ON QUASICONFORMAL DEFORMATIONS OF THE UNIVERSAL HYPERBOLIC SOLENOID

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Abstract. We investigate the Teichmüller metric and the complex structure on the Teichmüller space $\mathcal{T}(H_\infty)$ of the universal hyperbolic solenoid H_∞. In particular, a version of the Reich-Strebel inequality for H_∞ is obtained. As a consequence, we show that the Teichmüller type Beltrami coefficients determine unique geodesics in $\mathcal{T}(H_\infty)$, and we compute the infinitesimal form of the Teichmüller metric. In addition, we show that a Beltrami coefficient is Teichmüller extremal if and only if it is infinitesimally extremal. Finally, we show that the Kobayashi metric on $\mathcal{T}(H_\infty)$ equals the Teichmüller metric.

Introduction

Sullivan [31], motivated by the dynamics, introduced the universal hyperbolic solenoid H_∞ as a genus independent generalization of a closed surface (see Section 1.1). The solenoid H_∞ is the inverse limit space of the system of all finite, unbranched coverings of a closed surface of genus at least two. Since the coverings are unbranched, every point in the universal hyperbolic solenoid H_∞ has an open neighborhood homeomorphic to (2-disk)×(Cantor set). (For the local structure of the inverse limit space in the case of the branching see [19].) The global topology of the solenoid H_∞ is more complicated. Each path component of H_∞ is non-compact, simply connected and dense in the solenoid H_∞. Path components of H_∞ are homeomorphic to the unit disk with the non-standard topology (see Section 1.4).

We study the space of deformations of complex structures on H_∞, namely the Teichmüller space $\mathcal{T}(H_\infty)$ of the universal hyperbolic solenoid H_∞. Our focus is on the Teichmüller metric and on the natural complex structure of $\mathcal{T}(H_\infty)$. The Teichmüller space $\mathcal{T}(H_\infty)$ is particularly interesting because it is a "closure" of Teichmüller spaces of all closed surfaces of genus at least two [23]. Also, the statement that the action of the modular group of the solenoid H_∞ on $\mathcal{T}(H_\infty)$ has dense orbits is equivalent to the Ehrenpreis conjecture (see Section 10 and [23]). Recently, V. Markovic and the author [22] made a substantial progress in this direction (for the case of the punctured solenoid) by showing that the orbit of the basepoint accumulates at the basepoint and that the closure of the orbit is larger than the orbit. These results are corollaries of the following main result:

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Theorem. [22] *For every $\epsilon > 0$ there exist two finite index subgroups of $PSL_2(\mathbb{Z})$ which are conjugated by a $(1+\epsilon)$-quasisymmetric homeomorphism of the unit circle and this conjugation homeomorphism is not conformal.*

Algebraic properties of the modular group are of interest as well [5], [24], [21], [25], [6].

The solenoid H_∞ fibers over any closed surface S of genus at least two. If S is given a complex structure, then the fiber map $\pi_\infty : H_\infty \to S$ pulls back a complex structure on H_∞ which is constant in the Cantor set direction in the local charts; such complex structure is called a *transversely locally constant* (TLC) complex structure on H_∞. However, a large family of complex structures on H_∞ are not TLC. The holomorphic universal covering space and the covering group for a complex TLC solenoid were already defined in [24] and [4]. We introduce the holomorphic universal covering and the covering group for a non-TLC complex solenoid X (see Section 2). Using the universal covering group, we introduce the commensurable (Poincaré) theta series for the complex solenoid X(see Section 3) which is an analog of the Poincaré theta series for Riemann surfaces. The commensurable Poincaré theta series defines a surjective linear operator of the unit norm from the space of holomorphic functions on the holomorphic universal covering onto the space of holomorphic quadratic differentials on X (see Theorem 3.1).

The Reich-Strebel inequality is the essential tool in the quasiconformal Teichmüller theory of Riemann surfaces. We prove a version of the Reich-Strebel inequality for the universal hyperbolic solenoid (see Theorem 4.1). This inequality facilitates investigation of the Teichmüller metric on $\mathcal{T}(H_\infty)$. As a first consequence of the Reich-Strebel inequality, we prove a version of the Teichmüller theorem for the solenoid:

Theorem 5.1. *Let $f : H_\infty \to X$ be a quasiconformal map and let $\varphi \neq 0$ be a holomorphic quadratic differential on X. Then the path $t\frac{|\varphi|}{\varphi}$, $-1 < t < 1$, of Teichmüller type Beltrami coefficients on X gives a geodesic (in the Teichmüller metric) through the point $[f] \in \mathcal{T}(H_\infty)$. In addition, any two points on this geodesic have no other geodesics connecting them.*

The idea for proving Theorem 5.1 comes from the proof of the Teichmüller theorem for Riemann surfaces. However, there are additional difficulties arising from the need of continuity for the variation in the Cantor set (the transversal) direction.

A point in the Teichmüller space $\mathcal{T}(H_\infty)$ is a homotopy class of quasiconformal maps from H_∞ onto another complex solenoid X. A quasiconformal map $f : H_\infty \to X$ is called *extremal* if it has the least dilatation in its homotopy class. A *Teichmüller map* is a quasiconformal map whose Beltrami coefficient is of the type $k\frac{|\varphi|}{\varphi}$, for $0 < k < 1$ and $\varphi \neq 0$ a holomorphic quadratic differential on the solenoid. A major part in the proof of Theorem 5.1 is to show that Teichmüller maps are uniquely extremal maps in their homotopy classes. A natural question is whether each homotopy class of quasiconformal maps between solenoids contains a Teichmüller map. (This question is posed in a previous version of this paper.) In the case of a closed or a finite Riemann surface, the answer is positive. The extremality and the existence of the Teichmüller maps for closed surfaces is known as the Teichmüller theorem. In the case of a geometrically infinite Riemann surface R not every homotopy class contains a Teichmüller map, but an open dense subset of the Teichmüller space $\mathcal{T}(R)$ does contain Teichmüller maps (for example, see [17]). In a work after the appearance of this paper, the following surprising result is obtained:

Theorem. [12] *A generic point in the Teichmüller space $\mathcal{T}(H_\infty)$ of the universal hyperbolic solenoid H_∞ does not contain a Teichmüller map.*

The above theorem does not exclude the existence of extremal maps (not of Teichmüller type) in an arbitrary homotopy class. If such extremal maps always exist, then any two points in $\mathcal{T}(H_\infty)$ can be connected by a geodesic. Even if an extremal map for a certain homotopy class does not exist, it is not immediately implied that any two points cannot be joined by a geodesic. At this point we do not know whether each homotopy class of quasiconformal maps has an extremal map and whether any two points in $\mathcal{T}(H_\infty)$ are connected by a geodesic. A corollary to Theorem 5.1 is that the set of pairs of points connected by a unique geodesic is dense in $\mathcal{T}(H_\infty)\times\mathcal{T}(H_\infty)$ (see Corollary 5.1). The lifting of complex structures from closed Riemann surfaces to the solenoid H_∞ defines an embedding of the Teichmüller space of a closed surface into $\mathcal{T}(H_\infty)$. An important consequence of Theorem 5.1 is:

Corollary 5.2. *The Teichmüller spaces of closed surfaces of genus at least two isometrically embed into $\mathcal{T}(H_\infty)$, for their corresponding Teichmüller metrics.*

A Beltrami coefficient μ on the complex solenoid X is called *Teichmüller trivial* if the corresponding quasiconformal map $f : X \to X$ is homotopic to the identity. We say that a Beltrami differential μ is *infinitesimally trivial* if $\int_X \mu\varphi dm = 0$ for all holomorphic quadratic differentials φ on X, where m is the properly normalized transverse measure on X.

We show that the tangent space to $\mathcal{T}(H_\infty)$ at the marked solenoid X is isomorphic to the quotient of the space of smooth Beltrami differentials on X with the subspace of infinitesimally trivial smooth Beltrami differentials (see Theorem 7.1). Note that the space of smooth Beltrami differentials and the space of infinitesimally trivial smooth Beltrami differentials are not complete. However, their quotient is isomorphic to the space of holomorphic quadratic differentials on X with the Bers norm (see Corollary 7.1), which is a Banach space.

We described above the points in $\mathcal{T}(H_\infty)$ as homotopy classes of quasiconformal maps from a fixed complex solenoid H_∞ onto variable solenoids X. This implies that $\mathcal{T}(H_\infty)$ is isomorphic to the set of equivalence classes of Beltrami coefficients on H_∞, where two Beltrami coefficients are equivalent if the corresponding quasiconformal maps are homotopic. (The later description of $\mathcal{T}(H_\infty)$ is more convenient when considering the infinitesimal structure of $\mathcal{T}(H_\infty)$.) The Teichmüller distance from $[0]\in\mathcal{T}(H_\infty)$ to $[\mu]\in\mathcal{T}(H_\infty)$ is $1/2\log\frac{1+k_0(\mu)}{1-k_0(\mu)}$, where $k_0(\mu)=\inf_{\mu_1}\|\mu_1\|_\infty$ with the infimum over all Beltrami coefficients μ_1 Teichmüller equivalent to μ. We say that a Beltrami coefficient μ is *Teichmüller extremal* if $\|\mu\|_\infty = k_0(\mu)$. (Note that extremal maps have Teichmüller extremal Beltrami coefficients and Teichmüller extremal Beltrami coefficients correspond to extremal maps.) Therefore, to find the

Teichmüller distance it is enough to find the Teichmüller extremal Beltrami coefficient. A Beltrami differential μ is *infinitesimally extremal* if $k_1(\mu) = \|\mu\|_\infty$, where $k_1(\mu) = \inf_{\mu_1} \|\mu_1\|_\infty$ with the infimum over all μ_1 such that $\mu - \mu_1$ is infinitesimally trivial.

We show that a Beltrami coefficient μ on the solenoid X is Teichmüller extremal if and only if it is infinitesimally extremal (see Theorem 8.1). In addition, the infinitesimal form of the Teichmüller metric on the tangent space at $X \in \mathcal{T}(H_\infty)$ is given by $\sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$ (see Theorem 8.2), where a tangent vector at $X \in \mathcal{T}(H_\infty)$ is represented by the Beltrami differential μ and the supremum is over all holomorphic quadratic differentials φ on X with norm 1.

We also obtain the Teichmüller contraction principle (see [15]):

Corollary 8.1. *Let $0 < k < 1$ be fixed. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1(\|\mu\|_\infty - k_0(\mu)) \leq \|\mu\|_\infty - k_1(\mu) \leq C_2(\|\mu\|_\infty - k_0(\mu))$$

for any smooth Beltrami coefficient μ on the solenoid X with $\|\mu\|_\infty \leq k$.

In other words, the distance of an arbitrary Beltrami coefficient μ from the extremal value $k_0(\mu)$ of its Teichmüller class is comparable to its distance from the extremal value $k_1(\mu)$ of its infinitesimal class. We also show that $k_1(\mu) = \sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$ which is an exercise for Riemann surfaces and a non-trivial fact for the universal hyperbolic solenoid.

An analog of the classical Bers embedding gives a complex Banach manifold structure on $\mathcal{T}(H_\infty)$ (see [31]). The Kobayashi metric on $\mathcal{T}(H_\infty)$ is defined using the complex structure on $\mathcal{T}(H_\infty)$. It is equal to the Teichmüller metric (see Theorem 9.1), similarly to the Riemann surface case.

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1. The Universal Hyperbolic Solenoid H_∞

1.1. **Definition of the universal hyperbolic solenoid H_∞.** Fix a closed surface (S, x) of genus at least two with the base point x. Consider the family all finite sheeted unbranched pointed coverings $\pi_i : (S_i, x_i) \to (S, x)$ of (S, x) such that $\pi_i(x_i) = x$. There is a natural partial ordering defined by $\pi_i \leq \pi_j$ if there exists a finite covering $\pi_{j,i} : (S_j, x_j) \to (S_i, x_i)$ such that $\pi_j = \pi_i \circ \pi_{j,i}$ and $\pi_{j,i}(x_j) = x_i$. The family is inverse directed for this natural partial ordering between coverings, i.e. for any two coverings π_i and π_j there exists a third covering π_k such that $\pi_i, \pi_j \leq \pi_k$. The *universal hyperbolic solenoid* H_∞ is the inverse limit of this inverse directed system (see [31]). A point on H_∞ is given by a choice of one point y_i on each covering surface S_i such that $\pi_i(y_i) = y$ for a fixed $y \in S$ and that if $\pi_i \leq \pi_j$ then $\pi_{j,i}(y_j) = y_i$.

If we start the construction from a pointed closed surface (S', x') of genus at least two different from S, the inverse limit will be homeomorphic to H_∞. We see this by noting that (S, x) and (S', x') have a common covering closed surface (S'', x''). Thus the directed sets of coverings of S and of S' have a common cofinal subsystem of coverings of $S^{''}$. By the properties of inverse limits, the solenoid obtained by choosing $S^{'}$ as the base surface is homeomorphic to H_∞. Thus there is only one hyperbolic solenoid [23].

The (universal hyperbolic) solenoid H_∞ is a compact topological space which fibers over any unbranched finite sheeted covering S_j of the base surface S, including S itself. The natural projection map $\pi_\infty : H_\infty \to S_j$ is given by projecting any $\{y_i\}_{i\in I} \in H_\infty$ to its j-th coordinate. The fibers are homeomorphic to a Cantor set. The solenoid H_∞ is equipped with a canonical *basepoint* whose coordinates are the basepoints of surfaces in the coverings of S. The path components of H_∞ are called *leaves*; each leaf is dense in H_∞ and it is homeomorphic to a disk. The leaf which contains the basepoint of H_∞ is called the *baseleaf.* The restriction of the projection map to any leaf is the universal covering map of the base space (a closed surface covering S).

Any point $p \in H_\infty$ has an open neighborhood V with a chart map $\psi : V \to U \times T$, where U is a 2-disk and T is a Cantor set. The direction of the Cantor set is called the *transverse direction.* Given two overlapping charts $(U_1 \times T_1, \psi_1)$ and $(U_2 \times T_2, \psi_2)$, the transition map $\psi_2 \circ (\psi_1)^{-1}$ is required to map disks onto disks. For details see Sullivan [31] and Nag-Sullivan [23].

1.2. **Smooth and complex structures, and smooth conformal structures.**

A *smooth* structure on H_∞ is given by charts whose transition functions are C^∞ diffeomorphisms on leaves and the transversal variation is continuous in the C^∞-topology on maps. A *complex* structure on H_∞ subordinate to a fixed smooth structure consists of a sub-atlas such that the transition maps are holomorphic on disks [31]. As in [31], we consider only smooth complex structures for technical reasons.

A *smooth conformal structure* on H_∞ is the assignment of a smooth conformal structure on each leaf such that the transversal variation is continuous for the C^∞-topology. Sullivan [31] showed, using [1], that smooth conformal structures on H_∞ give complex structures and vice versa (see also [7]).

A homeomorphism $f : H_\infty \to X$ of two smooth solenoids is said to be a *diffeomorphism* if it is smooth in the disk direction and varies continuously in the transverse direction for the C^∞-topology on smooth maps.

1.3. **Beltrami differentials.**

We introduce Beltrami coefficients on a complex solenoid H_∞. Let $\|\cdot\|_\infty$ denote the essential supremum norm.

Definition 1.1. A *smooth Beltrami differential* μ on a complex solenoid H_∞ is an assignment of a smooth differential of type $(-1, 1)$ on each leaf which varies continuously in the transverse direction for the C^∞-topology. If μ is a smooth Beltrami differential and $\|\mu\|_\infty < 1$ then μ is called a *smooth Beltrami coefficient.*

To each smooth conformal structure on a complex solenoid H_∞ corresponds to a smooth Beltrami coefficient and conversely, a smooth Beltrami coefficient gives a smooth conformal structure on H_∞ (see [31], [7]).

A smooth Beltrami differential μ on H_∞ satisfies $\|\mu\|_\infty < \infty$ by the transversal continuity of μ and by the compactness of H_∞. If $\|\mu\|_\infty < 1$ then it gives a diffeomorphism of the complex solenoid H_∞ onto a complex solenoid X by solving the Beltrami equation (see [1]). Equivalently, a smooth Beltrami coefficient on H_∞ pulls-back a new smooth complex structure on H_∞. On the other hand, if $\|\mu\|_\infty < \infty$ then it corresponds to the tangent vector to a curve of smooth Beltrami coefficients and it is called a smooth Beltrami differential.

Two smooth Beltrami coefficients μ and ν on a complex solenoid H_∞ are *Teichmüller equivalent* if the corresponding diffeomorphisms $f^\mu : H_\infty \to X$ and $f^\nu : H_\infty \to Y$ have conformal images $c : X \to Y$ and if $c \circ f^\mu$ is homotopic to f^ν.

Definition 1.2. A *(generalized) Beltrami coefficient* μ on a complex solenoid H_∞ is an assignment of a Beltrami coefficient on each leaf such that there exists a sequence of Teichmüller equivalent smooth Beltrami coefficients μ_n on H_∞ which converges to μ on each leaf in an a.e. sense for the Lebesque measure on local leaves of a finite chart cover of H_∞ and $\|\mu\|_\infty \geq \liminf_{n\to\infty} \|\mu_n\|_\infty$.

A solution of the Beltrami equation for the Beltrami coefficient μ on H_∞ gives a homeomorphism $f^\mu : H_\infty \to X$, where X is a complex solenoid. Such f^μ is called a *quasiconformal map* of the complex solenoids H_∞ and X.

Definition 1.3. The Teichmüller space $\mathcal{T}(H_\infty)$ of a complex solenoid H_∞ is the space of all quasiconformal maps $f : H_\infty \to X$ up to an equivalence. Two maps $f_1 : H_\infty \to X_1$ and $f_2 : H_\infty \to X_2$ are *Teichmüller equivalent* if there is a transversely continuous (for the C^∞-topology) conformal map $c : X_1 \to X_2$ such that $f_2^{-1} \circ c \circ f_1$ is homotopic to the identity. We denote by $[f] \in \mathcal{T}(H_\infty)$ the Teichmüller equivalence class of $f : H_\infty \to X$.

The above definition is equivalent to:

Definition 1.4. The Teichmüller space $\mathcal{T}(H_\infty)$ is the space of Teichmüller equivalence classes of Beltrami coefficients on H_∞.

We write $[\mu]$ for the Teichmüller class of μ and we use definitions 1.3 and 1.4 simultaneously. The composition of two diffeomorphisms is a diffeomorphism. Therefore, our definition of $\mathcal{T}(H_\infty)$ is independent of the choice of the basepoint H_∞.

1.4. **The transversely locally constant (TLC) complex structures and the profinite completion group.** Let S be a closed Riemann surface of genus at least two. A complex structure on the solenoid H_∞ can be obtained by the pull-back of the complex structure on S using the projection map $\pi_\infty : H_\infty \to S$. The pull-back equips each leaf of H_∞ with a complex structure and the complex structure is locally constant in the transverse direction. The leaves become biholomorphically equivalent to the unit disk. The projection map π_∞ when restricted to any leaf is a holomorphic universal covering map for the Riemann surface S (see [31] and [23]).

The lifted complex structure on H_∞ can be described in terms of the uniformizing Fuchsian group G of the Riemann surface S. We define the *profinite group completion* $\widehat{G}$ of G: Let G_n be the intersection of all subgroups of G of index less than or equal to n. Each G_n is a characteristic finite index subgroup of G (see

[4] and [24]). Using the sequence G_n, we define the profinite metric ρ on G by $\rho(A,B) = \frac{1}{n}$ for $A, B \in G$ if $AB^{-1} \in G_n$ and $AB^{-1} \notin G_{n+1}$. The completion $\widehat{G}$ (called the profinite group completion) of G with respect to ρ is a compact topological group homeomorphic to the Cantor set (see [4] and [24]). Each $t \in \widehat{G}$ is an equivalence class of Cauchy sequences of elements of G in the ρ-metric.

Let Δ be the unit disk. We define $\Delta \times_G \widehat{G}$ to be the quotient of $\Delta \times \widehat{G}$ under the action of G. The action of $A \in G$ on $\Delta \times \widehat{G}$ is given by $A(z,t) = (Az, tA^{-1})$, where tA^{-1} is defined by the action of A^{-1} from the right on each term of a Cauchy sequence representing $t \in \widehat{G}$. The complex structure on H_∞ induced by S is obtained by a natural identification $H_\infty \equiv \Delta \times_G \widehat{G}$ (see [4] and [24]). This is a *transversely locally constant* (TLC) complex structure on H_∞ in the sense of Sullivan [31].

There are other complex structures on H_∞ coming from complex structures on finite sheeted unbranched topological coverings of S. These are also TLC structures. If a complex structure of a finite covering S_1 of S is not the lift of a complex structure on S, then the induced TLC structure on H_∞ by the complex structure on S_1 is different from all induced TLC structures on H_∞ by complex structures on S. Moreover, there are complex structures on H_∞ which are even not TLC, i.e. they are not lifts of complex structures of any finite unbranched covering of the base surface S.

From now on we consider only complex solenoids. Unless otherwise stated, we assume that the solenoid H_∞ has a fixed TLC complex structure $H_\infty \equiv \Delta \times_G \widehat{G}$ and that the solenoid X has an arbitrary complex structure.

2. Holomorphic quadratic differentials, transverse measure and universal covering space

We assume, as before, that $H_\infty \equiv \Delta \times_G \widehat{G}$, where G is a Fuchsian uniformizing group of a closed Riemann surface of genus at least two. Then $\Delta \times \widehat{G}$ is the holomorphic universal covering space of H_∞ and G is the covering group whose action on $\Delta \times \widehat{G}$ is given by $A(z,t) = (Az, tA^{-1})$, for $A \in G$. The universal covering projection $\pi : \Delta \times \widehat{G} \to \Delta \times_G \widehat{G} \equiv H_\infty$ is given by $\pi(z,t) = (z,t)/\sim$, where $(z,t)/\sim$ is the orbit of (z,t) under elements of G. We say that $\Delta \times \widehat{G}$ is a universal covering because it has simple global structure (the product) as opposed to $\Delta \times_G \widehat{G} \equiv H_\infty$.

Let ω be a fundamental polygon for the action of the group G on Δ. Then $\omega \times \widehat{G}$ is a fundamental set for the action of G on $\Delta \times \widehat{G}$. Namely, $\Delta \times_G \widehat{G}$ is isomorphic to the quotient of $\omega \times \widehat{G}$ under the action of G. The identifications on $\omega \times \widehat{G}$ under G are only on the sides of ω with other sides of ω on different levels in the direction of $\widehat{G}$ (the transverse direction).

Let X be the universal hyperbolic solenoid with an arbitrary complex structure. We say that φ is a *holomorphic quadratic differential* on X if it is a holomorphic quadratic differential on each leaf and continuous (in the local charts) in the transverse direction (the Cantor set direction) in the topology of the uniform convergence on compact sets of continuous map. We note that the uniform convergence on compact subsets for holomorphic maps implies the convergence in the C^∞-topology by the Cauchy integral formula for holomorphic functions.

Let φ be a holomorphic quadratic differential on a TLC complex solenoid $H_\infty \equiv \Delta \times_G \widehat{G}$. To lift φ to the universal covering space $\Delta \times \widehat{G}$, we fix a chart $U \times \widehat{G} \subset \Delta \times \widehat{G}$ of H_∞. The lift $\tilde{\varphi}(z,t)$ is defined on $U \times \widehat{G}$ by the values of $\varphi(z,t)$ in this chart and it is analytically extended to $\Delta \times \widehat{G}$. Then $\tilde{\varphi}(z,t)$ is a holomorphic function on each leaf, continuous in the transverse direction for the topology of the uniform convergence on compact subsets, and it satisfies $\tilde{\varphi} = A^*\tilde{\varphi}$ with $(A^*\tilde{\varphi})(z,t) = \tilde{\varphi}(Az, tA^{-1})A^{'}(z)^2$ for $A \in G$.

The profinite group completion $\widehat{G}$ of G is a compact topological group. Consequently, there exists unique left and right translation invariant Borel probability measure m (the Haar measure) on $\widehat{G}$ [30]. The product $|\tilde{\varphi}(z,t)|dxdydm$ introduces a measure on $\Delta \times \widehat{G}$. This measure is invariant under the action of G, and consequently it projects to a measure on H_∞. Recall that $\omega \times \widehat{G} \subset \Delta \times \widehat{G}$ is a fundamental set for the action of G on $\Delta \times \widehat{G}$, where ω is a fundamental polygon for G in Δ. Thus we replace the integration of this measure over H_∞ by the integration over $\omega \times \widehat{G}$.

If G_1 is a finite index subgroup of G, then $\Delta \times_{G_1} \widehat{G_1} \equiv H_\infty \equiv \Delta \times_G \widehat{G}$ (see [24] and [5]). The measure $|\tilde{\varphi}(z,t)|dxdydm$ on $\Delta \times \widehat{G_1}$ is locally given by the product of $|\tilde{\varphi}(z,t)|dxdy$ on leaves $\Delta \times \{t\} \subset \Delta \times \widehat{G_1}$ and the Haar measure m on $\widehat{G_1}$. Note that $m(\widehat{G_1}) = 1/[G : G_1]$, where $[G : G_1]$ is the index of G_1 in G [24]. A fundamental set ω_1 for G_1 in Δ is obtained by gluing $[G : G_1]$ translates of ω in Δ under elements of $G - G_1$. Then the measure on $\omega_1 \times \widehat{G_1}$ is the pull-back of the measure on $\omega \times \widehat{G}$ via their natural identification. Thus if we lift the absolute value of a quadratic differential from H_∞ to $\omega \times \widehat{G}$ or $\omega_1 \times \widehat{G_1}$ the integration will give the same result.

We denote the above defined measure on H_∞ by $|\varphi|dm$.

A leaf of $H_\infty \equiv \Delta \times_G \widehat{G}$ is identified with a G-orbit of one element $t \in \widehat{G}$. Let $K \subset \widehat{G}$ satisfy $m(K) = 0$. Since m is translation invariant, it follows that $m(\cup_{A \in G} A(K)) = 0$. Therefore, there is a well-defined notion of sets of leaves of H_∞ of measure zero.

We say that ψ is a *measurable holomorphic quadratic differential* on H_∞ if it is a holomorphic quadratic differential on almost all leaves (except possibly on a set of measure zero) and if it is measurable on each local chart of H_∞. In addition, if $|\psi|dm$ is a finite measure on H_∞ then we call it an *integrable holomorphic quadratic differential.*

2.1. **The universal covering.** We introduce the holomorphic universal covering for a non-TLC complex solenoid X. Let $f : H_\infty \to X$ be a leafwise diffeomorphism and a quasiconformal map of the fixed TLC solenoid H_∞ onto X. In particular, $[f] \in \mathcal{T}(H_\infty)$ (and every Teichmüller class of maps from H_∞ contains many leafwise diffeomorphisms).

Candel [7] showed that any smooth conformal structure on the solenoid X is represented by a Riemannian metric which is hyperbolic on all leaves (curvature equal to -1) and continuous for the transversal variation. From now on, we use the hyperbolic metric representative for a conformal (or complex) structure on X.

We fix a chart $(U \times T, \psi)$ of X such that $\psi \circ f((\{0\} \times \widehat{G}/\sim)) = \{0\} \times T$. To see that such a chart exists consider a compact set $K = f((\{0\} \times \widehat{G})/\sim)$ and charts $(V \times T^{'}, \chi)$ with $\chi^{-1}(V \times \{t\})$ containing exactly one point of K for each $t \in T^{'}$. By post-composing chart maps χ with a transversely continuous leafwise isometry and by possibly decreasing $T^{'}$ and V, we get new charts, called $(V \times T^{'}, \chi)$ again, such that $\chi^{-1}((0,t)) \in K$ where $0 \in V$ and $t \in T^{'}$. Consider the family of such chart coverings of K. Since K is compact, there exists a finite subcover $\{(V_1 \times T_1, \chi_1), \dots, (V_n \times T_n, \chi_n)\}$ of K. By decreasing slightly some of the T_i, $i = 1, \dots, n$, we arrange that $\{\chi_i^{-1}(V_i \times T_i); i = 1, 2, \dots n\}$ is a pairwise disjoint cover of K. Set $U = V_1 \cap \dots \cap V_n$, $T = T_1 \cup \dots \cup T_n$ and $\psi^{-1}|_{U \times T_i} = \chi_i^{-1}$.

We introduce a family of hyperbolic isometries $\pi_t^X : \Delta \to X$, for $t \in T$, from the unit disk Δ onto the leaves of X such that $\psi \circ \pi_t^X(0) = (0,t)$ and $(\psi \circ \pi_t^X)^{'}(0) > 0$ for the above constructed chart $(U \times T, \psi)$ of X.

Definition 2.1. The universal covering map $\pi^X : \Delta \times T \to X$ for the solenoid X is given by $\pi^X(z,t) = \pi_t^X(z)$.

The following proposition follows easily from the definition of π^X.

Proposition 2.1. The universal covering map $\pi^X : \Delta \times T \to X$ is a continuous surjective leafwise isometry which is a local homeomorphism. In particular, open subsets of $\Delta \times T$ on which π_X is bijective are local charts of X.

The definition of the covering space of X uses the map $f : H_\infty \to X$. Note that $\psi \circ f \circ \pi$ identifies $\widehat{G}$ with T. We induce a group structure on T from $\widehat{G}$, and we use the same letter t for the identified elements of $\widehat{G}$ and T. Their meaning should be read from the context.

Define $\tilde{f} : \Delta \times \widehat{G} \to \Delta \times T$ by

$$\tilde{f}(z,t) := (\pi_t^X)^{-1} \circ f \circ \pi(z,t).$$

Then $\tilde{f}$ is a transversely continuous lift of f such that $\pi^X \circ \tilde{f} = f \circ \pi$.

2.2. **The covering group.** The elements $A \in G$ are deck transformations for the universal covering map $\pi : \Delta \times \widehat{G} \to H_\infty$. They map $U \times \widehat{G}$ onto $A(U) \times \widehat{G}$, where U is an open subset of Δ.

Since $\pi^X \circ \tilde{f} = f \circ \pi$, it follows that $\pi^X \circ \tilde{f} \circ A = \pi^X \circ \tilde{f}$ for all $A \in G$. Let $(z,t) \in \Delta \times \widehat{G}$, $\tilde{f}(z,t) = (w_1, t_1) \in \Delta \times T$ and $(\tilde{f} \circ A)(z,t) = (w_2, t_2) \in \Delta \times T$. By the above, $\pi^X(w_2, t_2) = \pi^X(w_1, t_1)$. Consequently, $(\pi^X)^{-1}(\pi^X(w_1, t_1))$ contains (w_2, t_2). Note that $(\pi_{t_2}^X)^{-1} \circ \pi_{t_1}^X$ is an isometry of $\Delta \times \{t_1\}$ onto $\Delta \times \{t_2\}$ and $((\pi_{t_2}^X)^{-1} \circ \pi_{t_1}^X)(w_1) = w_2$. The action of $A \in G$ on T is induced by its natural action (by the left multiplication) on $\widehat{G}$ via the identification $\psi \circ f \circ \pi : \widehat{G} \equiv T$. We define the action A_X on the universal covering $\Delta \times T$ of X by

$$A_X(z,t) = ((\pi^X_{tA^{-1}})^{-1} \circ \pi^X_t(z), tA^{-1}),$$

where $t, tA^{-1} \in T \equiv \widehat{G}$. The map A_X is an isometry on each leaf. Moreover, A_X is continuous in the transverse direction T. To see this, consider the action of A_X on $\Delta \times \{t\}$ and $\Delta \times \{t_1\}$, for t_1 close to t. Then $A_X|_{\Delta\times\{t\}} = (\pi^X_{tA^{-1}})^{-1} \circ \pi^X_t$ and $A_X|_{\Delta\times\{t_1\}} = (\pi^X_{t_1A^{-1}})^{-1} \circ \pi^X_{t_1}$. Note that tA^{-1} and t_1A^{-1} are close in T, if t and t_1 are close. Since π^X is continuous, it follows that A_X is a leafwise isometry continuous in the transverse direction.

By the definition of A_X, we immediately obtain $\tilde{f} \circ A = A_X \circ \tilde{f}$. We consider the composition $A_X \circ B_X$ for $A, B \in G$. Let $(z, t_1) \in \Delta \times T$, and let $t_2 = t_1B^{-1}$ and $t_3 = t_2A^{-1}$. Then $(A_X \circ B_X)(z, t_1) = (A_X(B_X(z)), t_1B^{-1}A^{-1}) = ((\pi^X_{t_3})^{-1} \circ \pi^X_{t_2} \circ (\pi^X_{t_2})^{-1} \circ \pi^X_{t_1}(z), t_3) = ((\pi^X_{t_3})^{-1} \circ \pi^X_{t_1}(z), t_3) = (A \circ B)_X(z, t_1)$, where $(A \circ B)_X$ is the deck transformation corresponding to $A \circ B \in G$. Further, id_X acts by the identity on $\Delta \times T$ and $(A^{-1})_X = (A_X)^{-1}$ is the inverse of A_X. Thus, the set G_X of the deck transformations for $\pi^X : \Delta \times T \to X$ is a group isomorphic to G.

We note that the holomorphic universal covering $\pi^X : \Delta \times T \to X$ depends on the choice of $H_\infty = \Delta \times_G \widehat{G}$ and on the choice of $f : H_\infty \to X$. The deck transformations also depend on these choices. The existence of the holomorphic universal covering for X is a technical tool in our investigation. It is also used in [21] to identify the isometry group of a complex solenoid X.

Let $\tilde{\mu}$ be the Beltrami coefficient of the lift $\tilde{f}$ of a smooth quasiconformal map $f : H_\infty \to X$ as above. Then $\tilde{\mu}$ is invariant under the action of G on $\Delta \times \widehat{G}$. Namely, $\tilde{\mu}(z,t) = (A^*\tilde{\mu})(z,t) := \tilde{\mu}(Az, tA^{-1})\frac{\overline{A'(z,t)}}{A'(z,t)}$.

The lift $\tilde{f} : \Delta \times \widehat{G} \to \Delta \times T$ of $f : H_\infty \to X$ maps a fundamental set $\omega \times \widehat{G}$ for G onto a fundamental set $F = \tilde{f}(\omega \times \widehat{G}) = \cup_{t\in T}\omega_t \times \{t\}$ for G_X. Also the boundary of $\omega \times \widehat{G}$ is mapped onto a set of area zero because quasiconformal mappings map sets of area zero onto sets of area zero. Since $\tilde{f}$ is transversely continuous, the fundamental set F is compact.

2.3. **Holomorphic quadratic differentials on X.** Let μ be a smooth Beltrami differential on X. Denote by $\tilde{\mu}$ its lift to $\Delta \times T$. Then $\tilde{\mu}$ satisfies

$$\|\tilde{\mu}(z,t) - \tilde{\mu}(z,t_1)\|_\infty = \sup_{z\in\Delta} |\tilde{\mu}(z,t_1) - \tilde{\mu}(z,t)| \to 0 \tag{1}$$

as $t_1 \to t$ for all $t \in T$ because of the continuity of μ in the transverse direction and of the invariance under the covering maps in G_X. We use $\tilde{\mu}$ to construct a holomorphic quadratic differential on X.

Denote by $B(\Delta)$ the space of all holomorphic functions on Δ which have finite Bers norm $\|\varphi\|_B = \sup_{z\in\Delta} |\rho(z)^{-2}\varphi(z)| < \infty$, where $\rho(z) = \frac{1}{1-|z|^2}$ is the Poincaré (hyperbolic) density for the unit disk Δ. Bers' reproducing formula (see [2] and [17, page 139]) gives a continuous, linear, surjective map $P : L^\infty(\Delta) \to B(\Delta)$ which has a continuous right inverse. We define a holomorphic function $\tilde{\varphi}(z,t) = P(\tilde{\mu}(z,t))$ on each leaf $\Delta \times \{t\}$, for $t \in T$. By the continuity of $\tilde{\mu}$ in the transverse direction (see (1)) and by the continuity of P, the holomorphic function $\tilde{\varphi}(z,t)$ is continuous in the transverse direction in the Bers norm. Also, $(A_X)^*(\tilde{\varphi}) = \tilde{\varphi}$ because $(A_X)^*(\tilde{\mu}) = \tilde{\mu}$ for $A_X \in G_X$. Consequently, $\tilde{\varphi}$ projects to a leafwise holomorphic quadratic differential φ on X which is continuous in the transverse direction in the local charts. We call such φ a *holomorphic quadratic differential* on X.

In the opposite direction, a holomorphic quadratic differential φ on X lifts to a holomorphic function $\tilde{\varphi}$ on $\Delta \times T$ which satisfy $(A_X)^*\tilde{\varphi} = \tilde{\varphi}$ for all $A_X \in G_X$. Then $\tilde{\mu}(z,t) = \rho(z)^{-2}\overline{\tilde{\varphi}(z,t)}$ is a smooth Beltrami differential on $\Delta \times T$ which is invariant under the deck transformations. Thus $\tilde{\mu}$ projects to a smooth Beltrami differential μ on X. Note that $P(\tilde{\mu}(z,t)) = \tilde{\varphi}(z,t)$ (see [17]).

We showed

Proposition 2.5. Any holomorphic quadratic differential on X arises from Bers' reproducing formula. □

We use the notation $\bar{P}(\mu) = \varphi$ for this construction.

Recall that $f : H_\infty \to X$ gives an identification of $\widehat{G}$ with T. Denote by m, again, the push-forward measure on T of the Haar measure m on $\widehat{G}$ via this identification. The measure $|\tilde{\varphi}(z,t)|dxdydm$ on $\Delta \times T$ is invariant under the deck transformations. Denote the projection of this measure on X by $|\varphi|dm$. Define $\|\varphi\| = \int_X |\varphi|dm$.

We define measurable and integrable holomorphic quadratic differentials on X in the similar fashion as in the TLC solenoid case using the measure m on X.

3. Commensurable Poincaré Theta Series

We introduce the commensurable (Poincaré) theta series for a complex solenoid X which is an analog of the classical Poincaré theta series for closed Riemann surfaces. Consider the holomorphic universal covering space $\Delta \times T$ and the deck transformations $A_X \in G_X$ for the solenoid X. Given a measurable function $\varphi : \Delta \times T \to \mathbb{C}$ which is holomorphic on almost all leaves, we define the commensurable (Poincaré) theta series of φ by

$$\Theta(\varphi)(z,t) = \sum_{A_X \in G_X} \varphi(A_X(z,t))A_X^{'}(z,t)^2 \tag{2}$$

where $A_X^{'}(z,t)$ is the derivative, in the z variable, of $A_X(z,t)$ on the leaf $\Delta \times \{t\}$. At this point $\Theta(\varphi)$ is only a formal sum, which "defines" a quadratic differential on the quotient X.

We denote by $A(X)$ the space of all holomorphic quadratic differentials on X which are continuous in the transverse direction in the local charts. Denote by $A^1(X)$ the space of all integrable a.e. leafwise holomorphic quadratic differentials on X. Clearly, $A(X) < A^1(X)$.

Let $A(\Delta \times T)$ be the vector space of all bounded functions $\varphi : \Delta \times T \to \mathbb{C}$ that are holomorphic on the leaves $\Delta \times \{t\}$, $t \in T$, and continuous in the transverse direction in the Bers norm, namely

$$\|(\varphi(z,t) - \varphi(z,t_1))\rho(z)^{-2}\|_\infty \to 0$$

as $t \to t_1$. For example, let $g : \Delta \to \mathbb{C}$ be a bounded holomorphic function. Then $\varphi(z,t) := g(z)$ is an element of $A(\Delta \times T)$.

Let $A^1(\Delta \times T)$ be the vector space of all functions $\varphi : \Delta \times T \to \mathbb{C}$ which are holomorphic on almost all leaves and integrable, i.e.

$$\int_{\Delta\times T} |\varphi(z,t)|dxdydm < \infty.$$

Proposition 3.1. Let $\varphi \in A(\Delta \times T)$. Then the Poincaré theta series $\Theta(\varphi)$ converges on all leaves of $\Delta \times T$ to a holomorphic function which is continuous in the Bers norm in the transverse direction and which satisfies

$$(\Theta(\varphi) \circ A_X)(A_X^{'})^2 = \Theta(\varphi)$$

for all $A_X \in G_X$.

Proof. We use the standard area argument to show the convergence of the Poincaré series. However, the summation is over different leaves of $\Delta \times T$ and some care in the argument is needed.

Let $B_r(0) \subset \Delta$ denote the hyperbolic disk with the center $0 \in \Delta$ and the radius r. Choose $r > 0$ such that $\pi^X : B_r(0) \times T \to X$ is an injection into X, namely the deck transformations $A_X \in G_X$ do not identify any two points of $B_r(0) \times T$. Define $B_t := \pi^X(B_r(0) \times \{t\})$, $t \in T$, to be a disk in the hyperbolic metric of the leaf $l = \pi^X(\Delta \times \{t\})$ of X.

Identify l with Δ by an isometry h such that the point $\pi^X(0,t) \in B_t$ is mapped to $0 \in \Delta$. Consider the family of hyperbolic disks $B_{tA^{-1}} = \pi^X(B_r(0) \times \{tA^{-1}\})$, for $A \in G$, on the fixed leaf l of X. The family $\{B_{tA^{-1}}\}_{A\in G}$ is pairwise disjoint in l. Therefore, the sum of the Euclidean areas of the family $\{h(B_{tA^{-1}})\}_{A\in G}$ in Δ is less than π.

Note that $A_X(B_r(0) \times \{t\}) = (\pi^X_{tA^{-1}})^{-1} \circ \pi^X_t(B_r(0) \times \{t\}) = ((\pi^X_{tA^{-1}})^{-1} \circ h^{-1})(h(B_t))$. Since $(\pi^X_{tA^{-1}})^{-1} \circ h^{-1}$ maps the center of $h(B_{tA^{-1}})$ onto $0 \in \Delta \times \{tA^{-1}\}$ and since the center of $h(B_t)$ is $0 \in \Delta$, it follows that the Euclidean area of $((\pi^X_{tA^{-1}})^{-1} \circ h^{-1})(h(B_t)) = (\pi^X_{tA^{-1}})^{-1}(B_t)$ is equal to the Euclidean area of $h(B_{tA^{-1}})$. Consequently, the sum of Euclidean areas of $\{A_X(B_r(0) \times \{t\})\}_{A_X\in G_X}$ is less than π. Therefore Θ converges uniformly on compact subsets of $\Delta \times T$. □

The commensurable theta series for a complex solenoid X has similar properties to the Poincaré theta series for Riemann surfaces.

Theorem 3.1. Let $\varphi \in A(\Delta \times T)$. The commensurable theta series defines a surjective, continuous, linear operator $\Theta : A(\Delta \times T) \to A(X)$ of norm 1 such that the image under Θ of the unit ball contains a ball of radius $\frac{1}{3}$, for the L^1-norms on the above spaces.

Proof. By Proposition 3.1, $\Theta(\varphi)$ is holomorphic on leaves, continuous in the transverse direction in the Bers norm and invariant under the deck transformations. The induced map $\Theta : A(\Delta \times T) \to A(X)$ is a continuous linear map, and the standard argument shows that its norm less than or equal to 1.

Let $\varphi \in A(X)$ and let $\tilde{\varphi}$ be its lift to $\Delta \times T$. Since $\tilde{\varphi}$ is invariant under the covering maps and the fundamental set is compact, it follows that $|\tilde{\varphi}(z,t)\rho^{-2}(z)|$ is bounded, where $\rho(z) = \frac{1}{1-|z|^2}$. Therefore, the standard argument for the Poincaré theta series (see [17, page 50]) applied on the leaves shows that Θ is surjective and that the image of the unit ball in $A(\Delta\times T)$ contains the ball of radius $1/3$ in $A(X)$.

It remains to show that Θ has norm 1. Let G_n be a finite index subgroup of G of a large index k. Then $\widehat{G_n}<\widehat{G}$ and let $\tilde{f}:T_n\equiv\widehat{G_n}$. We get $\lambda_X(T_n)=\frac{1}{k}$. Define $\psi(z,t)=k$ for $(z,t)\in\Delta\times T_n$ and $\psi(z,t)=0$ otherwise. Then $\|\psi\|=\int_{\Delta\times T}|\psi(z,t)|dxdy=1$. Furthermore, $\Theta(\psi)(z,t)=k+\sum_{A_X\in(G_n)_X,\ A_X\neq id}kA_X'(z,t)^2$ for $(z,t)\in\Delta\times T_n$, where $(G_n)_X<G_X$ is isomorphic to G_n. Let $F_X=\cup_{t\in T_n}\omega_t\times\{t\}$ be the image under $\tilde{f}$ of the fundamental set $F=\omega\times\widehat{G_n}$ for the action of G_n on $\Delta\times\widehat{G_n}$, where $\omega\subset\Delta$ is the Dirichlet fundamental polygon for G_n centered at $0\in\Delta$. Then F_X is one choice of a fundamental set for G_X on $\Delta\times T$. Finally,

$$\|\Theta(\psi)\|\geq 1-C\cdot\sup_{t\in T_n}[area_E(\Delta-\omega_t)]\to 1$$

as $k\to\infty$. Therefore $\|\Theta\|=1$ □

Acknowledgement. Adam Epstein, in a private communication, informed me that $A(H_\infty)$, for H_∞ a TLC complex solenoid, is not closed for the L^1-norm.

We find the closure of $A(X)$ by extending Θ to $A^1(\Delta\times T)$. To do so, we show that $A(\Delta\times T)$ is dense in $A^1(\Delta\times T)$.

Proposition 3.2. *The closure of $A(\Delta\times T)$ in the L^1-norm is equal to $A^1(\Delta\times T)$.*

Proof. It is clear that the closure of $A(\Delta\times T)$ is contained in $A^1(\Delta\times T)$. We prove that any $\varphi\in A^1(\Delta\times T)$ can be approximated by elements of $A(\Delta\times T)$. Let T^1 be the set of all $t\in T$ such that $\int_\Delta|\varphi(z,t)|dxdy<\infty$. Then $m(T-T^1)=0$.

Let $B_n=\{t\in T^1;\ \int_\Delta|\varphi(z,t)|dxdy\geq n\}$. Then $\cap_{n=1}^\infty B_n=\emptyset$. It follows that $\int_{\Delta\times B_n}|\varphi(z,t)|dxdydm\to 0$ as $n\to\infty$.

We define

$$\varphi_1(z,t)=\begin{cases}0 & \text{, if } t\in B_n\\ \varphi(z,t) & \text{, otherwise}\end{cases}\tag{3}$$

Clearly $\varphi_1\in A^1(\Delta\times T)$ and it is close to φ for n large.

Since $\int_\Delta|\varphi_1(z,t)|dxdy\leq n$ for all $t\in T$, it follows that φ_1 is bounded in the sense of Bers. Let $\varphi_2(z,t)=\varphi_1(rz,t)$, $0<r<1$. Then φ_2 is close to φ_1 in the L^1-norm, for r close to 1. The absolute value of the integrable holomorphic quadratic differential φ_2 is uniformly bounded on $\Delta\times T$ with the bound depending on n and r.

We use the Bers' reproducing formula [2] to finish the proof. Let $\mu_2(z,t)=\varphi_2(z,t)\rho^{-2}(z)$, where $\rho(z)$ is the Poincaré density in the unit disk Δ. Let

$$\mu_3(z,t)=\begin{cases}\mu_2(z,t) & \text{, if } |z|\leq r_1<1\\ 0 & \text{, otherwise}\end{cases}\tag{4}$$

Let $\varphi_3=\frac{3}{\pi}\iint_\Delta\frac{\mu_3(z,t)}{(1-z\bar{w})^4}dudv$ be the holomorphic quadratic differential obtained from μ_3 using the Bers' reproducing formula on each disk $\Delta\times\{t\}$, $t\in T$ (see [17, page 50]). Then φ_3 is close to φ_2 for r_1 close to 1 by noting that $\int|\varphi_3-\varphi_2|=$

$\iiint_{\Delta\times T} |\iint_{\Delta-\Delta_{r_1}} \frac{\varphi_2(w)\rho(w)^{-2}}{(1-z\bar{w})^4} dudv| dxdydm \leq \sup_{w\in\Delta} |\varphi_2(w,t)|\pi(1-r_1^2)$, where Δ_{r_1} is the disk of radius r_1 and center 0.

By Lusin's theorem, there exists a continuous function μ_4 with compact support in $\Delta\times T$ which agrees with μ_3 up to a set of measure $\epsilon > 0$ for the measure $dxdydm$. Then μ_4 gives a holomorphic quadratic differential $\varphi_4 \in A(\Delta\times T)$ which is as close as we want to φ for n large, for r and r_1 close to 1, and for ϵ small. □

Theorem 3.2. *The commensurable Poincaré theta series extends to a continuous surjective linear operator of norm* 1 *from* $A^1(\Delta\times T)$ *onto* $A^1(X)$.

Proof. The theorem follows by Proposition 3.2 and by the continuity of Θ on $A(\Delta\times T)$. □

Corollary 3.1. *The closure of* $A(X)$ *in the* L^1*-norm is equal to* $A^1(X)$. □

4. The Reich-Strebel Inequality

The Reich-Strebel inequality is a fundamental tool in the quasiconformal Teichmüller theory of Riemann surfaces. It is used to analyze properties of the Teichmüller metric, including geodesics and the infinitesimal form of the metric.

We prove below a version of the Reich-Strebel inequality for the complex solenoid X. A Beltrami coefficient μ of a quasiconformal map $f : X \to X$ is said to be *Teichmüller trivial* if f is homotopic to the identity. Then μ is *Teichmüller equivalent* to the trivial Beltrami coefficient 0.

Theorem 4.1 (The Reich-Strebel inequality for H_∞). Let φ be a holomorphic quadratic differential on a complex solenoid X and let μ be a Teichmüller trivial Beltrami coefficient. Then

$$\|\varphi\| \leq \int_X \frac{\left|1+\mu\frac{\varphi}{|\varphi|}\right|^2}{1-|\mu|^2}|\varphi|dm. \tag{5}$$

Proof. The argument uses the Reich-Strebel inequality for closed Riemann surfaces (see [28] and [14]). It is enough to prove the inequality for smooth μ.

Assume first that $X \equiv \Delta\times_G \widehat{G}$, for some Fuchsian group G uniformizing a closed Riemann surface of genus at least two. Denote by $\pi_\infty : \Delta\times_G\widehat{G} \to \Delta/G$ the natural projection map. Let μ be a smooth Teichmüller trivial Beltrami coefficient on X obtained by lifting a smooth Teichmüller trivial Beltrami coefficient on Δ/G by the fiber map π_∞. Let φ be a holomorphic quadratic differential on X obtained by lifting a holomorphic quadratic differential on Δ/G via the fiber map π_∞. Let $\tilde{\mu}$ and $\tilde{\varphi}$ be lifts of μ and φ to the universal covering $\Delta\times\widehat{G}$ of X. In general, differentials μ and φ on $X \equiv \Delta\times_G\widehat{G}$ are lifts of differentials on Δ/G if and only if $\tilde{\mu}(z,t)$ and $\tilde{\varphi}(z,t)$ are constant in t, $\tilde{\mu}(Az,t)\frac{\overline{A'(z)}}{A'(z)} = \tilde{\mu}(z,t)$ and $\tilde{\varphi}(Az,t)A'(z)^2 = \tilde{\varphi}(z,t)$, for all $A\in G$. We allow μ to be the lift of a Beltrami coefficient on Δ/G_1 and φ to be the lift of a holomorphic quadratic differential on Δ/G_2, where G_1 and G_2 are finite index subgroups of G. Then, we define $G = G_1\cap G_2$ and arrive at the starting situation.

Consider the fundamental set $\omega \times \widehat{G}$ of X, where ω is a fundamental polygon for G in Δ. On each $\omega \times \{t\}$, $t \in \widehat{G}$, we have

$$\iint_{\omega\times\{t\}} |\tilde{\varphi}(z,t)|dxdy \leq \iint_{\omega\times\{t\}} \frac{\left|1+\tilde{\mu}(z,t)\frac{\tilde{\varphi}(z,t)}{|\tilde{\varphi}(z,t)|}\right|^2}{1-|\tilde{\mu}(z,t)|^2}|\tilde{\varphi}(z,t)|dxdy \tag{6}$$

by the Reich-Strebel inequality on Δ/G, where $z = x + iy$. Integrating both sides of (6) with respect to the Haar measure m on $\widehat{G}$ we obtain

$$\int_X |\varphi|dm \leq \int_X \frac{\left|1+\mu\frac{\varphi}{|\varphi|}\right|^2}{1-|\mu|^2}|\varphi|dm. \tag{7}$$

Assume now that a smooth Teichmüller trivial Beltrami coefficient μ and a holomorphic quadratic differential φ on the solenoid $X \equiv \Delta\times_G\widehat{G}$ are not necessarily lifts of a smooth Teichmüller trivial Beltrami coefficient and a holomorphic quadratic differential on Δ/G_1, where G_1 is a finite index subgroup of G. We approximate μ and φ by μ_n and φ_n coming from lifts, by the projection map π_∞, of differentials on a closed Riemann surface Δ/G_n, where G_n is a finite index subgroup of G.

Let $\tilde{\varphi}$ and $\tilde{\mu}$ be lifts to $\Delta \times \widehat{G}$ of the above differentials φ and μ on $\Delta \times_G \widehat{G}$, where φ and μ are not lifts to X of differentials on a closed Riemann surface of genus at least two. We approximate $\tilde{\varphi}$ first. Let $\tilde{\nu}$ be a smooth Beltrami differential on $\Delta\times\widehat{G}$ such that $P(\tilde{\nu}) = \tilde{\varphi}$. This implies that $\bar{P}(\nu) = \varphi$. If G_n is any finite index subgroup of G then $X \equiv \Delta \times_{G_n} \widehat{G_n}$ (see [24] and [5]). Given $\epsilon > 0$, there exists a large enough finite index subgroup G_n of G such that $\|\tilde{\nu}(z,t) - \tilde{\nu}(z,t_1)\|_\infty < \frac{\epsilon}{2}$ for all $t, t_1 \in \widehat{G_n}$. Fix a fundamental polygon ω_n for the action of G_n on Δ. Define

$$\tilde{\nu}_n(z,t) = \begin{cases} \tilde{\nu}(z,id), & \text{if } z \in \omega_n \\ \tilde{\nu}(A^{-1}(z),id)A'(A^{-1}(z))/\overline{A'(A^{-1}(z))}, & \text{if } z \in A(\omega_n) \text{ and } A \in G_n \end{cases}$$

for $(z,t) \in \Delta \times \widehat{G_n}$. By the definition, $\tilde{\nu}_n(A(z),t)\frac{\overline{A'(z)}}{A'(z)} = \tilde{\nu}_n(z,t)$ for all $A \in G_n$ and for all $t \in \widehat{G_n}$.

The Beltrami differential $\tilde{\nu}_n$ is constant in t and invariant with respect to G_n on each leaf $\Delta \times \{t\}$, $t \in \widehat{G_n}$. Consequently $\tilde{\nu}_n$ is a lift on $\Delta \times \widehat{G_n}$ of a Beltrami differential on Δ/G_n. Further, for all $t \in G_n$,

$$\|\tilde{\nu}(z,t) - \tilde{\nu}_n(z,t)\|_\infty < \epsilon \tag{8}$$

because $\|\tilde{\nu}(z,t)-\tilde{\nu}(z,t_1)\|_\infty < \frac{\epsilon}{2}$ for all $t, t_1 \in \widehat{G_n}$. We define $\tilde{\varphi}_n(z,t) = P(\tilde{\nu}_n(z,t))$ on $\Delta \times \widehat{G_n}$. Then $\tilde{\varphi}_n(z,t)$ is constant in t (the transverse direction) and it is invariant under the action of G_n on $\Delta\times\widehat{G_n}$. It projects to a holomorphic quadratic differential φ_n on X. By the continuity of $\bar{P}$ and by $\|\nu - \nu_n\|_\infty \to 0$ as $n \to \infty$, we get that $\varphi_n \to \varphi$ as $n \to \infty$ in the Bers norm uniformly on each leaf of X.

In the similar fashion as for $\tilde{\nu}$, we find an approximating sequence $\tilde{\mu}_n(z,t)$ for $\tilde{\mu}(z,t)$ such that, for all $(z,t), (z,t_1) \in \Delta \times \widehat{G_n}$,

$$\|\tilde{\mu}_n(z,t) - \tilde{\mu}(z,t)\|_\infty < \epsilon \tag{9}$$

and $\tilde{\mu}_n$ is the lift of a Beltrami coefficient on Δ/G_n. However, our method does not guarantee that $\tilde{\mu}_n(z,t)$ is Teichmüller trivial on the leaves of $\Delta \times \widehat{G_n}$. Since we want to apply the Reich-Strebel inequality for closed Riemann surfaces, we modify $\tilde{\mu}_n$ to be Teichmüller trivial on the leaves of $\Delta \times \widehat{G_n}$ and G_n-invariant on each leaf. Note that on each leaf the Teichmüller class of $f^{\tilde{\mu}} \circ (f^{\tilde{\mu}_n})^{-1}$ is the same as the Teichmüller class of $(f^{\tilde{\mu}_n})^{-1}$ because $\tilde{\mu}$ is Teichmüller trivial on each leaf of $\Delta \times \widehat{G_n}$. Thus, the Teichmüller class $[f^{\tilde{\mu}} \circ (f^{\tilde{\mu}_n})^{-1}]$ is constant in the leaf direction and on each leaf it belongs to the Teichmüller space of Δ/G_n. By (9), $[f^{\tilde{\mu}} \circ (f^{\tilde{\mu}_n})^{-1}]$ is close to the basepoint in the universal Teichmüller space $\mathcal{T}(\Delta)$ of each leaf $\Delta \times \{t\}$, for $t \in T$. The Beltrami coefficient $\tilde{\lambda}_n$ of the harmonic representative $f^{\tilde{\lambda}_n}$ of the Teichmüller class $[f^{\tilde{\mu}} \circ (f^{\tilde{\mu}_n})^{-1}]$ satisfies $\|\tilde{\lambda}_n\|_\infty \to 0$ as $n \to \infty$ by [16, Lemma 3], and it is G_n-invariant on each leaf. Consequently, the map $f^{\tilde{\lambda}_n} \circ f^{\tilde{\mu}_n}$ is Teichmüller trivial on each leaf of $\Delta \times \widehat{G_n}$ and its Beltrami coefficient $\tilde{\lambda}_n'$ satisfies $\|\tilde{\mu} - \tilde{\lambda}_n'\|_\infty \to 0$ as $n \to \infty$.

Denote by φ_n and λ_n' the projections of $\tilde{\varphi}_n$ and $\tilde{\lambda}_n'$ onto $\Delta \times_{G_n} \widehat{G_n} \equiv X$. We apply (7) with φ_n and λ_n', and obtain

$$\int_X |\varphi_n| dm \leq \int_X \frac{\left|1 + \lambda_n' \frac{\varphi_n}{|\varphi_n|}\right|^2}{1 - |\lambda_n'|^2} |\varphi_n| dm.$$

Note that even though λ_n' is not necessarily smooth everywhere the inequality (7) still holds. Letting $n \to \infty$ in the above inequality, we obtain (5) by the Lebesgue dominated convergence theorem.

We assume now that X is a solenoid with a non-TLC complex structure. Let μ be a smooth Teichmüller trivial Beltrami coefficient on X. Let φ be a holomorphic quadratic differential on X. Let ν be a smooth Beltrami differential on X such that $\varphi = \bar{P}(\nu)$. To apply the previous result, consider a smooth $(1+\frac{1}{n})$-quasiconformal map $f_n : \Delta \times_{G_n} \widehat{G_n} \to X$ continuous for the transversal variation, where G_n is a Fuchsian group uniformizing a closed Riemann surface Δ/G_n. Such G_n and f_n exist because TLC complex structures on the solenoid are dense among all complex structures (see Sullivan [31] or [23]). Denote by $f^\mu : X \to X$ Teichmüller trivial smooth quasiconformal map with the Beltrami coefficient μ. We form a Teichmüller trivial smooth quasiconformal self map $g_n = (f_n)^{-1} \circ f^\mu \circ f_n$ of $\Delta \times_{G_n} \widehat{G_n}$, and denote its Beltrami coefficient by μ_n. Let ν_n be the Beltrami coefficient of $f^\nu \circ f_n$ and define a holomorphic quadratic differential on $\Delta \times_{G_n} \widehat{G_n}$ by $\varphi_n = \bar{P}(\nu_n)$. Then φ_n is a holomorphic quadratic differential and μ_n is a Teichmüller trivial smooth Beltrami coefficient on $\Delta \times_{G_n} \widehat{G_n}$. Thus

$$\int_{\Delta \times_{G_n} \widehat{G_n}} |\varphi_n| dm \leq \int_{\Delta \times_{G_n} \widehat{G_n}} \frac{\left|1 + \mu_n \frac{\varphi_n}{|\varphi_n|}\right|^2}{1 - |\mu_n|^2} |\varphi_n| dm,$$

by the Reich-Strebel inequality for the TLC solenoids proved above. We change the domain of the integration in the above inequality to X using the map f_n^{-1} :

$X \to \Delta \times_{G_n} \widehat{G_n}$ to obtain

$$\int_X |\varphi_n \circ f_n^{-1}| J_{f_n^{-1}} dm \leq \int_X \frac{\left|1 + \mu_n \circ f_n^{-1} \frac{\varphi_n \circ f_n^{-1}}{|\varphi_n \circ f_n^{-1}|}\right|^2}{1 - |\mu_n \circ f_n^{-1}|^2} |\varphi_n \circ f_n^{-1}| J_{f_n^{-1}} dm, \tag{10}$$

where $J_{f_n^{-1}}$ is the Jacobian of f_n^{-1}.

We intend to apply the Lebesgue dominated convergence in (10) in order to prove (5). To do so, we need to arrange that $\varphi_n \circ f_n^{-1} \to \varphi$, $J_{f_n^{-1}} \to 1$, $\mu_n \circ f_n^{-1} \to \mu$, and that $|\varphi_n|$, $J_{f_n^{-1}}$ are bounded above by positive constants on X and that $\|\mu_n\|_\infty$ is bounded away from 1.

We define the lift of $(f_n)^{-1} : X \to \Delta \times_{G_n} \widehat{G_n}$ to corresponding universal coverings. Let $g : H_\infty \to X$ be a quasiconformal marking on X, and let $\pi^X : \Delta \times T \to X$ be the covering map constructed using g, and let $\tilde{g} : \Delta \times \widehat{G} \to \Delta \times T$ be the lift of g. Further, the map $k_n := (f_n)^{-1} \circ g : H_\infty \to \Delta \times_{G_n} \widehat{G_n}$ defines a non-standard covering $\pi^n : \Delta \times T \to \Delta \times_{G_n} \widehat{G_n}$. Denote by $\tilde{k}_n : \Delta \times \widehat{G} \to \Delta \times T$ the lift of $k_n : H_\infty \to \Delta \times_{G_n} \widehat{G_n}$ to the universal coverings. The lift $\widetilde{(f_n)^{-1}} : \Delta \times T \to \Delta \times T$ of $(f_n)^{-1}$ to universal coverings is defined by

$$\widetilde{(f_n)^{-1}} := \tilde{k}_n \circ (\tilde{g})^{-1}.$$

Using the commutativity of the covering diagrams $\pi^n \circ \tilde{k}_n = k_n \circ \pi^n$ and $(\tilde{g})^{-1} \circ \pi^X = \pi \circ (\tilde{g})^{-1}$ for $\tilde{k}_n$ and $(\tilde{g})^{-1}$ and their invariance under appropriate deck transformations, we obtain that $\pi^n \circ \widetilde{(f_n)^{-1}} = (f_n)^{-1} \circ \pi^X$ and that $\widetilde{(f_n)^{-1}}$ conjugates deck transformations of X to deck transformations of $\Delta \times_{G_n} \widehat{G_n}$.

The lift $\widetilde{(f_n)^{-1}} : \Delta \times T \to \Delta \times T$ extends by the continuity to a map $\tilde{h}_n : S^1 \times T \to S^1 \times T$ which is quasisymmetric on each $S^1 \times \{t\}$, for $t \in T$, and continuous in the transverse direction in the quasisymmetric topology. We define $\tilde{C}_n : \bar{\Delta} \times T \to \bar{\Delta} \times T$ such that $\tilde{C}_n|_{\bar{\Delta} \times \{t\}}$ is the unique Möbius map which maps $\tilde{h}_n(1,t)$, $\tilde{h}_n(i,t)$ and $\tilde{h}_n(-1,t)$ onto $(1,t)$, (i,t) and $(-1,t)$ for each $t \in T$. Since $\tilde{h}_n$ is continuous in the transverse direction, then $\tilde{C}_n$ is continuous in the transverse direction. Note that $\tilde{C}_n$ fixes the leaves of $\Delta \times T$. Then $\tilde{C}_n \circ \tilde{h}_n : S^1 \times T \to S^1 \times T$ fixes $(1,t)$, (i,t) and $(-1,t)$. We change the covering $\pi^n : \Delta \times T \to \Delta \times_{G_n} \widehat{G_n}$ into covering $\pi^n \circ \tilde{C}_n : \Delta \times T \to \Delta \times_{G_n} \widehat{G_n}$ with deck transformations being conjugates by $\tilde{C}_n$ of deck transformations for π^n. We rename $\tilde{C}_n \circ \tilde{h}_n$ and $\pi^n \circ \tilde{C}_n$ to $\tilde{h}_n$ and π^n.

Extend $\tilde{h}_n$ to $\bar{\Delta} \times T$ using the barycentric extension (see Douady-Earle [8]) on each leaf. Denote the extension of $\tilde{h}_n$ by $\tilde{f}_n^{-1}$. Note that $\tilde{h}_n \to id$ on $S^1 \times T$. By the properties of the barycentric extension, it follows that $\tilde{f}_n^{-1}$ converges to the identity uniformly on compact subsets of the type $K \times T$, where $K \subset \Delta$ is compact. Further, $\tilde{f}_n^{-1}$ is invariant under the action of the covering maps on $\Delta \times T$ by the conformal naturality of the barycentric extension. Also, $J_{\tilde{f}_n^{-1}}$ converges to 1 uniformly on compact subsets of the form $K \times T$ by properties of the barycentric extension.

Consider a fixed fundamental set F for X in $\Delta\times T$ constructed using the marking $g : H_\infty \to X$. The inequality (10) becomes

(11)
$$\int_F |\tilde{\varphi}_n\circ\tilde{f}_n^{-1}|J_{\tilde{f}_n^{-1}}dxdydm \leq \int_F \frac{|1+\tilde{\mu}_n\circ\tilde{f}_n^{-1}\frac{\tilde{\varphi}_n\circ\tilde{f}_n^{-1}}{|\tilde{\varphi}_n\circ\tilde{f}_n^{-1}|}|^2}{1-|\tilde{\mu}_n\circ\tilde{f}_n^{-1}|^2}|\tilde{\varphi}_n\circ\tilde{f}_n^{-1}|J_{\tilde{f}_n^{-1}}dxdydm.$$

Denote by $\tilde{\nu}$ and $\tilde{\nu}_n$ the lifts of the Beltrami coefficients ν and ν_n. Note that we implicitly identify the transverse sets of universal coverings of $\Delta\times_{G_n}\widehat{G_n}$ and X using the map $\tilde{f}_n^{-1}$. Then $\Delta\times T$ is considered as the universal covering of both $\Delta\times_{G_n}\widehat{G_n}$ and X. The Beltrami coefficients $\tilde{\nu}_n$ converge to $\tilde{\nu}$ uniformly on compact subsets of $\Delta\times T$ because f^ν and f_n are smooth maps, $\tilde{f}_n\to id$ uniformly on compact subsets and the quasiconformal constant of $\tilde{f}_n$ is converging to 1.

Since $\tilde{\nu}_n$ is converging to $\tilde{\nu}$ uniformly on compact subsets, then $\tilde{\varphi}_n\to\tilde{\varphi}$ uniformly on compact subsets. It follows that $\tilde{\varphi}_n\circ\tilde{f}_n^{-1}\to\tilde{\varphi}$ uniformly on compact subsets of $\Delta\times T$, because $\tilde{f}_n^{-1}\to id$ as $n\to\infty$. Similarly, $\mu_n\circ\tilde{f}_n^{-1}\to\mu$ uniformly on compact subsets. In addition, $\|\mu_n\|_\infty$ is uniformly bounded away from 1 and $|\tilde{\varphi}_n|$ is bounded on compact subsets of $\Delta\times T$. By letting $n\to\infty$ in (11), we obtain (5) for arbitrary complex solenoid X. $\square$

5. Teichmüller maps

The first application of the Reich-Strebel inequality for Riemann surfaces is a proof that Teichmüller type Beltrami differentials determine unique geodesics for the Teichmüller metric. We prove a similar fact for the Teichmüller space of the solenoid.

Let $\varphi\neq 0$ be a holomorphic quadratic differential on the solenoid X. We show that $k\frac{|\varphi|}{\varphi}$, for $0\leq k<1$, is a Beltrami coefficient on X in the sense of Definition 1.2 and it is said to be of *Teichmüller type*. The Teichmüller type Beltrami coefficient $k\frac{|\varphi|}{\varphi}$ is smooth in the complement of zeros of φ (a discrete subset of each leaf) and it is not defined at the zeros.

We approximate $k\frac{|\varphi|}{\varphi}$ with smooth Beltrami coefficients in the same (universal) Teichmüller class on each leaf such that the approximating coefficients are smooth and continuous for the transversal variation on X.

Proposition 5.1. Let $\varphi\neq 0$ be a holomorphic quadratic differential on the complex solenoid X. Then there exists a sequence of smooth Beltrami coefficients μ_n on X which are Teichmüller equivalent to $k\frac{|\varphi|}{\varphi}$ when restricted to each leaf of X such that μ_n converge to $k\frac{|\varphi|}{\varphi}$ uniformly on compact sets of the complement of zeros of φ and that $\sup_{p\in X}|\mu_n(p)|\to k$ as $n\to\infty$. In particular, $k\frac{|\varphi|}{\varphi}$, $0<k<1$, is a Beltrami coefficient on X.

Proof. Let $\Delta\times T$ be the universal covering space of X and let $\tilde{\varphi}(z,t)$ be the lift of φ. Then $\tilde{\varphi}(z,t)$ is invariant under the deck transformations on $\Delta\times T$ and it is continuous for the transverse direction in the Bers norm on $\Delta\times T$. By the continuity in the transverse direction and by the Rouché's theorem, given a zero

$(p,t) \in \Delta \times T$ of $\tilde{\varphi}$ there exists a small enough neighborhood of (p,t) such that $\tilde{\varphi}$ has a zero in each local leaf of the neighborhood. However, multiple zeros of $\tilde{\varphi}(z,t)$ on one local leaf can be close to two or more zeros of $\tilde{\varphi}(z,t)$ on the nearby local leaves.

Fix a zero of φ in X. Let $(z_0,t_0) \in \Delta \times T$ be a single lift of the fixed zero. Let δ denote the minimum of the number 1 and of the 1/3 of the hyperbolic distance of (z_0,t_0) to the closest zero of $\tilde{\varphi}(z,t_0)$ on the leaf $\Delta \times \{t_0\}$. Fix $n \geq 4$. Denote by $B_r(z)$ the hyperbolic disk with the center z and the radius r in the unit disk Δ. We choose $T' \subset T$ such that T' is a neighborhood of t_0 homeomorphic to T and such that $\tilde{\varphi}(z,t)$ does not have any zeros in $[B_\delta(z_0) - B_{\frac{\delta}{n+1}}(z_0)] \times T'$. The choice of T' is possible by the transversal continuity of zeros of $\tilde{\varphi}(z,t)$ and its size depends on n. The number of zeros counted with their multiplicity in each $B_{\frac{\delta}{n+1}}(z_0) \times \{t\}$, for $t \in T'$, equals to the multiplicity of the zero (z_0,t_0) by the Rouché's theorem. By decreasing T' if necessary, we arrange that no zero of $\tilde{\varphi}(z,t)$ in $B_{\frac{\delta}{n+1}}(z_0) \times \{t\}$, for $t \in T'$, is at distance less than $\frac{5}{2}\delta$ from any zero outside $B_{\frac{\delta}{n+1}}(z_0) \times \{t\}$. By further decreasing T' if necessary, we arrange that no point of $B_\delta(z_0) \times T'$ is identified with another point of $B_\delta(z_0) \times T'$ by a deck transformation. Then the covering map is a homeomorphisms from $B_\delta(z_0) \times T'$ onto a neighborhood $\pi^X(B_\delta(z_0) \times T')$ of $\pi^X(z_0,t_0)$ in X. Thus $B_\delta(z_0) \times T'$ is a chart around the zero $\pi^X(z_0,t_0)$ of φ.

For each zero $x \in X$ of φ, we choose its neighborhood $V(x,\delta) = \pi^X(B_\delta(z_0) \times T')$ in X as above. In other words, we fix $\delta > 0$ such that the restriction of the projection map π^X to $B_\delta(z_0) \times T'$ is a homeomorphism onto its image and such that the zeros of φ which are outside $V(x,\delta)$ are on the leafwise distance from the zeros in $V(x,\delta)$ at least $\frac{5}{2}\delta$ for the hyperbolic metric on leaves. Let $x_1, x_2 \in X$ be two zeros of φ, and let $V_1 = V_1(x_1,\delta_1)$ and $V_2 = V_2(x_2,\delta_2)$ be their neighborhoods as above. If the set of the zeros of φ in V_1 and the set of the zeros of φ in V_2 are disjoint then $V_1 \cap V_2 \neq \emptyset$. To see this, we assume on the contrary that V_1 and V_2 intersect and that no zeros of φ are in $V_1 \cap V_2$. Lift the situation to $\Delta \times T$. The lifts of V_1 and V_2 are given by $B_{\delta_1}(z_1) \times T^1$ and $B_{\delta_2}(z_2) \times T^2$. If $B_{\delta_1}(z_1) \times T^1$ and $B_{\delta_2}(z_2) \times T^2$ do not intersect then we can move $B_{\delta_2}(z_2) \times T^2$ by a covering map such that it intersects $B_{\delta_1}(z_1) \times T^1$. Call the translated neighborhood $B_{\delta_2}(z_2) \times T^2$ again. To be more precise, assume $B_{\delta_1}(z_1) \times \{t\} \cap B_{\delta_2}(z_2) \times \{t\} \neq \emptyset$ for some $t \in T^1 \cap T^2$ and $\delta_1 \geq \delta_2$. Then the distance between the zeros in $B_{\frac{\delta_1}{n+1}}(z_1) \times \{t\}$ and $B_{\frac{\delta_2}{n+1}}(z_2) \times \{t\}$ is less than $(2 + \frac{2}{n+1})\delta_1$. Thus, for $n \geq 4$, the distance between a zero in $B_{\delta_1}(z_1) \times \{t\}$ and a zero in $B_{\delta_2} \times \{t\}$ is less than $\frac{5}{2}\delta_1$ which is in the contradiction with the construction of V_1 and V_2.

Assume now that the above constructed neighborhoods V_1 and V_2 of two zeros of φ have zeros of φ in common. As above, we lift the situation to the universal cover. Let $B_{\frac{\delta_1}{n+1}}(z_1) \times \{t\}$ and $B_{\frac{\delta_2}{n+1}}(z_2) \times \{t\}$ contain common zeros of $\tilde{\varphi}(z,t)$ for a fixed $t \in T$ and $\delta_1 \geq \delta_2$. All zeros of $\tilde{\varphi}(z,t)$ in $B_{\frac{\delta_2}{n+1}}(z_2) \times \{t\}$ are contained in $B_{\frac{\delta_1}{n+1}}(z_1) \times \{t\}$ because $[B_{\delta_1}(z_1) - B_{\frac{\delta_1}{n+1}}(z_1)] \times \{t\}$ does not contain any zeros of $\tilde{\varphi}(z,t)$. This holds for all leaves of the intersection, because the inner disks always contain zeros of $\tilde{\varphi}$.

The set of zeros of $\varphi(z,t)$ in X is closed and consequently, compact. The above constructed neighborhoods make a covering of the set of zeros. By the compactness, there exists a finite subcovering of the set of zeros of φ. Neighborhoods of our finite covering can intersect each other. Every time two neighborhoods intersect, on each leaf of intersection one neighborhood contains all the zeros of φ that are in the corresponding leaf of the other neighborhood. We erase leaves of the neighborhood whose zeros are contained in the other. By this process of erasing leaves we may get at most finitely many neighborhoods out of one. Thus, we obtain again a finite covering $\{V_1, V_2, \dots, V_k\}$ by disjoint open neighborhoods of zeros of φ in X. Let $B_{\delta_i}(z_i) \times T^i$, $i = 1, 2, \dots, k$, be lifts in $\Delta \times T$ of V_i. Since $\{V_1, V_2, \dots, V_k\}$ covers the zeros of φ, the orbit of $\{B_{\delta_1}(z_1) \times T^1, B_{\delta_2}(z_2) \times T^2, \dots, B_{\delta_k}(z_k) \times T^k\}$ under all deck transformations $A_X \in G_X$ on $\Delta \times T$ covers each zero of $\tilde{\varphi}$ and each two sets in the orbit are disjoint.

We define $h_n(z,t)$ to be 0 in $B_{\frac{\delta_i}{n}}(z_i) \times T^i$, to be 1 on $(B_{\delta_i}(z_i) - B_{\frac{\delta_i}{n-1}}(z_i)) \times T^i$ and smoothly interpolate in $(B_{\frac{\delta_i}{n-1}}(z_i) - B_{\frac{\delta_i}{n}}(z_i)) \times T^i$ by a transversely constant interpolation, for $i = 1, 2, \dots k$. We extend $h_n(z,t)$ to orbits of $B_{\delta_i}(z_i) \times T^i$ under $A_X \in G_X$ by $h_n(A_X(z,t)) := h_n(z,t)$. Finally, define $h_n(z,t)$ to be equal to 1 in the complement of the orbits. Thus $h_n(z,t)$ is a smooth, deck transformations invariant function on $\Delta \times T$ which is locally constant in the transverse direction.

The Beltrami coefficient $\tilde{\nu}_n(z,t) = h_n(z,t)k\frac{|\tilde{\varphi}(z,t)|}{\tilde{\varphi}(z,t)}$ is smooth on leaves, continuous for the transversal variation in the C^∞-topology and invariant under the deck transformations. The Beltrami coefficient of $f^{k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}} \circ (f^{\tilde{\nu}_n})^{-1}$ is close to the identity in the universal Teichmüller space $\mathcal{T}(\Delta \times \{t\})$ of each leaf $\Delta \times \{t\}$, $t \in T$. This follows from Gardiner-Lakic [16, Theorem 3] and the fact that the modulus of each annulus $B_{\delta_i}(z_i) - B_{\frac{\delta_i}{n-1}}(z_i)$ is comparable to $\frac{1}{\log n}$. Consequently, the harmonic representative $\tilde{\nu}_n'(z,t)$ of the class

$$\Big[\Big(\frac{k\frac{|\tilde{\varphi}|}{\tilde{\varphi}} - \tilde{\nu}_n}{1 - k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}\bar{\tilde{\nu}}_n}\theta\Big) \circ (f^{\tilde{\nu}_n})^{-1}\Big]$$

with $\theta = \partial_z f^{\tilde{\nu}_n}/\overline{\partial_z f^{\tilde{\nu}_n}}$ exists on each leaf. The harmonic Beltrami coefficient $\tilde{\nu}_n'(z,t)$ is smooth on leaves and continuous in the transverse direction. It is invariant under deck transformations and $\sup_{(z,t)\in\Delta\times T} |\tilde{\nu}_n'(z,t)| \to 0$ as $n \to \infty$.

Let $\tilde{\mu}_n(z,t)$ be the Beltrami coefficient of $f^{\tilde{\nu}_n'} \circ f^{\tilde{\nu}_n}$. Clearly, $\tilde{\mu}_n(z,t)$ is smooth on leaves and continuous in the transverse direction in the C^∞-topology. Further, $\tilde{\mu}_n(z,t)$ is in same Teichmüller class as $k\frac{|\varphi|}{\varphi}$ on each leaf and it is invariant under deck transformations. Finally, $\tilde{\mu}_n(z,t)$ is converging to $k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}$ uniformly on compact subsets of the complement of zeros of $\tilde{\varphi}(z,t)$ in $\Delta\times T$ and $\sup_{(z,t)\in\Delta\times T} |\tilde{\mu}_n(z,t)| \to k$ as $n \to \infty$. $\square$

Let $k_0(\mu) = \inf_{\mu_1\in[\mu]} \|\mu_1\|_\infty$. We say that a sequence of smooth Beltrami coefficients μ_n is *minimizing* for the Teichmüller class $[\mu]$ if $\mu_n \in [\mu]$ and $\lim_{n\to\infty} \|\mu_n\|_\infty = k_0(\mu)$. A Beltrami coefficient $\mu_1 \in [\mu]$ is *extremal* if $\|\mu_1\|_\infty = k_0(\mu)$.

Theorem 5.1. Let $f : H_\infty \to X$ be a quasiconformal map and let $\varphi \neq 0$ be a holomorphic quadratic differential on X. Then the path $t\frac{|\varphi|}{\varphi}$, $-1 < t < 1$, of Teichmüller type Beltrami coefficients on X gives a geodesic (in the Teichmüller metric) through the point $[f] \in \mathcal{T}(H_\infty)$. In addition, any two points on this geodesic have no other geodesics connecting them.

Proof. To simplify the notation, we assume that $f : H_\infty \to X$ is the base point $[id : H_\infty \to H_\infty] = [0]$ of $\mathcal{T}(H_\infty)$ and that H_∞ has an arbitrary, not necessarily TLC, complex structure.

Fix $0 < k < 1$. By Proposition 5.1, there exists a sequence μ_n of smooth Beltrami coefficients Teichmüller equivalent to $k\frac{|\varphi|}{\varphi}$ on each leaf of X such that $\mu_n \to k\frac{|\varphi|}{\varphi}$ uniformly on compact subsets of the complement of zeros of φ in H_∞ and that $\|\mu_n\|_\infty \to k$ as $n \to \infty$. Let μ be a fixed Beltrami coefficient on X such that $[\mu] = [\mu_n]$. Then by the chain rule and by the Reich-Strebel inequality for the solenoid H_∞ (see Theorem 4.1), we obtain

$$\|\varphi\| \leq \int_{H_\infty} \frac{|1 - \mu_n \frac{\varphi}{|\varphi|}|^2}{1 - |\mu_n|^2} \frac{|1 + \mu \frac{\varphi}{|\varphi|}\theta|^2}{1 - |\mu|^2} |\varphi| dm$$

where $\theta = \frac{1 - \overline{\frac{\mu_n \varphi}{|\varphi|}}}{1 - \frac{\mu_n \varphi}{|\varphi|}}$ (see [28] and [14, page 120]). If we normalize φ such that $\|\varphi\| = 1$ and let $n \to \infty$ in the above inequality, we get

$$K \leq \int_{H_\infty} \frac{|1 + \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi| dm \tag{12}$$

where $K = \frac{1+k}{1-k}$.

We claim that $\|\mu\|_\infty \geq k$. To see this, assume on the contrary that $\|\mu\|_\infty < k$. Then, from (12), we obtain the double inequality

$$K \leq \int_{H_\infty} \frac{|1 + \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi| dm < K$$

which gives a contradiction. Thus $\|\mu\|_\infty \geq k$.

Since each $\mu_1 \in [\mu]$ satisfies $\|\mu_1\|_\infty \geq k$ we conclude that $k\frac{|\varphi|}{\varphi}$ is an extremal Beltrami coefficient.

Assume that ν_n is a minimizing sequence of smooth Beltrami coefficients Teichmüller equivalent to μ on H_∞. Namely, $\nu_n \in [\mu]$ and $\lim_{n\to\infty} \|\nu_n\|_\infty = k = k_0(\mu)$. Using the inequality (12), we get

$$K \leq \int_{H_\infty} \frac{1 + |\nu_n|}{1 - |\nu_n|} |\varphi| dm \leq \frac{1 + \|\nu_n\|_\infty}{1 - \|\nu_n\|_\infty} = K + \epsilon_n \tag{13}$$

where $\epsilon_n \to 0$ as $n \to \infty$. Since $\|\varphi\| = 1$, the above inequality implies that $|\nu_n|$ has a subsequence which converges to k in a bounded pointwise a.e. sense on X. To see this, it is enough to show that $\frac{1+|\nu_n|}{1-|\nu_n|}$ converges to K in the measure $\alpha(*) = \int_* |\varphi| dm$ since the convergence in the measure of functions in $L^1(H_\infty)$ implies the existence of a subsequence converging in the a.e. sense (see [29, page 74]). Assume on the contrary that $\frac{1+|\nu_n|}{1-|\nu_n|}$ does not converge to K in the measure α. Then there exist

$\delta > 0$ and a subsequence $\frac{1+|\nu_{n_i}|}{1-|\nu_{n_i}|}$ such that $\frac{1+|\nu_{n_i}|}{1-|\nu_{n_i}|} \leq K - \delta$ on sets $D_{n_i} \subset H_\infty$ with $\alpha(D_{n_i}) \geq \delta$. This implies that

$$\int_{H_\infty} \frac{1+|\nu_{n_i}|}{1-|\nu_{n_i}|}|\varphi| dm \leq K + \epsilon_{n_i}(1-\delta) - \delta^2$$

which, for i large enough, gives a contradiction with (13).

Denote the subsequence of $|\nu_n|$ which converges a.e. to k by $|\nu_n|$ again. Then inequality (12) gives

$$K \leq \int_{H_\infty} \frac{1+|\nu_n|^2 + 2Re(\nu_n \frac{\varphi}{|\varphi|})}{1-|\nu_n|^2}|\varphi| dm \leq K + \epsilon_n.$$

Similarly as above, $Re(\nu_n \frac{\varphi}{|\varphi|})$ converges to k in a.e. sense. Thus ν_n converges to $k\frac{|\varphi|}{\varphi}$ in a bounded pointwise a.e. sense.

Any minimizing sequence ν_n has a subsequence which converges pointwise a.e. to $k\frac{|\varphi|}{\varphi}$. Therefore, there is a unique extremal Beltrami coefficient $k\frac{|\varphi|}{\varphi}$ in the class $[\mu] \in \mathcal{T}(H_\infty)$ by the standard argument (see [14]).

To prove the uniqueness of the Teichmüller geodesic between $[0]$ and $[\mu]$ (where μ is Teichmüller equivalent to $k\frac{|\varphi|}{\varphi}$ on each leaf of H_∞), we use the idea of L. Zhong (see [33]). Assume that there exists another geodesic $\gamma(t)$, $0 \leq t \leq k$, connecting $[0]$ and $[\mu]$ different from $[\mu_n^t]$. Let $[\nu_1] = \gamma(t_1)$ be a point on $\gamma(t)$ which is not on $[\mu_n^t]$, where ν_1 is a smooth Beltrami coefficient on H_∞. Denote by ν_2 the Beltrami coefficient of $f^\mu \circ (f^{\nu_1})^{-1}$. Let ν_1^n and ν_2^n be minimizing sequences of smooth Beltrami coefficients equivalent to ν_1 and ν_2, respectively. Let ξ_n be the sequence of the Beltrami coefficients of maps $f^{\nu_2^n} \circ f^{\nu_1^n}$. Since $[\nu_1]$ is a point on the geodesic $\gamma(t)$, the sequence ξ_n is minimizing for the class $[\mu]$. By the above, ξ_n converges to $k\frac{|\varphi|}{\varphi}$ in a.e. sense.

Let $k_1 = \lim_{n\to\infty} \|\nu_1^n\|_\infty$ and let $k_2 = \lim_{n\to\infty} \|\nu_2^n\|_\infty$. Because $[\nu_1^n]$ is on the geodesic $\gamma(t)$, we get

$$d([0],[\mu]) = d([0],[\nu_1^n]) + d([\nu_1^n],[\xi_n]),$$

where d is the Teichmüller distance.

Since $d([\nu_1^n],[\xi_n]) = d([0],[\nu_2^n])$ we get

$$d([0],[\mu]) = d([0],[\nu_1^n]) + d([0],[\nu_2^n]).$$

From the above equality we obtain $k = \frac{k_1+k_2}{1+k_1k_2}$. By the chain rule

$$|\xi_n| = \Big| \frac{\nu_1^n + (\nu_2^n \circ f^{\nu_1^n})\theta}{1+\overline{\nu_1^n}(\nu_2^n \circ f^{\nu_1^n})\theta} \Big| \leq \frac{|\nu_1^n| + |\nu_2^n \circ f^{\nu_1^n}|}{1+|\nu_1^n||\nu_2^n \circ f^{\nu_1^n}|} \tag{14}$$

a.e. on H_∞, where $\theta = \overline{\partial_z f^{\nu_1^n}}/\partial_z f^{\nu_1^n}$.

We claim that $\lim_{n\to\infty} |\nu_1^n(p)| = k_1$ and $\lim_{n\to\infty} |\nu_2^n \circ f^{\nu_1^n}(p)| = k_2$ for almost all points p of the solenoid H_∞. Assume this is not true. Since ν_1^n and ν_2^n are minimizing, it follows that the sequences $|\nu_1^n(p)|$ and $|\nu_2^n(p)|$, for a.e. $p \in X$, have no accumulation points larger than k_1 and k_2, respectively. The remaining possibility is that there exists a set $D \subset H_\infty$ with $\alpha(D) > 0$ and a constant $\delta > 0$ such that $\liminf_{n\to\infty} |\nu_1^n(p)| \leq k_1 - \delta$ for $p \in D$, or similarly for $\nu_2^n \circ f^{\nu_1^n}$. By taking $\liminf_{n\to\infty}$ in (14) we obtain

$$k = \liminf_{n\to\infty} \Big| \frac{\nu_1^n + (\nu_2^n \circ f^{\nu_1^n})\theta}{1 + \overline{\nu_1^n}(\nu_2^n \circ f^{\nu_1^n})\theta} \Big| (p) \leq \liminf_{n\to\infty} \frac{|\nu_1^n| + |\nu_2^n \circ f^{\nu_1^n}|}{1 + |\nu_1^n||\nu_2^n \circ f^{\nu_1^n}|}(p) < k$$

for $p \in D$ which gives a contradiction.

Since $\lim_{n\to\infty} |\nu_1^n(p)| = k_1$, $\lim_{n\to\infty} |\nu_2^n \circ f^{\nu_1^n}(p)| = k_2$, and $k = \frac{k_1+k_2}{1+k_1k_2}$, the inequality (14) becomes an equality when $n \to \infty$. Namely,

$$\lim_{n\to\infty} \Big| \frac{\nu_1^n + (\nu_2^n \circ f^{\nu_1^n})\theta}{1 + \overline{\nu_1^n}(\nu_2^n \circ f^{\nu_1^n})\theta} \Big| = \lim_{n\to\infty} \frac{|\nu_1^n| + |\nu_2^n \circ f^{\nu_1^n}|}{1 + |\nu_1^n||\nu_2^n \circ f^{\nu_1^n}|} \qquad a.e.$$

which implies the existence of $s > 0$ such that $\nu_1^n = s(\nu_2^n \circ f^{\nu_1^n})\theta + \epsilon_n$ with $\epsilon_n \to 0$ as $n \to \infty$. By the chain rule

$$\xi_n = \frac{\nu_1^n + (\nu_2^n \circ f^{\nu_1^n})\theta}{1 + \overline{\nu_1^n}(\nu_2^n \circ f^{\nu_1^n})\theta}$$

and by $\xi_n \to k\frac{|\varphi|}{\varphi}$ in the a.e. sense as $n \to \infty$, we get that $\nu_1^n \to s_1 \cdot k\frac{|\varphi|}{\varphi}$ for fixed $0 < s_1 < 1$ in the a.e. sense as $n \to \infty$. Since all ν_1^n are Teichmüller equivalent on H_∞, it follows that ν_1^n are Teichmüller equivalent to $s_1 \cdot k\frac{|\varphi|}{\varphi}$ on each leaf. This is a contradiction with the fact that $[\nu_1^n]$ does not belong to the geodesic $[\mu_n^t]$. Consequently, there exists a unique Teichmüller geodesic between $[0]$ and $[\mu]$. □

Let H_∞ denote a fixed TLC complex solenoid. The points in $\mathcal{T}(H_\infty)$ which are represented by TLC smooth Beltrami coefficients are dense (see Sullivan [31]). Each TLC complex solenoid is obtained by the lift of the complex structure of a closed Riemann surface.

In the case of a closed Riemann surface, it is known that each Teichmüller class has a unique extremal Beltrami coefficient of Teichmüller type $k\frac{|\varphi|}{\varphi}$, where φ is a holomorphic quadratic differential on the surface. A holomorphic quadratic differential on a closed Riemann surface lifts to a transversely locally constant (TLC) holomorphic quadratic differential on H_∞. Together with the above theorem, we obtained:

Corollary 5.1. There exists a dense subset of points in $\mathcal{T}(H_\infty)$ such that any two can be connected by a unique Teichmüller geodesic. This dense subset is strictly larger than the set of all $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$ with X having a TLC complex structure. □

The set of pairs of points in $\mathcal{T}(H_\infty)$ that can be joined by a geodesic is larger than the set of TLC structures. The reason is that there exist holomorphic quadratic differentials on H_∞ which are not TLC (see Section 2.3). However, A. Epstein, V. Markovic and the author [12] have recently showed that not every Teichmüller class has a Teichmüller-type Beltrami coefficient representative.

An important corollary of Theorem 5.1 is the fact that the embedding of the Teichmüller space of any closed Riemann surface into $\mathcal{T}(H_\infty)$ is distance preserving for their Teichmüller metrics. We see this by noting that a Beltrami coefficient $k\frac{|\varphi|}{\varphi}$, with φ a holomorphic quadratic differential on a closed Riemann surface, lifts to the Beltrami coefficient $k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}$ on the solenoid H_∞, with $\tilde{\varphi}$ being the lift of φ on H_∞. Since $k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}$ is a generalized extremal Beltrami coefficient the distance in the lifting is preserved.

Corollary 5.2. An embedding of the Teichmüller space of a closed Riemann surface into $\mathcal{T}(H_\infty)$ induced by lifting complex structures on Riemann surfaces to TLC complex structures on H_∞ is an isometry for the corresponding Teichmüller metrics. $\square$

6. An Approximation Theorem for Holomorphic Quadratic Differentials

We prove a form of Bers' theorem on the density of rational functions in the space of holomorphic functions. We show that any $\varphi \in A(\Delta \times T)$ can be approximated in the L^1-norm by holomorphic functions in $A(\Delta \times T)$ which are rational and integrable on each leaf, and are locally constant in the transverse direction.

Theorem 6.1. Given $\varphi \in A(\Delta \times T)$ and $\epsilon > 0$, there exists $r \in A(\Delta \times T)$ such that

$$\int_{\Delta \times T} |\varphi - r| dxdydm < \epsilon,$$

where $r(z,t)$ is a rational holomorphic function on each leaf of $\Delta \times T$ which is locally constant in the transversal direction.

Proof. Since φ is uniformly bounded on $\Delta \times T$, it follows that $\varphi(\cdot, t) \in L^1(\Delta \times \{t\})$ for all $t \in T$. Let U_t be an open neighborhood of $t \in T$ consisting of all t_1 such that $\iint_\Delta |\varphi(z,t) - \varphi(z,t_1)| dxdy < \frac{\epsilon}{2}$. The family $\{U_t\}_{t \in T}$ covers T and we choose a finite subcover $\{U_{t_1}, U_{t_2}, \dots, U_{t_n}\}$. We arrange that the elements of the finite cover are pairwise disjoint by possibly decreasing some of them. For each t_i, we choose an integrable rational function $r_i(z)$ on $\widehat{\mathbb{C}}$ which is holomorphic on Δ such that

$$\iint_\Delta |\varphi(z,t_i) - r_i(z)| dxdy < \frac{\epsilon}{3}.$$

This choice is possible by the Bers' approximation theorem for the unit disk (see [17]). If $r_i(z)$ has zeros on the unit circle $S^1 \equiv \partial\Delta$ then we move the zeros to $\mathbb{C} - \bar{\Delta}$ to obtain a new rational function which is holomorphic in $S^1 \cup \Delta$, called $r_i(z)$ again, with

$$\iint_\Delta |\varphi(z,t_i) - r_i(z)| dxdy < \frac{\epsilon}{2}.$$

We define $r(z,t) = r_i(z)$, for $t \in U_i$. Note that $r(z,t)$ is locally constant in the transverse direction and it belongs to $A(\Delta \times T)$. By integrating the inequality

$$\iint_{\Delta \times \{t\}} |\varphi(z,t) - r(z,t)| dxdy < \epsilon$$

with respect to t, we get

$$\int_{\Delta \times T} |\varphi(z,t) - r(z,t)| dxdydm < \epsilon.$$

$\square$

Remark 6.1. Consider the universal covering $\mathbb{H}\times T$ of X, where $\mathbb{H}$ is the upper half-plane. The space $A(\mathbb{H}\times T)$ consists of all uniformly bounded holomorphic quadratic differentials on $\mathbb{H}\times T$ which are pull-backs of elements in $A(\Delta\times T)$. An element $\varphi\in A(\mathbb{H}\times T)$ is uniformly bounded and $|\varphi(z,t)|$ is of the order $|z|^{-4}$, as $|z|\to\infty$. The statement of Theorem 6.1 is unchanged if we replace $A(\Delta\times T)$ with $A(\mathbb{H}\times T)$.

7. Infinitesimal Beltrami differentials

We apply the Reich-Strebel inequality to study the infinitesimal structure of the Teichmüller space $\mathcal{T}(H_\infty)$. Recall that a smooth Beltrami differential μ is *infinitesimally trivial* if $\int_X \mu\varphi dm = 0$ for all holomorphic quadratic differentials $\varphi\in A(X)$. By Proposition 3.2, μ is infinitesimally trivial if and only if $\int_X \mu\varphi dm = 0$, for all $\varphi\in A^1(X)$. The following theorem is completely analogous to the case of a Riemann surface (see [14]).

Theorem 7.1. A smooth Beltrami differential ν on the solenoid X is infinitesimally trivial if and only if there exists a holomorphic curve μ_s of Teichmüller trivial smooth Beltrami coefficients such that $\mu_s = s\nu + O(s^2)$ uniformly on X.

Proof. Suppose $\mu_s = s\nu + O(s^2)$ is a holomorphic curve of Teichmüller trivial smooth Beltrami coefficients, where ν is smooth. The Reich-Strebel inequality is equivalent to

$$\Big|Re\int_X \frac{\mu_s\varphi}{1-|\mu_s|^2}dm\Big| \le \int_X \frac{|\mu_s|^2|\varphi|}{1-|\mu_s|^2}dm.$$

From $\mu_s = s\nu + O(s^2)$ and the above inequality we obtain

$$\Big|Re\int_X s\nu\varphi dm\Big| \le O(s^2)$$

for all $\varphi\in A(X)$ and for all s small, which shows that ν is infinitesimally trivial.

To show the converse, suppose that ν is infinitesimally trivial. Let $\tilde\nu$ be the lift of ν to the universal cover $\mathbb{H}\times T$ of X, where $\mathbb{H}$ is the upper half-plane. We define the Bers map Φ for the solenoid X (for a similar construction see Sullivan [31]). It maps smooth Beltrami coefficients on $\mathbb{H}\times T$ onto holomorphic functions on $\mathbb{H}_-\times T$, where $\mathbb{H}_-$ is the lower half plane. Given a smooth Beltrami coefficient $\tilde\mu$ on $\mathbb{H}\times T$, $\Phi(\tilde\mu)$ is defined on each leaf by taking the Schwarzian derivative in $\mathbb{H}_-\times T$ of the solution of the Beltrami equation with the Beltrami coefficient $\tilde\mu$ on $\mathbb{H}\times T$ and 0 on $\mathbb{H}_-\times T$. Since $A_X^*(\tilde\mu)=\tilde\mu$ for $A_X\in G_X$, then $A_X^*(\Phi(\tilde\mu))=\Phi(\tilde\mu)$.

Consider $s\in\mathbb{C}$ such that $|s|<\frac{1}{\|\nu\|_\infty}$. Then $\tilde\mu_s = s\tilde\nu$ is a smooth Beltrami coefficient on $\mathbb{H}\times T$. Let μ_s be the projection of $\tilde\mu_s$ on X. We show that the derivative $\frac{d}{ds}\Phi(\tilde\mu_s)|_{s=0} = \dot\Phi(\tilde\nu)$ satisfies $\dot\Phi(\tilde\nu)=0$. It is enough to show that $\iint_{\mathbb{H}\times\{t\}}\frac{\tilde\nu(z,t)}{(z-w)^4}dxdy = 0$ for all $t\in T$ and for all $w\in\mathbb{H}_-$ by the Bers' representation formula [2]. Assume on the contrary that there exists $t_0\in T$ and $w_0\in\mathbb{H}_-$ such that $\iint_{\mathbb{H}\times\{t_0\}}\frac{\tilde\nu(z,t)}{(z-w_0)^4}dxdy = d\neq 0$. Fix ϵ such that $0<\epsilon<\frac{|d|}{2}$. There exists a neighborhood U_{t_0} of t_0 such that $\iint_{\mathbb{H}\times\{t\}}|\frac{\tilde\nu(z,t_0)-\tilde\nu(z,t_1)}{(z-w_0)^4}|dxdy<\epsilon$ for all $t_1\in U_{t_0}$. We choose U_{t_0} to be homeomorphic to T. Consider a bounded holomorphic function $\phi(z,t)=\frac{1}{(z-w_0)^4}$ for $t\in U_{t_0}$ and for a fixed $w_0\in\mathbb{H}_-$, and $\phi(z,t)=0$ for

$t \in T - U_{t_0}$. Then

$$\Big| \int_{\mathbb{H}\times T} \tilde{\nu}(z,t)\phi(z,t)dxdydm \Big| \geq \frac{|d|}{2} m(U_{t_0}) > 0.$$

We show that the above inequality is impossible. The holomorphic function ϕ satisfies conditions of Remark 6.1. By Theorem 3.1, the holomorphic quadratic differential $\tilde{\varphi}(z,t) = \Theta(\phi)(z,t)$ projects to an element of $A(X)$, where Θ is the commensurable theta series for X. Let F be the fundamental domain for X on $\mathbb{H}\times T$. Then $\int_{\mathbb{H}\times T} \tilde{\nu}\phi(z,t)dxdydm = \int_F \tilde{\nu}\Theta(\phi)dxdydm = \int_F \tilde{\nu}\tilde{\varphi}dxdydm \neq 0$. This contradicts the fact that $\tilde{\nu}$ is infinitesimally trivial. The contradiction proves that $\dot{\Phi}(\tilde{\nu}) = 0$.

Let $\tilde{\varphi}^s(z,t) = \Phi(\tilde{\mu}_s)(z,t)$, for $z \in \mathbb{H}_-$ and let $\tilde{\xi}_s(z,t) = -2y^2\tilde{\varphi}^s(\bar{z},t)$, for $z \in \mathbb{H}$. Then $\Phi(\tilde{\xi}_s) = \Phi(\tilde{\mu}_s)$ by [2], which implies that $\tilde{\mu}_s$ is Teichmüller equivalent to $\tilde{\xi}_s$ on each leaf of $\mathbb{H}\times T$. Note that $\tilde{\varphi}^s$ and $\tilde{\xi}_s$ are invariant under covering maps. Since $\dot{\Phi}(\tilde{\nu}) = 0$ and by the holomorphicity of Φ, we get that $\|\tilde{\xi}_s\|_\infty \leq Cs^2$ for a fixed constant C. Let $\tilde{\eta}_s$ be the Beltrami coefficient of the quasiconformal map $(f^{\tilde{\xi}_s})^{-1} \circ f^{\tilde{\mu}_s}$ on $\mathbb{H}\times T$. The Beltrami coefficient $\tilde{\eta}_s$ is trivial on each leaf $\mathbb{H}\times T$. Further, $\tilde{\eta}_s$ is invariant under deck transformations and it satisfies $\tilde{\eta}_s = s\tilde{\nu} + O(s^2)$ (see [14]). Thus, it projects to a smooth Beltrami coefficient η_s on X with $\eta_s = s\nu + O(s^2)$. $\square$

Remark. Sullivan [31] defines a Beltrami coefficient ν to be infinitesimally trivial if $\int_X \nu\varphi = 0$ for all integrable quadratic differentials φ which are holomorphic on almost all leaves of X and carry transverse measures as a part of their definition. Our result shows that it is enough to consider the space $A(X)$ of transversely continuous holomorphic quadratic differentials with a fixed transverse measure (Haar measure) for the definition of the infinitesimally trivial Beltrami differentials. The space $A(X)$ of holomorphic quadratic differentials which are continuous for the transversal variation is not complete in the L^1-norm.

Denote by $L^\infty_s(X)$ the space of all smooth Beltrami differentials on X. The space $N(X)$ of infinitesimally trivial smooth Beltrami differentials consists of all $\nu \in L^\infty_s(X)$ such that $\int_X \nu\varphi dm = 0$, for all $\varphi \in A(X)$. The above theorem identifies the space of tangent vectors at $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$ with $L^\infty_s(X)/N(X)$.

However, the situation is not as nice as in the case of the Teichmüller space of a Riemann surface. Namely, $L^\infty_s(X)$ and $N(X)$ are not Banach spaces, because the smoothness is not preserved in the limit for the essential supremum (L^∞) topology. We show that the quotient space $L^\infty_s(X)/N(X)$ is Banach in the quotient norm (see corollary below).

In section 2, we introduced a continuous linear map $P : L^\infty_s(X) \to A(X)$, where $A(X)$ is equipped with the Bers norm. We show that this map induces a linear isomorphism $\bar{P}$ from the tangent space at the point $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$ onto $A(X)$.

Corollary 7.1. *The map $P : L^\infty_s(X) \to A(X)$ induces a continuous linear isomorphism from the normed space $L^\infty_s(X)/N(X)$ onto the Banach space $A(X)$ equipped* *with the Bers norm. Consequently, the tangent space* $L_s^\infty(X)/N(X)$ *at any point* $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$ *is a Banach space.*

Proof. As in the classical case of a Riemann surface (for example, see [17]), the linear map P has norm less than or equal to 3 for the essential supremum norm on $L_s^\infty(X)$ and for the Bers norm on $A(X)$.

It is enough to show that the kernel of P equals to $N(X)$. Note that P is a constant multiple of the derivative of the Bers map Φ pre- and post-composed with the inversion in the real line for the z-variable. In the proof of Theorem 7.1, we showed that $N(X)$ is a subset of the kernel of $\dot{\Phi}$.

It remains to show that if $\dot{\Phi}(\tilde{\nu}) = 0$ for the lift $\tilde{\nu}$ of $\nu \in L_s^\infty(X)$ then $\nu \in N(X)$. The fact that $\dot{\Phi}(\tilde{\nu}) = 0$ is equivalent to $\iint_{\mathbb{H}\times\{t\}} \frac{\tilde{\nu}(z,t)}{(z-w)^4} dxdy = 0$ for all $w \in \mathbb{H}_-$ and for all $t \in T$. By integrating the above equation three times with respect to w, we get

$$\iint_{\mathbb{H}\times\{t\}} \frac{w(w-1)\tilde{\nu}(z,t)}{z(z-1)(z-w)} dxdy = 0$$

for all $w \in \mathbb{H}_-$ and for all $t \in T$. We divide T into finitely many closed subsets T^i, $i = 1, 2, \ldots, k$ and form constant rational functions on $\Delta \times T^i$, $i = 1, 2, \ldots, k$, using linear combinations of $\varphi_w(z,t) = \frac{w(w-1)}{z(z-1)(z-w)}$, for $t \in T^i$. The above identity generalizes by the linearity to

$$\iint_{\mathbb{H}\times\{t\}} \tilde{\nu}(z,t)r(z,t)dxdy = 0,$$

for all $t \in T$, where $r(z,t)$ is any integrable rational holomorphic function on $\mathbb{H} \times T$ which is locally constant in the transverse direction.

By integrating the above identity with respect to the Haar measure λ_X, we get

$$\int_{\mathbb{H}\times T} \tilde{\nu}(z,t)r(z,t)dm = 0,$$

for any rational $r \in A(\mathbb{H} \times T)$ which is locally constant in the transverse direction.

We change the upper half plane $\mathbb{H}$ to the unit disk Δ. Then

$$\int_{\Delta\times T} \tilde{\nu}(z,t)r(z,t)dxdydm = 0,$$

where $r(z,t)$ is a locally constant, rational element of $A(\Delta \times T)$.

Then

$$\int_F \tilde{\nu}\Theta(r)dxdydm = \int_{\Delta\times T} \tilde{\nu}(z,t)r(z,t)dxdydm = 0,$$

where $\Theta(r)$ is the Poincaré theta series for r and F is a fundamental set for X on $\Delta \times T$. Since Θ is a continuous surjective map from $A(\Delta \times T)$ onto $A(X)$ and by Theorem 6.1, we get that $\int_X \nu\varphi dm = 0$ for all φ in a dense subset of $A(X)$. Consequently, $\int_X \nu\varphi dm = 0$ for all $\varphi \in A(X)$. Thus $\nu \in N(X)$. □

Remark. The pairing $\int_X \nu\varphi dm = 0$ identifies elements of $L_s^\infty(X)/N(X)$ with elements of the dual $A^*(X)$ of $A(X)$ with respect to the L^1-norm. Note that $(A^1(X))^* = A^*(X)$ by Proposition 3.2. In the Teichmüller theory of Riemann surfaces, it is standard that the dual to the space of integrable holomorphic quadratic differentials is isometrically identified with the tangent space.

Acknowledgement. The following theorem is due to V. Markovic.

Theorem 7.2. The dual $A^*(X)$ of $A(X)$ is strictly larger than the tangent space at $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$.

Proof. Choose $\psi \in A^1(X) - A(X)$. We define a continuous linear functional $l_\psi : A(X) \to \mathbb{C}$ by $l_\psi(\varphi) = \int_X \rho^{-2}\bar{\psi}\varphi dm$. Clearly, $l_\psi \neq 0$ because $l_\psi(\psi) \neq 0$. There exists a Beltrami differential μ on X (not necessarily smooth or the limit of smooth Beltrami differentials) such that $l_\psi(\varphi) = \int_X \mu\varphi dm$. The differential μ can be chosen to be a smooth Beltrami differential if and only if l_ψ represents a tangent vector.

Assume that μ is smooth. Then $\psi_1 = \bar{P}(\mu) \in A(X)$ and $l_{\psi_1} = l_\psi$. This implies that $\psi_1 = \psi$ almost everywhere. But this is a contradiction with our choice of ψ. Therefore $l_\psi \in A^*(X)$ does not represent a tangent vector. □

8. Extremal maps

A family of K-quasiconformal maps of the complex solenoid X onto another complex solenoid is not necessarily compact (as far as we know the first example of such family is obtained by Adam Epstein). Therefore, the existence of an extremal Beltrami coefficient in an arbitrary Teichmüller class is not guaranteed. Nonetheless, we give a necessary and sufficient condition for a Beltrami coefficient to be extremal. This condition is of particular interest because of the recent result of A. Epstein, V. Markovic and the author [12] which shows that not every Teichmüller class has a Teichmüller-type Beltrami coefficient representative (which is necessarily uniquely extremal).

We say that smooth Beltrami differentials ν and ν_1 on X are in the same *infinitesimal class* if $\nu - \nu_1 \in N(X)$ (see Theorem 7.1). The *norm* of the infinitesimal class of ν is $k_1(\nu) = \inf_{\nu_1} \|\nu_1\|_\infty$, where the infimum is over all ν_1 in the infinitesimal class of ν. A Beltrami differential $\nu \in L_s^\infty(X)$ is *infinitesimally extremal* if $\|\nu\|_\infty = k_1(\nu)$.

A Beltrami differential μ on X defines a linear functional on $A(X)$ via the natural pairing $\int_X \mu\varphi dm$, for $\varphi \in A(X)$. By the definition, Beltrami differentials in the same infinitesimal class define the same linear functional on $A(X)$. The norm of the linear functional on $A(X)$ given by a smooth Beltrami differential μ is less than or equal to $\|\mu\|_\infty$. Thus the norm of the linear functional on $A(X)$ given by μ is less than or equal to the norm $k_1(\mu)$ of the infinitesimal class of μ.

For a Riemann surface, the Reich-Strebel-Hamilton-Krushkal theorem says that a Beltrami coefficient is extremal if and only if it is infinitesimally extremal. We show that the similar result is true for a complex solenoid X.

Theorem 8.1. A Beltrami coefficient μ on the solenoid X is Teichmüller extremal if and only if it is infinitesimally extremal.

Proof. We develop two inequalities from the Reich-Strebel inequality (analogous to [15]) by following the exposition in [17, pages 101 and 102]. For the moment we do not assume the existence of Teichmüller extremal or infinitesimally extremal Beltrami coefficients in the given Teichmüller or infinitesimal class. The inequalities that we obtain are independent of this existence.

Let μ be a smooth Beltrami coefficient on the solenoid X. Let $K_0 = \frac{1+k_0}{1-k_0}$, where $k_0 = k_0(\mu) = \inf_{\mu_1 \in [\mu]} \|\mu_1\|_\infty$. We choose a minimizing sequence $\mu_n \in [\mu]$ and define $K_n = \frac{1+k_n}{1-k_n}$, where $k_n = \|\mu_n\|_\infty$. Then $K_n \to K_0$ as $n \to \infty$. Since μ and μ_n are in the same Teichmüller class, we can apply the Reich-Strebel inequality to the Beltrami coefficient of the Teichmüller trivial map $(f^{\mu_n})^{-1} \circ f^\mu$. Using the chain rule for the Beltrami coefficient of $(f^{\mu_n})^{-1} \circ f^\mu$ in the Reich-Strebel inequality, we obtain

$$1 \leq \int_X \frac{|1 - \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} \cdot \frac{|1 + \mu_n \theta \frac{\varphi}{|\varphi|}|^2}{1 - |\mu_n|^2} |\varphi| dm \tag{15}$$

where $\theta = \frac{1 - \overline{\mu \varphi}/|\varphi|}{1 - \mu\varphi/|\varphi|}$, $\varphi \in A(X)$ and $\|\varphi\| = \int_X |\varphi| dm = 1$.

The second fraction on the right side in (15) is less than or equal to K_n. By substituting K_n in (15) and letting $n \to \infty$, we obtain the first inequality

$$\frac{1}{K_0} \leq \int_X \frac{|1 - \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi| dm \tag{16}$$

where $\varphi \in A(X)$ and $\|\varphi\| = 1$.

To obtain the second inequality, assume for the moment that $[\mu] \in \mathcal{T}(H_\infty)$ has a Teichmüller type Beltrami coefficient representative $k_0 \frac{|\varphi_1|}{\varphi_1}$, $0 < k_0 < 1$. If we replace φ by φ_1, μ by $k_0 \frac{|\varphi_1|}{\varphi_1}$ and μ_n by μ in (15), we obtain

$$K_0 \leq \int_X \frac{|1 + \mu \frac{\varphi_1}{|\varphi_1|}|^2}{1 - |\mu|^2} |\varphi_1| dm$$

where $K_0 = \frac{1+k_0}{1-k_0}$. This implies the second inequality

$$K_0 \leq \sup_{\|\varphi\|=1} \int_X \frac{|1 + \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi| dm \tag{17}$$

for Beltrami coefficients with a Teichmüller type representative in their Teichmüller classes.

Not all points have Teichmüller type representatives. However, if $[f : H_\infty \to X]$ represents a TLC complex structure, then the set of points $[g : H_\infty \to Y]$ in $\mathcal{T}(H_\infty)$ such that $g \circ f^{-1}$ has a Teichmüller representative is dense in $\mathcal{T}(H_\infty)$. By an approximation argument (see [17, page 102]), the inequality (17) is correct for all μ on any X with TLC complex structure. To show (17) for the solenoid X with non-TLC complex structure, a continuity argument is needed. Since TLC-complex structures approximate non-TLC complex structures and their corresponding commensurable theta series approximate theta series for non-TLC complex solenoid (17) holds everywhere. A similar approximation argument is given in the proof of Theorem 4.1 (see also [14]).

The rest of the proof follows the steps in the proof of the principle of Teichmüller contraction (see [17, page 103] or [15]). Namely, using only formal manipulations of inequalities (16) and (17) we obtain the inequality

$$C_1(\|\mu\|_\infty - k_0) \leq \|\mu\|_\infty - \sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm \leq C_2(\|\mu\|_\infty - k_0) \tag{18}$$

where $\varphi \in A(X)$, $k_0 = \inf_{\mu_1 \in [\mu]} \|\mu_1\|_\infty$ and C_1 and C_2 are positive constants which depend only on $\|\mu\|_\infty$. A version of the inequality (18) for Riemann surfaces and Beltrami coefficients of constant absolute value appears first in [28].

From inequality (18), it follows that $\|\mu\|_\infty = \sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$ if and only if $\|\mu\|_\infty = k_0$. Since the norm $k_1(\mu)$ of the infinitesimal class of μ is greater than or equal to the norm of the linear functional on $A(X)$ defined by μ, we conclude that $\|\mu\|_\infty = \sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$ implies that μ is infinitesimally extremal.

Given a TLC complex solenoid X and a locally transversely constant Beltrami differential μ on X, the norm $k_1(\mu)$ of the infinitesimal class of μ is equal to $\sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$ by the corresponding equality for closed Riemann surfaces. Since the infinitesimal classes of Beltrami differentials which are locally constant in the transverse direction are dense in the tangent space of $\mathcal{T}(H_\infty)$ at X, a continuity argument shows that the above holds for all Beltrami differentials on X. Further, a continuity argument similar to the one in Proposition 8.1 below shows that $k_1(\mu) = \sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$ for non-TLC complex solenoids X as well. Thus the theorem follows. □

The above theorem states that the Teichmüller and the infinitesimal extremality are equivalent. However, we do not know whether each (Teichmüller or infinitesimal) class has an (generalized) extremal representative. By [12], there are Teichmüller and infinitesimal classes without Teichmüller-type Beltrami coefficient representatives (which are uniquely extremal) but the existence of extremal maps of a different type is not excluded.

The quantity $\|\mu\|_\infty - k_0(\mu)$ measures how much μ is away from being extremal in its Teichmüller class and the quantity $\|\mu\|_\infty - \sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$ measures how much μ is away from being Riesz representative of the functional $Re \int_X \mu\varphi dm$ on $A(X)$. In the proof of Theorem 8.1 (see (18)) we also showed:

Corollary 8.1. Let $0 < k < 1$ be fixed. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1(\|\mu\|_\infty - k_0(\mu)) \leq \|\mu\|_\infty - \sup_{|\varphi\|=1} Re \int_X \mu\varphi dm \leq C_2(\|\mu\|_\infty - k_0(\mu))$$

for any smooth Beltrami coefficient μ on X with $\|\mu\|_\infty \leq k$. □

This fact is known as the Teichmüller contraction principle (see [15] or [17, page 103]). Note that even we do not know whether there is an extremal map in each Teichmüller class the principle of Teichmüller contraction still holds.

In the course of the proof of Theorem 8.1 we also showed the following:

Corollary 8.2. The norm $k_1(\mu)$ of the infinitesimal class of a smooth Beltrami differential μ on the solenoid X is equal to $\sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm$, where $\varphi \in A(X)$.

As a consequence of the inequalities (16) and (17) we obtain the infinitesimal form of the Teichmüller metric similarly to the Teichmüller space of a Riemann surface (see [28] and [17, page 106]).

Theorem 8.2. Let μ be a smooth Beltrami coefficient on the solenoid X. Then the Teichmüller distance satisfies

$$d([0],[t\mu]) = t\Big(\sup_{\|\varphi\|=1} Re \int_X \mu\varphi dm\Big) + O(t^2)$$

for $t>0$ as $t\to 0$, where $\varphi\in A(X)$. □

9. The Kobayashi Metric

The Teichmüller space $\mathcal{T}(H_\infty)$ of the solenoid H_∞ is embedded via the Bers map into an open subset of the complex Banach space $A(X)$ equipped with the Bers norm. This embedding gives a global chart for the complex structure on $\mathcal{T}(H_\infty)$. The projection map $\Pi : M(H_\infty)\to\mathcal{T}(H_\infty)$ is a holomorphic surjection, where $M(H_\infty)$ is the unit ball in $L_s^\infty(H_\infty)$ and $\Pi(\mu)=[\mu]$. For details see Sullivan [31].

The Kobayashi pseudometric δ is the largest pseudometric which makes all holomorphic maps from the unit disk Δ into $\mathcal{T}(H_\infty)$ weakly contracting with respect to the Poincaré metric on Δ (for more details, see [14], [17], [27]). We show that the Kobayashi pseudometric δ is equal to the Teichmüller metric d. This fact is known for Teichmüller spaces of Riemann surfaces [27], [14].

The argument in [14, page 138] gives that $\delta([\mu],[\nu])\le d([\mu],[\nu])$ for each $[\mu],[\nu]\in\mathcal{T}(H_\infty)$. It remains to show the opposite inequality.

Theorem 9.1. On the Teichmüller space $\mathcal{T}(H_\infty)$ of the universal hyperbolic solenoid H_∞, the Kobayashi pseudometric δ equals the Teichmüller metric d. In particular, the Kobayashi pseudometric is a metric.

Proof. We need to show the inequality $d([\mu],[\nu])\le\delta([\mu],[\nu])$ for $[\mu],[\nu]\in\mathcal{T}(H_\infty)$. It is enough to show it on a dense subset of $\mathcal{T}(H_\infty)\times\mathcal{T}(H_\infty)$ by continuity of pseudometrics. The dense subsets on which we show the inequality is the set of all pairs of TLC complex structures.

We define a holomorphic projection of $\mathcal{T}(H_\infty)$ onto $\mathcal{T}(\Delta/G_n)$, where G_n is a finite index subgroup of G. Since $H_\infty=\Delta\times_G\widehat{G}$, we can lift any Beltrami differential μ on H_∞ to a Beltrami differential $\tilde{\mu}$ on $\Delta\times\widehat{G}$. Then we define a Beltrami coefficient $\tilde{\mu}_n$ on Δ to be the restriction of $\tilde{\mu}$ to a fundamental polyhedron ω_n for G_n on the leaf $\Delta\times\{id\}$. Extend the Beltrami coefficient $\tilde{\mu}_n$ to Δ by the push forward with elements of G_n. The projection μ_n of $\tilde{\mu}_n$ to Δ/G_n represents a point in the Teichmüller space $\mathcal{T}(\Delta/G_n)$. This defines a surjective holomorphic map from $\mathcal{T}(H_\infty)$ onto $\mathcal{T}(\Delta/G_n)$.

Let $[\mu],[\nu]\in\mathcal{T}(H_\infty)$ be lifts of points $[\mu_1],[\nu_1]\in\mathcal{T}(\Delta/G_n)$. Denote by δ_1 the Kobayashi metric and by d_1 the Teichmüller metric on $\mathcal{T}(\Delta/G_n)$. It is known that $d_1=\delta_1$. Since holomorphic maps weakly contract in the corresponding Kobayashi metrics, it follows that $\delta_1([\mu_1],[\nu_1])\leq\delta([\mu],[\nu])$. By Corollary 5.2, we know that $d([\mu],[\nu])=d_1([\mu_1],[\nu_1])$ because $[\mu],[\nu]\in\mathcal{T}(H_\infty)$ represent TLC complex structures. From $d([\mu],[\nu])=d_1([\mu_1],[\nu_1])=\delta_1([\mu_1],[\nu_1])$ and the above inequality we obtain

$$d([\mu],[\nu])\leq\delta([\mu],[\nu]).$$

The above inequality holds for an arbitrary pair of points in $\mathcal{T}(H_\infty)$ which are lifts of points in $\mathcal{T}(\Delta/G_n)$. Thus we obtain the inequality $d\leq\delta$ on the set of all pairs of points with TLC complex structures. These are dense in $\mathcal{T}(H_\infty)\times\mathcal{T}(H_\infty)$ and the theorem follows. □

A direct consequence of the above theorem is that any biholomorphic isomorphism of $\mathcal{T}(H_\infty)$ is an isometry for the Teichmüller metric.

10. Conclusions and Open Problems

In the Teichmüller spaces of finite surfaces every two points are connected by a unique geodesic determined by the Beltrami coefficient of the Teichmüller-type. In the infinite-dimensional non-separable Teichmüller spaces of open (geometrically infinite) Riemann surfaces every two points are connected by a geodesic while the uniqueness of the geodesic is false. The Teichmüller space $\mathcal{T}(H_\infty)$ of the universal hyperbolic solenoid H_∞ is an infinite-dimensional separable Banach manifold. However, we do not know whether there is a geodesic connecting any two points in $\mathcal{T}(H_\infty)$ due to the non-compactness of a family of K-quasiconformal maps.

We showed (Theorem 5.1) that there is dense set in $\mathcal{T}(H_\infty)\times\mathcal{T}(H_\infty)$ connected by unique geodesics described by holomorphic quadratic differentials on H_∞ (Teichmüller-type Beltrami coefficients determine these unique geodesics). This dense set contains all pairs of marked TLC complex structures on H_∞, but it is strictly larger. A. Epstein, V. Markovic and the author [12], showed that not every Teichmüller class has a Teichmüller-type Beltrami coefficient representative. However, it is possible to have extremal maps of a different type (and even if extremal maps do not exist, it is still possible to have geodesics) and we ask:

Question 1. Are any two points in $\mathcal{T}(H_\infty)$ connected by a (unique) geodesic?

We showed that infinitesimally extremal is equivalent to being extremal in the Teichmüller class of maps (Theorem 8.1). However, if any infinitesimal class has an extremal representative it is not guaranteed that any Teichmüller class has an extremal representative due again to the non-compactness of a family of K-quasiconformal maps. However, an implicit function argument could presumably be used to give a positive answer locally.

The Teichmüller space $\mathcal{T}(H_\infty)$ has a natural complex structure which induces the Kobayashi metric. We proved that the Kobayashi metric equals the Teichmüller metric on $\mathcal{T}(H_\infty)$.

Question 2. Is it true that each complex biholomorphic map of $\mathcal{T}(H_\infty)$ is geometric, i.e. comes from a self-map of H_∞?

Since the Kobayashi metric equals the Teichmüller metric, one needs to show that a linear isometry of the tangent spaces at X and Y produces a linear isometry of $A(X)$ and $A(Y)$ in the L^1-norm and that such isometry induces a complex isomorphism between X and Y. This is an outline due to Royden for closed surfaces which is extended to arbitrary Riemann surfaces by several authors [27],[10], [9], [18], [11] and [20]. The second step seems easier to extend due to the analysis in [20]. The first step seems harder. In the finite surface case, it follows by the duality between integrable holomorphic differentials and Beltrami differentials on a surface. For geometrically infinite surfaces, this was obtained by producing a predual to the space of integrable holomorphic quadratic differentials which consists of asymptotically trivial Beltrami differentials [9]. Since H_∞ is compact such space does not exist.

The modular group $Mod(H_\infty)$ consists of baseleaf preserving quasiconformal maps of H_∞ up to homotopy [24] and it acts on $\mathcal{T}(H_\infty)$.

Question 3. Is true that the action of the modular group $Mod(H_\infty)$ has dense orbits?

It is known that this action has orbits with accumulation points [21], but it is much harder to completely understand the quotient space $\mathcal{T}(H_\infty)/Mod(H_\infty)$. This question is equivalent to the conjecture of Ehrenpreis which states that any two closed Riemann surfaces have large finite covers that are almost conformal [5] (Sullivan made this observation).

There was a recent progress in understanding the quotient $\mathcal{T}(H_\infty)/Mod(H_\infty)$ for the case of the punctured solenoid. Namely, V. Markovic and the author [22] showed that the orbit of the basepoint in $T(H_\infty)$ accumulates to the basepoint and that the closure of the orbit is uncountable. In [21] we take the orbit of a non-TLC points and show that they accumulate on points not in the orbit, while in [22] we consider the orbit of a special TLC point which is a substantial progress.

It is also interesting to understand properties of the modular group. For example, any finite subgroup is cyclic [21]. Another question is to understand the dynamics of their action on $\mathcal{T}(H_\infty)$ similar to the closed surface case [32], [13], [3]. A positive answer to Question 1 could help in the view of the approach of Bers [3].

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