bounded measures. This allows us to consider earthquakes as elements of the universal Teichmüller space $\mathcal{T}(\mathbb{H}^2)$.

Proof of Theorem 1. Let $E^\lambda|_{S^1}$ be quasisymmetric and assume on the contrary that λ is not bounded. Thus, there exists a sequence of closed geodesic arcs I_n such that the length of I_n converges to 0 and $\lambda(I_n) \to \infty$ as $n \to \infty$. This follows by the use of Cantor diagonalization process.

Choose an arbitrary orientation of I_n. Let l_1^n and l_2^n be geodesics of $|\lambda|$ which intersect I_n closest to or at its left and right endpoints, respectively. Post compose E^λ with a Möbius map such that its restriction to l_1^n is the identity and call it E^λ again. We conjugate E^λ with a Möbius map γ_n such that $\gamma_n(l_1^n)$ is the geodesic with endpoints 0 and ∞, and $\gamma_n(l_2^n)$ is the geodesic with endpoints 1 and c_n, $c_n > 1$. We obtain a new earthquake $E^n = \gamma_n \circ E^\lambda \circ \gamma_n^{-1}$. Let f be a quasiconformal extension of $E^\lambda|_{S^1}$ to $\mathbb{H}^2$. Then $f_n = \gamma_n \circ f \circ \gamma_n^{-1}$ is a quasiconformal extension of $E^n|_{S^1} = \gamma_n \circ E^\lambda|_{S^1} \circ \gamma_n^{-1}$ to $\mathbb{H}^2$. The quasiconformal constant of f_n is equal to the quasiconformal constant of f. Consequently, $f_n|_{S^1}$ is quasisymmetric with the same constant as $f|_{S^1}$ and each E^n is the identity on the geodesic with endpoints 0 and ∞. Thus $f_n(\infty) = \infty$ and f_n restricts to a map of $\mathbb{R}$ onto itself.

We prove that the quasisymmetric constant of $f_n : \mathbb{R} \to \mathbb{R}$ tends to infinity as $n \to \infty$ which is a contradiction to the above, see [1]. By our normalization, $\gamma_n(l_2^n)$ has fixed points 1 and c_n such that $c_n \to \infty$ as $n \to \infty$. Clearly, $f_n(0) = 0$ and $-2 \leq f_n(-2) < 0$ because E^n is a left earthquake. Since $\lambda(I_n) \to \infty$, it is also true that $f_n(2) \to \infty$ as $n \to \infty$. Thus the ratio

$$\frac{f_n(2) - f_n(0)}{f_n(0) - f_n(-2)} \to \infty$$

as $n \to \infty$. This shows that the constant of quasisymmetry of f_n tends to infinity. The contradiction proves that λ is bounded.

To prove the opposite direction we assume that λ is bounded. In Proposition 1, we showed that E^λ is an earthquake. It remains to show that $E^\lambda|_{S^1}$ is quasisymmetric. A direct proof of this fact is certainly possible. However, our results in what follows establish the proof so we omit it at this point. $\square$

Let λ be a bounded earthquake measure. The corresponding earthquake E^λ when restricted to S^1 is a quasisymmetric map. An earthquake measure corresponds to the class of quasisymmetric maps where two maps are equivalent if they differ by a post composition with a Möbius map. Our interest is $\mathcal{T}(\mathbb{H}^2)$ and we do not distinguish between equivalent quasisymmetric maps. Consequently to each earthquake measure there corresponds a unique point of $\mathcal{T}(\mathbb{H}^2)$. By the abuse of notation, we say that two earthquakes are the same if their continuous extensions to S^1 differ by a post composition with a Möbius map. Multiplying λ with a positive parameter t, we obtain a path of bounded earthquake measures. The corresponding earthquakes $E^{t\lambda}$ form an earthquake path $t \mapsto [E^{t\lambda}|_{S^1}]$ in $\mathcal{T}(\mathbb{H}^2)$.

We note that for arbitrary hyperbolic surfaces X, there is a covering group equivariance of earthquake measures and corresponding homeomorphisms of S^1. Thus we obtain an earthquake path $t \mapsto [E^{t\lambda}|_{S^1}]$ in $\mathcal{T}(X)$.

3. Bending measure and cocycle

We extend the positive parameter t for earthquake path $t \mapsto [E^{t\lambda}|_{S^1}]$ into a complex parameter τ. For an earthquake measure λ_0 and a complex parameter $\tau \in \mathbb{C}$, $\tau\lambda_0$ is a complex measured lamination of special form (positive measure times complex number). We consider a general complex measured lamination.

Definition 3.1. A *bending measure* λ is a complex Borel transverse measure to a fixed geodesic lamination $|\lambda|$, called the support of λ.

Thurston introduced bending cocycle $E^\lambda : \mathbb{H}^2 \to \mathbb{H}^3$ for a bending measure λ. Following Epstein-Marden [5] we take the upper half-space model $\{(z,t) : z \in \mathbb{C}, \ t > 0\}$ of the hyperbolic 3-space $\mathbb{H}^3$. The hyperbolic plane $\mathbb{H}^2$ is embedded in $\mathbb{H}^3$ as the euclidean half-plane orthogonal to the x-axis in $\mathbb{C}$. The boundary of $\mathbb{H}^3$ is $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the boundary of $\mathbb{H}^2$ is $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Denote by T_{l*}^a a hyperbolic isometry of $\mathbb{H}^3$ given by the matrix

$$\left(\begin{array}{cc} e^{\frac{a}{2}} & 0 \\ 0 & e^{-\frac{a}{2}} \end{array} \right) \in SL(2,\mathbb{C}),$$

where l^* denotes the hyperbolic geodesics in $\mathbb{H}^3$ with endpoints 0 and ∞, and a is the translation length. For any oriented geodesic l, denote by $M \in SL(2,\mathbb{C})$ a hyperbolic isometry which maps the initial point of l to 0 and the terminal point of l to ∞. Define $T_l^a = M^{-1} \circ T_{l*}^a \circ M \in SL(2,\mathbb{C})$.

Assume that $|\lambda|$ is finite. Then λ is given by an assignment of complex numbers to each geodesic of $|\lambda|$. Let A be a fixed flat piece of $|\lambda|$. We define $E^\lambda|_A = id$. Let B be any other flat piece. We connect A to B by a geodesic arc c. Let $l_0, l_1, \ldots, l_n$ be geodesics of $|\lambda|$ which intersect c in the given order from A to B. We orient them to the left as seen from A. Let $a_0 = \lambda(l_0), a_1 = \lambda(l_1), \ldots, a_n = \lambda(l_n)$. Let $A_0 = A, A_1, \ldots, A_n, A_{n+1} = B$ be flat pieces along c in the given order. Define a bending cocycle $E^\lambda|_B = T_{l_0}^{a_0} \circ T_{l_1}^{a_1} \circ \cdots \circ T_{l_n}^{a_n}$ and $E^\lambda|_{l_n} = T_{l_0}^{a_0} \circ T_{l_1}^{a_1} \circ \cdots \circ T_{l_n}^{\frac{a_n}{2}}$. We consider E^λ and $M \circ E^\lambda$, where M is a Möbius map, as equivalent cocycles and do not distinguish between them.

Let

$$M = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2,\mathbb{C}).$$

Define the *norm* of M by $\|M\| = \max\{|a| + |b|, |c| + |d|\}$.

Assume that λ is an arbitrary bending measure. Then $|\lambda|$ could fill out $\mathbb{H}^2$. We fix a facet A of $|\lambda|$ such that A is either a flat piece or a geodesic of $|\lambda|$ with $\lambda(A) = 0$. Define $E^\lambda|_A = id$. Let B be an arbitrary facet of $|\lambda|$ and c be a geodesic arc connecting A to B.

We choose a sequence of finite approximations to λ on the part which intersects c. Fix n and choose points $C_0, C_1, \ldots C_{k(n)}$ on c with the following properties. Points C_0 and C_k are the endpoints of c, the distance between C_{i-1} and C_i is less than $\frac{1}{n}$, and $\lambda(C_i) = 0$ for $i = 1, 2, \ldots, k-1$. If $\lambda([C_{i-1}, C_i]) \neq 0$ for $i = 1, 2, \ldots, k-1$ then we choose one geodesic of $|\lambda|$ which intersects $[C_{i-1}, C_i]$ and assign to it weight $\lambda([C_{i-1}, C_i])$. If $\lambda([C_{k-1}, C_k]) \neq 0$ then we choose one geodesic which intersects

$[C_{k-1}, C_k]$ and assign to it weight $\lambda([C_{k-1}, C_k)) + \frac{1}{2}\lambda(C_k)$. We constructed a finite approximation λ_n of λ. Let E^{λ_n} be a bending cocycle for λ_n which is the identity on the facet containing C_0.

Epstein-Marden [5] showed that $E^{\lambda_n}|_B$ converges in the norm as n tends to infinity. The convergence is independent of the choice of a sequence. We define $E^\lambda|_B = \lim_{n\to\infty} E^{\lambda_n}|_B$.

4. HOLOMORPHIC MOTIONS

We describe needed results on holomorphic motions and apply them to our situation. To begin, we give the definition of a holomorphic motion of a closed subset of $\mathbb{C}$. Let Δ be the open unit disk and Λ be a fixed closed subset of $\mathbb{C}$.

Definition 4.1. *A map $F : \Delta \times \Lambda \to \mathbb{C}$, given by $F(\tau, z) = f_\tau(z)$ for $\tau \in \Delta$ and $z \in \Lambda$, is a holomorphic motion of Λ in $\mathbb{C}$ if the following are satisfied:*
1. $f_0(z) = z$ for all z in Λ,
2. for a fixed $\tau \in \Delta$, the map $z \mapsto f_\tau(z)$ is injective on Λ,
3. for a fixed $z \in \Lambda$, the map $\tau \mapsto f_\tau(z)$ is holomorphic in Δ.

Note that in the above definition we require that the parameter for a holomorphic motion is in the unit disk Δ. In Theorem 2, the parameter is not in the unit disk but in a neighborhood S_λ of the real line. By pre-composition of F with a Riemann mapping for S_λ which maps 0 to 0, we can arrange that the parameter is in Δ. Thus we do not have to require this in the definition.

When $\tau = 0$ in the situation in Theorem 2, the earthquake with zero measure $E^{0\lambda} = E^0$ is the identity on $\mathbb{R}$. Thus to show Theorem 2, we need to show that the bending cocycle extends to an injective map on $\mathbb{R}$, for a fixed parameter τ, and the extension is a holomorphic map in the parameter $\tau \in \Delta$, for a fixed $x \in \mathbb{R}$. We give the proof of these properties in the next two sections. In the rest of this section we prove Corollary 1.

Proof of Corollary 1. Slodkowski [17] showed that any holomorphic motion of a closed set Λ in $\mathbb{C}$ can be extended to a holomorphic motion of $\mathbb{C}$. Mañé, Sad and Sullivan [15] proved that if $f_\tau(z)$ is a holomorphic motion of $\mathbb{C}$ then f_{τ_0} is a quasiconformal map of $\mathbb{C}$, for any fixed $\tau_0 \in \Delta$. Further, if μ_τ is the Beltrami differential of f_τ, then the map $\tau \mapsto \mu_\tau$ from Δ into the open unit ball of $L^\infty(\Delta)$ is holomorphic. By taking Teichmüller class of μ_τ the above map gives a holomorphic map of Δ into the universal Teichmüller space $\mathcal{T}(\mathbb{H}^2)$.

Note that we assume $E^{\tau\lambda}(\infty) = \infty$ and consequently $E^{\tau\lambda}$ maps $\mathbb{R}$ into $\mathbb{C}$. By Theorem 2, $(x, \tau) \mapsto E^{\tau\lambda}(x)$ is a holomorphic motion of $\mathbb{R}$ with the parameter τ in a neighborhood S_λ of the real line. If we take an open simply connected neighborhood of $\mathbb{R}$ which is contained in S_λ, then by Slodkowski's theorem there exists an extension to the holomorphic motion of $\mathbb{C}$. Further, by theorem of Mañé, Sad and Sullivan, the map $\tau \mapsto [\mu_\tau]$ is holomorphic, where μ_τ is the Beltrami coefficient of the extension of the holomorphic motion restricted to the upper half-plane $\mathbb{H}^2$. Consequently, the restriction to the real parameter $t = Re(\tau)$ is analytic map into the Teichmüller space. Namely, real earthquake paths are real analytic paths in the Teichmüller space. □

Remark 4.1. We claim that Corollary 1 remains true for earthquake paths in Teichmüller spaces of arbitrary hyperbolic surfaces. Assume that λ is invariant for

the covering group Γ of a hyperbolic surface X. Then for any $\gamma \in \Gamma$ there exists a Möbius map γ_τ such that

$$E^{\tau\lambda}(\gamma(x)) = \gamma_\tau(E^{\tau\lambda}(x)) \tag{2}$$

for all $x \in \widehat{\mathbb{R}}$, see [5]. It is enough to show that there exists an extension of the holomorphic motion $(x, \tau) \mapsto E^{\tau\lambda}(x)$ of $\widehat{\mathbb{R}}$ to a holomorphic motion $(z, \tau) \mapsto E^{\tau\lambda}(z)$ of $\widehat{\mathbb{C}}$ which satisfies (2). This is proved by Earle-Kra-Krushkal [4] using Slodkowski's theorem.

5. Geometry of small bendings

In this section we investigate the properties of bending cocyles when the bending measure is purely imaginary with small norm. The main result in this section describes the image of a ray under the bending cocycle as seen from the image of the initial point of the ray.

It will be convenient to consider a unit tangent bundle $T^1\mathbb{H}^3$ of $\mathbb{H}^3$ given in coordinates (ζ, v), where $\zeta \in \mathbb{H}^3$ and v is a unit tangent vector at ζ. The distance between two points (ζ_1, v_1) and (ζ_2, v_2) in $T^1\mathbb{H}^3$ is given by

$$d_{T^1\mathbb{H}^3}((\zeta_1, v_1), (\zeta_2, v_2)) = d_{\mathbb{H}^3}(\zeta_1, \zeta_2) + \|v_1' - v_2\|_{T^1_{\zeta_2}\mathbb{H}^3}$$

where v_1' is the parallel transport of v_1 at point ζ_2 along the geodesic connecting ζ_1 and ζ_2.

Epstein-Marden [5] relate the distance in the norm between $A \in SL(2, \mathbb{C})$ and the identity $I \in SL(2, \mathbb{C})$ to the action of A on $T^1\mathbb{H}^3$. The lemma is originally stated for $\mathbb{H}^2$ but the same proof works for $\mathbb{H}^3$.

Lemma 5.1. *For a compact subset K of $\mathbb{H}^3$, there exists a constant C which satisfies the following. Let $\zeta_0 \in K$ and $A \in SL(2, \mathbb{C})$ such that $A(\zeta_0) \in K$. Then*

$$\frac{\|A - I\|}{C} \leq d_{T^1\mathbb{H}^3}((\zeta_0, v_0), A(\zeta_0, v_0)) \leq C\|A - I\|$$

for any unit tangent vector v_0 at ζ_0.□

We consider a neighborhood of a geodesic in $\mathbb{H}^3$ given by all points at the distance less than a fixed constant. For each point in the neighborhood we take the circle of unit tangent vectors and call the obtained set a *crescent*. More precisely,

Definition 5.1. *Let l be a geodesic in $\mathbb{H}^3$ and $d > 0$ be given. A crescent of width d around geodesic l is*

$$\begin{aligned} D(l; d) \quad &= \quad \{(\zeta, v) \in T^1\mathbb{H}^3 | \text{distance from } \zeta \text{ to } l \text{ is} \\ &\qquad \text{less than or equal to } d \text{ in the hyperbolic metric of } \mathbb{H}^3\}. \end{aligned}$$

We estimate the action on a crescent around geodesic l of an element of $SL(2, \mathbb{C})$ with axis l.

Lemma 5.2. *Given $d > 0$ and $\delta > 0$ there exists a constant M which satisfies the following. Let $T = T_l^a \in SL(2, \mathbb{C})$ be a hyperbolic isometry with axis l and with the translation length a, $|a| \leq \delta$. Then*

$$d_{T^1\mathbb{H}^3}(p, T(p)) \leq M|a|$$

for all $p \in D(l; d)$.

Proof. Since the metric $d_{T^1\mathbb{H}^3}$ is invariant under the action of elements of $SL(2,\mathbb{C})$, we can assume that l is a geodesic with endpoints 0 and ∞, and that $p = (\zeta, t)$ with $\zeta = (z, 1) \in \mathbb{H}^3$. Let $B(l; d) = \{\zeta \in \mathbb{H}^3; \text{distance from } \zeta \text{ to } l \text{ is less than or equal to } d \text{ in the hyperbolic metric of } \mathbb{H}^3\} \subset \mathbb{H}^3$. Note that $D(l; d) \subset T^1\mathbb{H}^3$ is obtained by taking the unit circle of tangent vectors at each point of $B(l; d)$. Define $K = B(l; d) \cap \{(z, t); e^{-|a|} \leq t \leq e^{|a|}\}$. Then $\zeta = (z, 1)$ and $T(\zeta)$ are in K. Because of the normalization, it is clear that $\|T - I\|$ is less than or equal to a constant times $|a|$. The lemma follows by an application of Lemma 5.1. $\square$

Let r be a geodesic ray which starts at $\zeta \in \mathbb{H}^3$ and let α be a real number such that $0 < \alpha < \pi$. A *hyperbolic cone* $C(\zeta, r, \alpha)$ is the set of all $\eta \in \mathbb{H}^3$ such that the geodesic ray starting at ζ through η makes an angle less than α with r. The angle is measured from 0 to π. The *boundary* of $C(\zeta, r, \alpha)$ consists of all $\eta \in \mathbb{H}^3$ such that the angle between r and the geodesic ray starting at ζ through η equals α. The *shadow* of $C(\zeta, r, \alpha)$ consists of all $z \in \widehat{\mathbb{C}}$ such that the geodesic ray connecting ζ to z lies in $C(\zeta, r, \alpha)$. The shadow of a hyperbolic cone is a circle on $\widehat{\mathbb{C}}$.

A point $(\zeta, v) \in T^1\mathbb{H}^3$ uniquely determines a geodesic ray r_v starting at ζ and tangent to v. Conversely, a geodesic ray r starting at ζ gives a point $(\zeta, v) \in T^1\mathbb{H}^3$ where v is the tangent vector to r at ζ. Then we define $C(\zeta, v, \alpha) = C(\zeta, r_v, \alpha)$.

We generalize a cone argument of Keen-Series [11] to our situation. They used the geometry of cones for finite surfaces and we use similar argument for general hyperbolic surfaces. Our goal is to prove that the image of a ray under the bending map is contained in a hyperbolic cone for small enough pure imaginary bending measure. We divide the argument into several lemmas.

The following lemma relies on elementary hyperbolic geometry.

Lemma 5.3. *Given α, $0 < \alpha < \pi$, and a geodesic ray r starting at $\zeta \in \mathbb{H}^3$ there exists a number β, $\beta > \alpha$, such that cone $C(\zeta_1, r_1, \beta)$ is contained in $C(\zeta, r, \alpha)$, where $\zeta_1 \in r$ is a point at the distance 1 from ζ and r_1 is a sub-ray of r starting at ζ_1. Moreover, the difference $\beta - \alpha > 0$ depends only on α.*
Proof. Let P be an arbitrary geodesic plane in $\mathbb{H}^3$ which contains r. Denote by l one of the two geodesic rays of the boundary of $C(\zeta, r, \alpha)$ which are contained in P. Let $z \in \widehat{\mathbb{C}}$ be the endpoint of l. Form a hyperbolic triangle T with vertices ζ, ζ_1 and z. The triangle T has a zero angle at z. We choose β to be equal to the exterior angle of T at ζ_1. Consequently $C(\zeta_1, r_1, \beta) \subset C(\zeta, r, \alpha)$.

By the Gauss-Bonnet theorem the sum of the the angles in T is less than π, namely $\alpha + (\pi - \beta) + 0 < \pi$. This gives $\beta > \alpha$. The length of the finite side $[\zeta, \zeta_1]$ is 1. Thus, by a formula from hyperbolic trigonometry, the difference $\beta - \alpha > 0$ depends only on α. $\square$

Let v be the tangent vector at ζ to a geodesic ray r with the initial point ζ. Take $\zeta_1 \in r$ at the distance 1 from ζ. Denote by r_1 a sub-ray of r starting at ζ_1 and by v_1 a tangent vector to r_1 at ζ_1. Given a bending measure λ, let A and A_1 be facets of $|\lambda|$ which contain ζ and ζ_1, respectively. Maps $E^\lambda|_A$ and $E^\lambda|_{A_1}$ are the restrictions of the bending cocycle to A and A_1. Thus they are Möbius maps and we can consider their actions on $T^1\mathbb{H}^3$, not only on A and A_1.

Lemma 5.4. *Given $\delta > 0$ there exists a constant M such that for all bending measures λ with $\|\lambda\| < \delta$,*

$$d_{T^1\mathbb{H}^3}(E^\lambda|_A(\zeta_1, v_1), E^\lambda|_{A_1}(\zeta_1, v_1)) \leq M|\lambda([\zeta, \zeta_1])|$$

where (ζ, v), (ζ_1, v_1), A and A_1 are as above.

Proof. Without loss of generality, we assume that $E^\lambda|_A = id$. Then $E^\lambda|_{A_1}$ is the limit of cocyles with finite bending measure approximations λ_n of λ along $[\zeta, \zeta_1]$ which are the identity on the facet containing A. Let $\{l_1, l_2, \ldots, l_k\}$ be the support of λ_n and $a_i = \lambda_n(l_i)$. Then $E^{\lambda_n}|_{A_1} = T_{l_1}^{a_1} \circ T_{l_2}^{a_2} \circ \cdots \circ T_{l_k}^{a_k}$ and $E^\lambda|_{A_1} = \lim_{n \to \infty} E^{\lambda_n}|_{A_1}$.

By the invariance of the metric on $T^1\mathbb{H}^3$ under Möbius maps and using Lemma 5.2 with $d = 1$, we get

$$d_{T^1\mathbb{H}^3}(T_{l_1}^{a_1} \circ T_{l_2}^{a_2} \circ \cdots \circ T_{l_{i-1}}^{a_{i-1}}(\zeta_1, v_1), T_{l_1}^{a_1} \circ T_{l_2}^{a_2} \circ \cdots \circ T_{l_i}^{a_i}(\zeta_1, v_1)) \leq M|a_i|$$

for a fixed constant M and for each $i = 2, 3, \ldots, k$.

The triangle inequality combined with the above inequality gives

$$d_{T^1\mathbb{H}^3}((\zeta_1, v_1), E^{\lambda_n}(\zeta_1, v_1)) \leq M \sum_{i=1}^{k} |a_i|.$$

Since the inequality holds for each finite approximation and $\sum_{i=1}^{k} |a_i| = \lambda([\zeta, \zeta_1])$ by the definition of λ_n, the lemma follows. $\square$

We prove the main result of this section. A purely imaginary bending measure λ has *constant sign* if either $Im(\lambda(I)) \geq 0$ or $Im(\lambda(I)) \leq 0$ for each closed finite length geodesic arc I.

Proposition 5.1. *Given α, $0 < \alpha < \pi$, there exists $\epsilon > 0$ which satisfies the following. For any purely imaginary bending measure λ of constant sign with $\|\lambda\| < \epsilon$, the image under the bending cocycle E^λ of a geodesic ray r_v, starting at $\zeta \in \mathbb{H}^3$ and tangent to $v \in T_\zeta^1\mathbb{H}^3$, is contained in cone $C(E^\lambda(\zeta, v), \alpha)$.*

Proof. We assume without loss of generality that the bending cocycle restricted to the facet containing ζ equals the identity.

Let ζ_i, $i = 0, 1, 2, \ldots$ be points on the ray r in the given order such that $\zeta_0 = \zeta$ and the distance between ζ_i and ζ_{i+1} equals 1. Denote by v_i the tangent vector to r at ζ_i and by A_i the facet of $|\lambda|$ which contains ζ_i.

Then $E^\lambda|_{A_{i+1}} = E^\lambda|_{A_i} \circ E_i$, where E_i is the restriction to A_{i+1} of a bending cocycle for λ normalized to be the identity on A_i. By Lemma 5.4, the distance between $E^\lambda|_{A_{i+1}}(\zeta_{i+1}, v_{i+1})$ and $E^\lambda|_{A_i}(\zeta_{i+1}, v_{i+1})$ is less than $M\|\lambda\|$, for all bending measures λ such that $\|\lambda\| < \delta$.

We show that the cone $C(E_i(\zeta_{i+1}, v_{i+1}), \alpha)$ is inside the cone $C(\zeta_i, v_i, \alpha)$. This implies $C(E^\lambda(\zeta_{i+1}, v_{i+1}), \alpha) \subset C(E^\lambda(\zeta_i, v_i), \alpha)$, because above cones are mapped onto $C(E^\lambda(\zeta_{i+1}, v_{i+1}), \alpha)$ and $C(E^\lambda(\zeta_i, v_i), \alpha)$ by $E^\lambda|_{A_i}$.

By the invariance under Möbius maps, we can assume that $\zeta_i = (0, 0, 1) \in \mathbb{H}^3$, $\zeta_{i+1} = (0, 0, e^{-1}) \in \mathbb{H}^3$ and v_i, v_{i+1} are unit tangent vectors to the oriented geodesic connecting ∞ to 0 at points ζ_i, ζ_{i+1}. The shadow of cone $C(\zeta_{i+1}, v_{i+1}, \beta)$ is a disk with center at the origin of $\mathbb{C}$. The shadow of $C(\zeta_{i+1}, v_{i+1}, \alpha)$ is contained in the interior of the shadow of $C(\zeta_{i+1}, v_{i+1}, \beta)$. In particular, the boundary circle C_α of the shadow of $C(\zeta_{i+1}, v_{i+1}, \alpha)$ is at the positive distance from the boundary circle C_β of the shadow of $C(\zeta_{i+1}, v_{i+1}, \beta)$. Since $d_{T^1\mathbb{H}^3}(E_i(\zeta_{i+1}, v_{i+1}), (\zeta_{i+1}, v_{i+1})) \leq M\|\lambda\|$, by choosing $\epsilon > 0$ small enough we arrange for E_i to be as close as we want to the identity for $\|\lambda\| < \epsilon$. Because C_α and C_β are compact sets, we choose ϵ such that $E_i(C_\alpha) \cap C_\beta = \emptyset$. Consequently the shadow of $C(E_i(\zeta_{i+1}, v_{i+1}), \alpha)$ is contained in the shadow of $C(\zeta_{i+1}, v_{i+1}, \beta)$. The above inclusion, $E^\lambda(\zeta_{i+1}) \in C(\zeta_i, v_i, \alpha)$ and

$C(\zeta_{i+1}, v_{i+1}, \beta) \subset C(\zeta_i, v_i, \alpha)$ implies

$$C(E_i(\zeta_{i+1}, v_{i+1}), \alpha) \subset C(\zeta_i, v_i, \alpha)$$

for ϵ small enough.

We prove that $E^\lambda([\zeta_i, \zeta_{i+1}])$ is contained in cone $C(E^\lambda(\zeta_i, v_i), \alpha)$. A cone is a convex set and the endpoints of $E^\lambda([\zeta_i, \zeta_{i+1}])$ are in the cone $C(E^\lambda(\zeta_i, v_i), \alpha)$. Since the bending is in only one direction (λ has constant sign), we get $E^\lambda([\zeta_i, \zeta_{i+1}]) \subset C(E^\lambda(\zeta_i, v_i), \alpha)$.

Each cone $C(E^\lambda(\zeta_{i+1}, v_{i+1}), \alpha)$ is contained in the previous cone $C(E^\lambda(\zeta_i, v_i), \alpha)$ and all of them are contained in the first cone. Consequently, bent geodesic ray $E^\lambda(r)$ is contained in the first cone $C(E^\lambda(\zeta, v), \alpha)$. □

Remark 5.1. The image of a ray r under E^λ is a bent geodesic. If r_i is a sub-ray of r with initial point ζ_i and with the unit tangent vector v_i, then $E^\lambda(r_i)$ is contained in cone $C(E^\lambda(\zeta_i, v_i), \alpha)$. The sequence of cones $C(E^\lambda(\zeta_i, v_i), \alpha)$, $i = 0, 1, 2, \ldots$ is nested. If we allow for λ to change sign, we can still get Proposition 5.1 by further decreasing $\epsilon > 0$.

A bending cocycle is defined on the boundary of facets on $S^1 = \widehat{\mathbb{R}}$ by a simple extension. Assume that the endpoint of r is not on a facet of $|\lambda|$. In order to show that the bending cocycle continuously extends to the endpoint of r, we could show that vertices of nested cones leave any compact subset of $\mathbb{H}^3$. Then the shadows of the cones would converge to a single point in $\widehat{\mathbb{C}}$. Such property is not hard to prove [11].

Instead of proving it, in the next section we use properties of holomorphic maps to show that bending cocycle extends to all points of $S^1 = \widehat{\mathbb{R}}$.

6. Conclusions

We use Proposition 5.1 to prove Theorem 2. In order to do that we divide the bending cocycle E^λ into earthquake cocycle and pure bending cocycle.

A hyperbolic isometry T_l^a with the axis l and with the translation length $a = Re(a) + iIm(a)$ can be written as a composition $T_l^a = T_l^{iIm(a)} \circ T_l^{Re(a)}$ of a hyperbolic translation and rotation. Assume that λ is a finite bending measure. Let A be a facet of $|\lambda|$ such that $E^\lambda|_A = id$. Let B be any other facet of $|\lambda|$ and $\{l_1, l_2, \ldots, l_k\}$ be geodesics of $|\lambda|$ which separate A from B in the given order. Define $a_i = \lambda(l_i)$. Then

$$E^\lambda|_B = T_{l_1}^{a_1} \circ T_{l_2}^{a_2} \circ \cdots \circ T_{l_k}^{a_k}.$$

If $a, b \in \mathbb{R}$ and l_1, l_2 are two geodesics, then it is elementary to show that

$$T_{l_1}^a \circ T_{l_2}^{ib} = T_{l_2^*}^{ib} \circ T_{l_1}^a$$

where $l_2^* = T_{l_1}^a(l_2)$. We use the above to write

$$E^\lambda|_B = (T_{l_1^*}^{iIm(a_1)} \circ T_{l_2^*}^{iIm(a_2)} \circ \cdots \circ T_{l_k^*}^{iIm(a_k)}) \circ (T_{l_1}^{Re(a_1)} \circ T_{l_2}^{Re(a_2)} \circ \cdots \circ T_{l_k}^{Re(a_k)})$$

where $l_i^* = T_{l_1}^{Re(a_1)} \circ T_{l_2}^{Re(a_2)} \circ \cdots \circ T_{l_{i-1}}^{Re(a_{i-1})}(l_i)$.

If λ is a non-finite bending measure, then $E^\lambda|_B$ is the limit of finite bending cocycle approximation. By taking limit in the above equality, we get

$$E^\lambda|_B = E^{iIm(\lambda^*)}|_{B'} \circ E^{Re(\lambda)}|_B$$

where $\lambda^* = (E^{Re(\lambda)})_*(\lambda)$ is the push-forward of λ by $E^{Re(\lambda)}$ and $B' = E^{Re(\lambda)}(B)$.

Since λ^* has different support from λ, its norm is changed. We give an upper bound estimate on the norm of λ^*.

Lemma 6.1. *Let λ be a bounded bending measure and let $\lambda^* = (E^{Re(\lambda)})_*(\lambda)$ be the push-forward of λ. There exists $C > 0$ such that*

$$\|\lambda^*\| \leq Ce^{\|\lambda\|}\|\lambda\|.$$

Proof. Let $\{l_i\}_{i \in Ind}$ be geodesics of $|\lambda|$ which have one endpoint in common. Then $|\lambda(\cup_{i \in Ind} l_i)| < \infty$ because λ is a bounded measure. The push-forward geodesics $\{l_i^* = E^{Re(\lambda)}(l_i)\}_{i \in Ind}$ of $|\lambda^*|$ also have one endpoint in common and

(3) $$\lambda^*(\cup_{i \in Ind} l_i^*) = \lambda(\cup_{i \in Ind} l_i).$$

We estimate the norm of λ^*. Let I^* be a closed geodesic arc of length 1. Denote by A^* and B^* facets of $|\lambda^*|$ which contain endpoints of I^*. Let A and B be corresponding facets of $|\lambda|$ and take a closed geodesic arc I which connects A to B. There are two possibilities for I. Either all geodesics of $|\lambda|$ which intersect I have one common endpoint or at least two of them are at a positive distance apart.

In the first case, by (3) we get $|\lambda^*(I^*)| \leq \|\lambda\|$.

Assume that we are in the second case. Give an arbitrary orientation to I. Let l and r be geodesics of $|\lambda|$ which intersect I closest to its left and right endpoints, respectively. Choose geodesic $\{l_1 = l, l_2, \ldots, l_k = r\}$ of $|\lambda|$ intersecting I in the given order and which satisfy the following. Each l_i and l_{i+1} are at the positive distance apart. Denote by I_i the closed geodesic arc orthogonal to l_i and l_{i+1}. We require that the length of each I_i is at least 1 and $|\lambda(I_i)| \leq 2\|\lambda\|$. Let $l_i^* = E^{Re(\lambda)}(l_i)$ and I_i^* be the closed geodesic arc orthogonal to l_i^* and l_{i+1}^*.

Denote by $dist(l^*, r^*)$ the distance between geodesics $l^* = l_1^*$ and $r^* = l_k^*$, and by $length(I_i^*)$ the length of a closed geodesic arc I_i^*. Then, by Lemma 2.1, we get

$$dist(l^*, r^*) \geq \sum_{i=1}^{k-1} length(I_i^*) \geq (k-1)C(1)e^{-\|\lambda\|}$$

where $C(1) \geq \frac{1}{2}$ and $dist(l^*, r^*) \leq 1$. Consequently, we get $k - 1 \leq 2e^{\|\lambda\|}$ because $dist(l^*, r^*) \leq length(I^*) = 1$.

By the definition of push-forward

$$\lambda^*(I^*) \leq \sum_{i=1}^{k-1} \lambda(I_i) \leq 2(k-1)\|\lambda\|$$

which together with above gives

$$\|\lambda^*\| \leq 4e^{\|\lambda\|}\|\lambda\|.$$

□

We are ready to prove Theorem 2.

Proof of Theorem 2. From now on λ is a bounded earthquake measure. In particular, it is positive. We define a neighborhood of the real line where the complex parameter $\tau = t + is$ will take its values.

By the previous remarks, we can write $E^{\tau\lambda} = E^{is\lambda^*} \circ E^{t\lambda}$ where $\lambda^* = (E^{t\lambda})_*(\lambda)$. Bending cocycle $E^{\tau\lambda}$ is normalized to be the identity on a facet A whose boundary

contain ∞. In particular, $E^{\tau\lambda}(\infty) = \infty$. We fix α, $0 < \alpha < \frac{\pi}{2}$, and take $\epsilon > 0$ from Proposition 5.1. By Lemma 6.1, we have $\|is\lambda^*\| \le |s|Ce^{\|t\lambda\|}\|\lambda\|$. Define

$$S_\lambda = \{\tau = t + is; |s| < \frac{\epsilon}{Ce^{\|t\lambda\|}\|\lambda\|}\}.$$

We fix a point $x \in \mathbb{R}$ and define $E^{\tau\lambda}(x)$. If x is on the boundary of a facet then $E^{\tau\lambda}(x)$ is given by the cocycle and by Epstein-Marden [5] it is holomorphic in τ.

Assume that x is not on a facet of λ. Then there exists a sequence of nested geodesics l_i such that they converge to x. We connect a fixed facet A of λ, where $E^{\tau\lambda}|_A = id$, to $E^{t\lambda}(x) = x_t$ by a family of geodesic rays r_t with initial point $\zeta \in A$. Define an open set V in $\mathbb{C}$ to consists of the union of all shadows of cones $C(\zeta, r_t, \alpha)$ for $t \in \mathbb{R}$. Since $0 < \alpha < \frac{\pi}{2}$ the complement of V in $\mathbb{C}$ contains at least three points. Let λ_n be the restriction of λ to geodesic of $|\lambda|$ which intersect r_0 between ζ and l_n. By Proposition 5.1, $E^{is\lambda_n^*}(r_t^n)$ is contained in the cone $C(\zeta, r_t^n, \alpha)$ where r_t^n is the ray with initial point ζ and endpoint $x_t^n = E^{\tau\lambda_n}(x) \in \mathbb{R}$. Thus $E^{is\lambda_n^*}(x_t^n)$ is in V. For each n, point x_t^n is on a facet of λ_n^* and consequently $E^{\tau\lambda_n}(x)$ is holomorphic in τ by [5]. In addition, by the above $E^{\tau\lambda_n}(x) \in V$. The family of holomorphic maps $\tau \mapsto E^{\tau\lambda_n}(x)$ is normal by Montel's theorem and it has a convergent subsequence. For τ real, $E^{\tau\lambda}$ is an earthquake. Thus, for any two convergent subsequences, their limits agree for τ real. By the uniqueness of holomorphic maps, the limit is unique for the whole sequence and we denote it by $E^{\tau\lambda}(x)$.

It remains to show that $E^{\tau\lambda} : \mathbb{R} \to \mathbb{R}$ is injective for any $\tau \in S_\lambda$. This is again a consequence of Proposition 5.1. Let $x \neq y$ be two points of $\mathbb{R}$ and define $x_1 = E^{t\lambda}(x)$ and $y_1 = E^{t\lambda}(y)$. Then $x_1 \neq y_1$ because earthquakes are injective. Take a geodesic l connecting x_1 and y_1, and fix $\zeta \in l$. Let v_x and v_y be the unit tangent vectors to l at ζ in the directions of x_1 and y_1, respectively. By Proposition 5.1 and above, points $E^{\tau\lambda}(x)$ and $E^{\tau\lambda}(y)$ are in the shadows of cones $C(E^{\tau\lambda}(\zeta), v_x, \alpha)$ and $C(E^{\tau\lambda}(\zeta), v_y, \alpha)$, respectively. But these shadows are disjoint since we chose α such that $0 < \alpha < \frac{\pi}{2}$. Thus $E^{\tau\lambda}$ is injective which together with previous results finishes the proof. $\square$

Remark 6.1. By Epstein-Marden [5], the bending cocycle is holomorphic in the parameter. This allows us to claim that the bending map is holomorphic on the boundary of the facets on S^1. We showed that the bending map extends to all of S^1 and that it is holomorphic for the parameter in a neighborhood of the real line. In addition, we showed that the bending map is injective on S^1, for a fixed parameter in the above set. These two new results allowed us to use theory of holomorphic motions.

Remark 6.2. We showed that the bending cocycle extends to $\mathbb{R}$ and is holomorphic in the parameter, when the parameter is restricted to a neighborhood of the real line. Also, our proof shows that the extension of a bending cocycle on the whole of $\widehat{\mathbb{R}}$ is obtained as the limit of finite approximations.

Gardiner [6] gave expressions for the tangent vectors of earthquake and bending maps. The formulas are obtained by taking the limit of tangent vectors of finite approximations. Our results rigorously show that these formulas are correct.

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