LIMITS OF TEICHMÜLLER GEODESICS IN THE UNIVERSAL TEICHMÜLLER SPACE

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Abstract. Thurston's boundary to the universal Teichmüller space $T(\mathbb{H})$ is the set of asymptotic rays to the embedding of $T(\mathbb{H})$ in the space of geodesic currents; the boundary is identified with the projective bounded measured laminations $PML_{bdd}(\mathbb{H})$ of $\mathbb{H}$. We prove that each Teichmüller geodesic in $T(\mathbb{H})$ has a unique limit point in the Thurston's boundary to $T(\mathbb{H})$ unlike in the case of closed surfaces.

1. Introduction

The Teichmüller space $T(\mathbb{D})$ of the unit disk $\mathbb{D}$, called the *universal* Teichmüller space, consists of all quasisymmetric maps $h : S^1 \to S^1$ which fix 1, i and -1 on the unit circle S^1. The Teichmüller space of an arbitrary hyperbolic surface embeds in $T(\mathbb{D})$ as a complex Banach submanifold. Some properties of Teichmüller spaces can be easier explained by studying the universal Teichmüller space $T(\mathbb{D})$. One example where studying universal Teichmüller space $T(\mathbb{D})$ simplifies the considerations is the closure of Teichmüller spaces in terms of degenerations of hyperbolic structures-the Thurston's boundary.

Thurston's boundary to the universal Teichmüller space $T(\mathbb{D})$ is identified with the space of projective bounded measured laminations $PML_{bdd}(\mathbb{D})$ of the unit disk $\mathbb{D}$, where $\mathbb{D}$ is identified with the hyperbolic plane (cf. [?], [?]). Bonahon [?] used geodesic currents to give an alternative description of Thurston's boundary to Teichmüller spaces of closed surfaces. Namely, the Teichmüller space of a closed surface embeds into the space of geodesic currents (equipped with the weak* topology) and the asymptotic rays to the image of the embedding are boundary points of the Teichmüller space (cf. [?]). The universal Teichmüller space $T(\mathbb{D})$ embeds into the space of geodesic currents of $\mathbb{D}$ when an appropriate topology on geodesic currents is introduced (cf. [?], [?], [?]) and this embedding is real analytic (cf. Otal [?]). The image of $T(\mathbb{D})$ in the space of geodesic currents is closed and unbounded, and the space of its asymptotic rays-the Thurston's boundary- is identified with the projective bounded measured laminations $PML_{bdd}(\mathbb{D})$ (cf. [?], [?]). In particular, the earthquake paths $t \mapsto E^{t\mu}$ as $t \to \infty$ accumulate to their corresponding projective earthquake measures $[\mu] \in PML_{bdd}(\mathbb{D})$ (cf. [?], [?]). The construction of the Thurston's boundary works for all hyperbolic surfaces simultaneously since any invariance under a Fuchsian group is preserved under the construction.

In the case of closed surfaces, Masur [?] proved that the Teichmüller geodesic rays obtained by shrinking the vertical directions of holomorphic quadratic differentials

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with uniquely ergodic vertical foliations converge to the projective classes of their vertical foliations in the Thurston's boundary. On the other hand, if the vertical foliation of a holomorphic quadratic differential consists of finitely many cylinders then the limit in the Thurston's boundary is the projective class of the measured lamination supported on finitely many simple closed geodesics homotopic to the cylinders with equal weights (cf. [**?**]). In both cases the Teichmüller geodesic rays have unique endpoints on Thurston's boundary and the endpoints depend only on the vertical foliations. However, when the vertical foliations of holomorphic quadratic differentials on closed surfaces are not uniquely ergodic then the limit sets of the corresponding Teichmüller rays consist of more than one point (cf. [**?**], [**?**]).

We consider the limits of Teichmüller geodesic rays in the universal Teichmüller space $T(\mathbb{D})$ corresponding to integrable holomorphic quadratic differentials. When the holomorphic quadratic differential has no zeroes in $\mathbb{D}$ and the natural parameter maps the unit disk into a domain in $\mathbb{C}$ between the graphs of two functions, then the Teichmüller geodesic ray has a unique endpoint in the Thurston's boundary of $T(\mathbb{D})$ but the endpoint depends on both vertical and horizontal foliations of φ (cf. [**?**]). We extend this result to all integrable holomorphic quadratic differentials.

The hyperbolic plane is identified with the unit disk $\mathbb{D}$ and the visual boundary of the hyperbolic plane is identified with the unit circle S^1. A (hyperbolic) geodesic in $\mathbb{D}$ is uniquely determined by it endpoints; the space of geodesics of $\mathbb{D}$ is identified with $S^1 \times S^1 - diag$. Let φ be an arbitrary integrable holomorphic quadratic differential on the unit disk $\mathbb{D}$. Each vertical trajectory of φ has two distinct endpoints on the boundary circle S^1 of the unit disk $\mathbb{D}$ (cf. [**?**]). Thus each vertical trajectory of φ is homotopic to a unique geodesic of $\mathbb{D}$ relative ideal endpoints on S^1. Let v_φ be the set of the geodesics in $\mathbb{D}$ homotopic to the vertical trajectories of φ. Given a box of geodesics $[a,b]\times[c,d]\subset S^1\times S^1 - diag$, denote by $I_{[a,b]\times[c,d]}$ a subarc of a horizontal trajectory that intersects each vertical trajectory of φ with one endpoint in $[a,b]$ and the other endpoint in $[c,d]$.

We define a measured lamination μ_φ of $\mathbb{D}$ supported on v_φ by

$$\mu_\varphi([a,b]\times[c,d]) = \int_{I_{[a,b]\times[c,d]}} \frac{1}{l(x)}dx$$

where $l(x)$ is the length of a vertical trajectory through $x\in I_{[a,b]\times[c,d]}$ and the integration is in the natural parameter of φ. We obtain (cf. Proposition **??** and Theorem **??**)

Proposition 1. *Let μ_φ be the measured lamination homotopic to the vertical foliation of an integrable holomorphic quadratic differential φ on $\mathbb{D}$ defined above. Then*

$$\|\mu_\varphi\|_{Th} = \sup_{[a,b]\times[c,d]} \mu_\varphi([a,b]\times[c,d]) < \infty$$

where the supremum is over all boxes $[a,b]\times[c,d]$ with $cr(a,b,c,d)=2$.

The measured lamination μ_φ satisfies

$$\mu_\varphi(\{a\}\times[c,d]) = 0$$

for all $a\in S^1$ and $[c,d]\subset S^1$, and in particular it does not have atoms.

For $\epsilon > 0$, let T_ϵ be the Teichmüller mapping that shrinks the vertical trajectories by a multiplicative constant ϵ. The path $\epsilon \mapsto T_\epsilon$ as $\epsilon \to 0^+$ leaves every bounded subset of the universal Teichmüller space $T(\mathbb{D})$. We obtain

Theorem 1. *Let*

$$\epsilon \mapsto T_\epsilon$$

be the Teichmüller ray in $T(\mathbb{D})$ that shrinks the vertical trajectories of an integrable holomorphic quadratic differential φ by a multiplicative constant $\epsilon > 0$. Then

$$T_\epsilon \to [\mu_\varphi] \in PML_{bdd}(\mathbb{D})$$

as $\epsilon \to 0^+$, where μ_φ is the measured lamination defined above and the convergence is in the weak topology.*

In particular, the limit set of any Teichmüller ray in $T(\mathbb{D})$ consists of a unique point.

Remark 1. The limit point μ_φ depends on the vertical foliation and the lengths of the vertical trajectories unlike for closed surfaces. The lengths of vertical trajectories are given by the transverse measure to the horizontal foliation. Therefore the limit points depend on both vertical and horizontal foliations of φ.

Remark 2. The measure $\mu_\varphi([a,b]\times[c,d])$ is the modulus of the vertical trajectories of φ with one endpoint in $[a,b]$ and another endpoint in $[c,d]$ (cf. Proposition **??**).

2. Thurston's boundary via geodesic currents

We identify the unit disk $\mathbb{D}$ with the hyperbolic plane; the visual boundary to $\mathbb{D}$ is the unit circle S^1. A homeomorphism $h : S^1 \to S^1$ is said to be *quasisymmetric* if there exists $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{|h(I)|}{|h(J)|} \leq M$$

for all circular arcs I, J with a common boundary point and disjoint interiors such that $|I| = |J|$, where $|I|$ is the length of I. A homeomorphism is quasisymmetric if and only if it extends to a quasiconformal map of the unit disk.

Definition 2.1. The universal Teichmüller space $T(\mathbb{D})$ consists of all quasisymmetric maps $h : S^1 \to S^1$ that fix $-i, 1, i \in S^1$.

If $g : \mathbb{D} \to \mathbb{D}$ is a quasiconformal map, denote by $K(g)$ its quasiconformal constant. The Teichmüller metric on $T(\mathbb{D})$ is given by $d(h_1, h_2) = \inf_g K(g)$, where g runs over all quasiconformal extensions of the quasisymmetric map $h_1 \circ h_2^{-1}$. The Teichmüller topology is induced by the Teichmüller metric.

Bonahon's approach [**?**] to Thurston's boundary of the Teichmüller space $T(S)$ of a closed surface S is to embed $T(S)$ into the space of geodesic currents on S. A *geodesic current* on S is a positive Borel measure on the space of geodesics $(S^1 \times S^1 \setminus diag)/\mathbb{Z}_2$ of the universal covering $\mathbb{D}$ of S that is invariant under the action of the covering group $\pi_1(S)$. Each point in the Teichmüller space $T(S)$ is a quasisymmetric map

$$h : S^1 \to S^1$$

that conjugates the covering Fuchsian group $\pi_1(S)$ onto another Fuchsian group.

The *Liouville measure* $\mathcal{L}$ on the space of geodesic of $\mathbb{D}$ is given by

$$\mathcal{L}(A) = \int_A \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2}$$

for any Borel set $A \subset S^1 \times S^1 - diag$. If $A = [a,b] \times [c,d]$ then

$$\mathcal{L}([a,b] \times [c,d]) = \log \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

To each quasisymmetric map of $h : S^1 \to S^1$ that conjugates $\pi_1(S)$ onto another Fuchsian group, we assign the pull back $h_*(\mathcal{L})$ of the Liouville measure. This assignment is a homeomorphism of $T(S)$ onto its image in the space of geodesic currents for S when equipped with the weak* topology (cf. [**?**]). The set of the asymptotic rays to the image of $T(S)$ is identified with the projective measured laminations on S-the Thurston's boundary of $T(S)$ (cf. [**?**]).

The universal Teichmüller space $T(\mathbb{D})$ maps into geodesic currents by taking the pull backs by quasisymmetric maps of the Liouville measure. There is no invariance condition on the quasisymmetric maps or on the pull backs of the Liouville measure. A geodesic current α is *bounded* if

$$\sup_{[a,b]\times[c,d]} \alpha([a,b] \times [c,d]) < \infty$$

where the supremum is over all $[a,b] \times [c,d]$ with $\mathcal{L}([a,b] \times [c,d]) = \log 2$. The pull backs $h_*(\mathcal{L})$ for h quasisymmetric are bounded geodesic currents.

The space of bounded geodesic currents is endowed with the family of Hölder norms parametrized with the Hölder exponents $0 < \nu \leq 1$ (cf. [**?**]). The pull backs of the Liouville measure define a homeomorphism of $T(\mathbb{D})$ onto its image in the bounded geodesic currents; the homeomorphism is differentiable with a bounded derivative given by a Hölder distribution (cf. [**?**]) and, in fact, Otal [**?**] proved that it is real-analytic. The asymptotic rays to the image of $T(\mathbb{D})$ are identified with the space of projective bounded measured laminations (cf. [**?**]). Thus Thurston's boundary of $T(\mathbb{D})$ is the space $PML_{bdd}(\mathbb{D})$ of all projective bounded measured laminations on $\mathbb{D}$ (and an analogous statement holds for any hyperbolic Riemann surface). Alternatively, the space of geodesic currents can be endowed with the uniform weak* topology (cf. [**?**]) and Thurston's boundary for $T(\mathbb{D})$ is again $PML_{bdd}(\mathbb{D})$ (cf. [**?**]).

3. The asymptotics of the modulus

Let (a,b,c,d) be a quadruple of distinct points on S^1 given in the counter-clockwise order. Denote by $\Gamma_{[a,b]\times[c,d]}$ the family of all differentiable curves whose interiors are in $\mathbb{D}$ that have one endpoint on the arc $[a,b] \subset S^1$ and the other endpoint on the arc $[c,d] \in S^1$. An *admissible metric* ρ for the family $\Gamma_{[a,b]\times[c,d]}$ is a non-negative measurable function on $\mathbb{D}$ such that the ρ-length of each $\gamma \in \Gamma_{[a,b]\times[c,d]}$ is at least one, namely

$$l_\rho(\gamma) = \int_\gamma \rho(z)|dz| \geq 1.$$

The *modulus* $mod(\Gamma_{[a,b]\times[c,d]})$ of the family $\Gamma_{[a,b]\times[c,d]}$ is given by

$$mod(\Gamma_{[a,b]\times[c,d]}) = \inf_\rho \int_{\mathbb{D}} \rho(z)^2 dxdy$$

where the infimum is over all admissible metrics ρ.

The following lemma is an easy consequence of the asymptotic properties of the moduli (cf. [**?**]).

Lemma 3.1 (cf. [**?**])**.** *Let (a,b,c,d) be a quadruple of points on S^1 in the counterclockwise order. Let $\Gamma_{[a,b]\times[c,d]}$ consist of all differentiable curves γ in $\mathbb{D}$ which connect $[a,b]\subset S^1$ with $[c,d]\subset S^1$. Then*

$$mod(\Gamma_{[a,b]\times[c,d]})-\frac{1}{\pi}\mathcal{L}([a,b]\times[c,d])-\frac{2}{\pi}\log 4\to 0$$

as $mod(\Gamma_{[a,b]\times[c,d]})\to\infty$, where $\mathcal{L}$ is the Liouville measure.

Remark 3.2. Note that simultaneously $mod(\Gamma_{[a,b]\times[c,d]})\to\infty$ and $\mathcal{L}([a,b]\times[c,d])\to\infty$.

4. The convergence of Teichmüller rays

Let φ be an integrable holomorphic quadratic differential on the unit disk $\mathbb{D}$. In other words, $\varphi:\mathbb{D}\to\mathbb{C}$ is holomorphic and

$$\|\varphi\|=\iint_{\mathbb{D}}|\varphi(z)|dxdy<\infty.$$

We define the *width* of a curve γ in $\mathbb{D}$. By Strebel [**?**], the unit disk $\mathbb{D}$ can be decomposed into countably many disjoint open strips $S(\beta_i)$ up to a countable family of singular vertical trajectories, where β_i is an open interval on a horizontal trajectory and $S(\beta_i)$ is the union of maximal non-singular intervals of vertical trajectories intersecting β_i. The strips $S(\beta_i)$ are open and simply connected. The quantity

$$\sqrt{\varphi(z)dz^2}$$

is well defined on each $S(\beta_i)$ since it is simply connected and does not contain any zeroes of φ. Any Borel $A\subset\beta_i$ has well-defined width

$$width(A)=\int_A|Re\sqrt{\varphi(z)dz^2}|.$$

If $\gamma\subset S(\beta_i)$, denote by $\pi_{\beta_i}(\gamma)$ the projection of γ onto β_i along the vertical trajectories. Then the width of γ is defined by

$$width(\gamma)=\int_{\pi_{\beta_i}(\gamma)}|Re\sqrt{\varphi(z)dz^2}|.$$

Assume that γ is not contained in a single strip. Consider the collection of Borel sets $\pi_{\beta_i}(\gamma\cap S(\beta_i))$ for all i with $\gamma\cap S(\beta_i)\neq\emptyset$. If a subarc of β_i is homotopic relative vertical trajectories to a subarc of β_{i_1}, then we identify them and assume this identification without further mention. We define the *width* by

$$width(\gamma)=\sum_{i=1}^{\infty}width(\gamma\cap[S(\beta_i)-\cup_{k=0}^{i-1}S(\beta_k)]).$$

The definition $width(\gamma)$ is given in terms of the strips $S(\beta_i)$. To see that $width(\gamma)$ is independent of the choice of the strips, let $S(\beta_j')$ be another countable collection of disjoint open strips that covers $\mathbb{D}$ up to countable union of singular vertical trajectories. The two strips $S(\beta_i)$ and $S(\beta_j')$ are either disjoint or they intersect in an open strip $S(\beta_{i,j})$, where $\beta_{i,j}$ is an open subinterval on β_i which can be homotoped modulo vertical trajectories to subinterval of β_j'. The homotopy is measure preserving for $\int_* |Re\sqrt{\varphi(z)dz^2}|$. Since $\beta_i - \cup_j \beta_{i,j}$ is at most countable (which is of measure zero), it follows that

$$width(\gamma \cap S(\beta_i)) = \sum_j width(\gamma \cap [S(\beta_{i,j}) - \sum_{k=0}^{j-1} S(\beta_{i,k})]).$$

This implies that $width(\gamma)$ is independent of the choice of the covering by the strips.

Proposition 4.1. *Let $\Gamma = \Gamma([a,b] \times [c,d])$ be the family of rectifiable arcs in $\mathbb{D}$ with one endpoint in $[a,b] \subset S^1$ and the other endpoint in $[c,d] \subset S^1$. Denote by T_ϵ the Teichmüller map of $\mathbb{D}$ that shrinks the vertical trajectories of φ by the multiplicative constant $\epsilon > 0$. Then*

$$\limsup_{\epsilon \to 0^+} \epsilon \cdot mod(T_\epsilon(\Gamma)) \leq mod(\Gamma_v([a,b],[c,d]))$$

where $\Gamma_v([a,b],[c,d])$ is the set of vertical trajectories with one endpoint in $[a,b]$ and the other endpoint in $[c,d]$.

Proof. By Strebel [**?**], almost every point of S^1 is on a finite distance from an interior point of $\mathbb{D}$ in the metric $\sqrt{|\varphi(z)|}|dz|$. Let $a', b', c', d' \in S^1$ be on the finite distance from an interior point such that $[a,b] \subset [a',b']$ and $[c,d] \subset [c',d']$. Let $\Gamma' = \Gamma([a',b'] \times [c',d'])$. Namely, let

$$\Gamma' = \{\gamma | \gamma \text{ is rectifiable and has endpoints in } [a',b'] \text{ and } [c',d']\}.$$

Since $\Gamma \subset \Gamma'$, we have $mod(T_\epsilon(\Gamma)) \leq mod(T_\epsilon(\Gamma'))$. Let $l_{a',b'}$ and $l_{c',d'}$ be two simple non-intersecting differentiable arcs in $\mathbb{D}$ with endpoints a', b' and c', d', respectively. Let $\mathbb{D}'$ be the subset of $\mathbb{D}$ with boundary consisting of arcs $l_{a',b'}$, $[b',c'] \subset S^1$, $l_{c',d'}$ and $[d',a'] \subset S^1$. Let $\Gamma'' = \Gamma(l_{a',b'} \times l_{c',d'})$ be the family of rectifiable curves in $\mathbb{D}'$ that connect $l_{a',b'}$ and $l_{c',d'}$. Then each curve in Γ' contains a curve in Γ'' and we have

$$mod(T_\epsilon(\Gamma')) \leq mod(T_\epsilon(\Gamma'')). \tag{1}$$

Fix $\eta > 0$ and define

$$\Gamma''_{>\eta} = \{\gamma \in \Gamma'' | w(\gamma) > \eta\}$$

and

$$\Gamma''_{\leq\eta} = \{\gamma \in \Gamma'' | w(\gamma) \leq \eta\}.$$

By the subadditivity of the modulus

$$mod(T_\epsilon(\Gamma'')) \leq mod(T_\epsilon(\Gamma''_{>\eta})) + mod(T_\epsilon(\Gamma''_{\leq\eta})).$$

We first consider $mod(T_\epsilon(\Gamma''_{>\eta}))$. Define the metric $\rho_\epsilon(w) = \frac{1}{\eta}|\sqrt{\varphi_\epsilon(w)dw^2}|$ for $w \in \mathbb{D}'_\epsilon$, where φ_ϵ is the terminal holomorphic quadratic differential on $T_\epsilon(\mathbb{D}') = \mathbb{D}'_\epsilon$. Recall that the terminal quadratic differential on $T_\epsilon(\mathbb{D}')$ is obtained as follows. Let ζ be the natural parameter of φ on $\mathbb{D}'$, i.e. $d\zeta^2 = \varphi(z)dz^2$; let $\omega = T_{\epsilon,\zeta}(\zeta)$, where $T_{\epsilon,\zeta}$ shrinks the vertical direction of ζ by the multiplicative constant $\epsilon > 0$. Then define $\varphi_\epsilon(\omega)d\omega^2 = d\omega^2$. If $w = T_\epsilon(z)$ then $\varphi_\epsilon(w)dw^2 = d\omega^2$.

The metric ρ_ϵ is allowable for $T_\epsilon(\Gamma''_{>\eta})$ since $width(T_\epsilon(\gamma)) > \eta$ for all $\epsilon > 0$ and all $\gamma \in T_\epsilon(\Gamma''_{>\eta})$. Then

$$mod(T_\epsilon(\Gamma''_{>\eta})) \leq \iint_{T_\epsilon(\mathbb{D}')} \rho_\epsilon(w)^2 dA = \frac{\epsilon}{\eta^2} \iint_{\mathbb{D}'} |\varphi(w)| dA$$

which gives

$$\limsup_{\epsilon\to 0^+} \epsilon\cdot mod(T_\epsilon(\Gamma''_{>\eta}))=0. \tag{2}$$

We estimate $mod(T_\epsilon(\Gamma''_{\leq\eta}))$. Let $z_0\in\mathbb{D}'$ be fixed. Denote by d^φ the path metric defined by integrating $|\sqrt{\varphi(z)dz^2}|$. Let $d_0=\max_{z\in l_{a',b'}\cup l_{c',d'}} d^\varphi(z_0,z)$. For $R>0$ define $\mathbb{D}'_R=\{z\in\mathbb{D}'|d^\varphi(z_0,z)\leq R\}$. Given $\epsilon_1>0$ there exists $R>2d_0$ such that

$$\iint_{\mathbb{D}'-\mathbb{D}'_R}|\varphi(z)|dA<\epsilon_1.$$

Denote by $\Gamma_v(l_{a',b'},l_{c',d'})$ the set of vertical trajectories γ connecting $l_{a',b'}$ with $l_{c',d'}$. The choice $R>2d_0$ and the fact that the vertical trajectories are geodesics for d^φ implies that $\Gamma_v(l_{a',b'},l_{c',d'})\subset\mathbb{D}'_R$. From now on we choose $R=R(\epsilon_1)$ as above.

For $M>0$, define $(\Gamma''_{\leq\eta})_M=\{\gamma\in\Gamma''_{\leq\eta}|\gamma\subset\mathbb{D}'_M\}$. Note that

$$\Gamma''_{\leq\eta}=(\Gamma''_{\leq\eta})_{R+1}\cup[\Gamma''_{\leq\eta}-(\Gamma''_{\leq\eta})_{R+1}]$$

which gives

$$mod(T_\epsilon(\Gamma''_{\leq\eta}))\leq mod(T_\epsilon((\Gamma''_{\leq\eta})_{R+1})+mod(T_\epsilon(\Gamma''_{\leq\eta}-(\Gamma''_{\leq\eta})_{R+1})).$$

Since T_ϵ is ϵ^{-1}-quasiconformal, we have

$$\epsilon\cdot mod(T_\epsilon(\Gamma''_{\leq\eta}-(\Gamma''_{\leq\eta})_{R+1}))\leq\epsilon\cdot\epsilon^{-1}\cdot mod(\Gamma''_{\leq\eta}-(\Gamma''_{\leq\eta})_{R+1})=mod(\Gamma''_{\leq\eta}-(\Gamma''_{\leq\eta})_{R+1}).$$

Define metric $\rho(z)=\sqrt{|\varphi(z)dz^2|}$ for $z\in\mathbb{D}'-\mathbb{D}'_R$ and $\rho(z)=0$ otherwise. Then $\rho(z)$ is allowable for the family $\Gamma''_{\leq\eta}-(\Gamma''_{\leq\eta})_{R+1}$. Thus

$$\limsup_{\epsilon\to 0^+}\epsilon\cdot mod(T_\epsilon(\Gamma''_{\leq\eta}-(\Gamma''_{\leq\eta})_{R+1}))\leq\iint_{\mathbb{D}'-\mathbb{D}'_R}|\varphi(z)|dA<\epsilon_1. \tag{3}$$

We estimate $mod(T_\epsilon((\Gamma''_{\leq\eta})_{R+1})$. Note that $\mathbb{D}'_{R+1}$ is a compact metric space for the distance d^φ. Similar to the above

$$\epsilon\cdot mod(T_\epsilon((\Gamma''_{\leq\eta})_{R+1}))\leq mod((\Gamma''_{\leq\eta})_{R+1}).$$

By Keith [**?**], we have that

$$\limsup_{\eta\to 0^+} mod((\Gamma''_{\leq\eta})_{R+1})\leq mod(\limsup_{\eta\to 0^+}(\Gamma''_{\leq\eta})_{R+1}).$$

We establish that

$$\limsup_{\eta\to 0^+}(\Gamma''_{\leq\eta})_{R+1}=\Gamma_v(l_{a',b'},l_{c',d'}). \tag{4}$$

Let $\gamma_n:I\to\mathbb{D}'_{R+1}$ be a sequence of uniformly Lipschitz parametrizations of curves in $(\Gamma''_{\leq\eta_n})_{R+1}$ with $\eta_n\to 0$ as $n\to\infty$ that converges to $\gamma:I\to\mathbb{D}'_{R+1}$. Then

$$width(\gamma)=0.$$

Indeed, $width(\gamma)=c>0$ implies that $width(\gamma_n)>c/2>0$ for all n large enough. This contradicts $\gamma_n\in(\Gamma''_{\leq\eta)n})_{R+1}$.

Since $width(\gamma)=0$, this implies $\gamma\in\Gamma_v(l_{a',b'},l_{c',d'})$. Since $\Gamma_v(l_{a',b'},l_{c',d'})\subset(\Gamma''_{\leq\eta})_{R+1}$ by our choice of $R>0$, we obtain (**??**). Then (**??**), (**??**), (**??**) and (**??**) imply that

$$\limsup_{\epsilon\to 0^+}\epsilon\cdot mod(T_\epsilon(\Gamma))\leq mod(\Gamma_v(l_{a',b'},l_{c',d'})). \tag{5}$$

It remains to be proved that $\Gamma_v(l_{a',b'},l_{c',d'})$ can be replaced by $\Gamma_v([a,b]\times[c,d])$ in (**??**). Note that (**??**) is true for all $l_{a',b'}$ and $l_{c',d'}$. Choose a sequence $l^k_{a',b'}$ and $l^k_{c',d'}$ such that $l^k_{a',b'}\to[a',b']\subset S^1$ and $l^k_{c',d'}\to[c',d']\subset S^1$ as $k\to\infty$ in the Hausdorff topology on closed subsets of $\bar{\mathbb{D}}=\mathbb{D}\cup S^1$. Denote by $\mathbb{D}'_k$ the subset of $\mathbb{D}$ corresponding to $l^k_{a',b'}$ and $l^k_{c',d'}$. Define

$$\Gamma^k_v([a',b'],[c',d'])=\Gamma_v([a',b'],[c',d'])\cap\mathbb{D}'_k.$$

We claim that

$$\lim_{k\to\infty} mod(\Gamma_v(l^k_{a',b'},l^k_{c',d'})-\Gamma^k_v([a',b'],[c',d']))=0. \tag{6}$$

Indeed, let $C>0$ be the lower bound on the distance d^{φ} between $l^k_{a',b'}$ and $l^k_{c',d'}$ over all k. Then $\rho(z)=\frac{1}{C}\sqrt{|\varphi(z)|}|dz|$ is admissible for $\Gamma_v(l^k_{a',b'},l^k_{c',d'})$. Let A_k be the union of the maximal vertical trajectories in $\mathbb{D}$ that connect $l^k_{a',b'}$ and $l^k_{c',d'}$ and do not connect $[a',b']$ and $[c',d']$. Then $A_k\supset A_{k+1}$ for all k (since we can choose $l^k_{a',b'}$ and $l^k_{c',d'}$ such that $\mathbb{D}'_k\subset\mathbb{D}'_{k+1}$).

We claim that $\cap_{k=1}^{\infty}A_k=\emptyset$. Assume that a horizontal trajectory γ belongs to the union that makes A_k. Then there exists either a Euclidean neighborhood of $[a',b']$ or a Euclidean neighborhood of $[c',d']$ in $\bar{\mathbb{D}}=\mathbb{D}\cup S^1$ such that γ is disjoint from this neighborhood. There exists $k'>k$ such that γ does not intersect either $l^{k'}_{a',b'}$ or $l^{k'}_{c',d'}$. Thus γ does not belong to $\cap_{k=1}^{\infty}A_k$ and $\cap_{k=1}^{\infty}A_k=\emptyset$. This gives

$$\iint_{A_k}|\varphi(z)|dxdy\to 0$$

as $k\to\infty$ and (**??**) follows. From (**??**) and (**??**) we get

$$\limsup_{\epsilon\to 0^+}\epsilon\cdot mod(T_\epsilon(\Gamma)\le\lim_{k\to\infty} mod(\Gamma^k_v([a',b'],[c',d'])). \tag{7}$$

By Keith [**?**], we have that

$$\lim_{k\to\infty} mod(\Gamma^k_v([a',b'],[c',d']))\le mod(\limsup_{k\to\infty}\Gamma^k_v([a',b'],[c',d']))$$

where $\limsup_{k\to\infty}\Gamma^k_v([a',b'],[c',d'])$ is computed with respect to the Euclidean metric on $\bar{\mathbb{D}}=\mathbb{D}\cup S^1$. Assume the sequence subarc $\gamma_k\in\Gamma^k_v([a',b'],[c',d'])$ converges to γ. Let z_1 and z_2 be two arbitrary points on γ. Then there exists k large enough such that γ_k has points as close as we want to z_1 and z_2. This implies that the horizontal distance between z_1 and z_2 is arbitrary small, and therefore z_1 and z_2 lie on the same vertical trajectory. Thus γ is a subset of a single vertical trajectory and we can assume that the vertical trajectory is non singular. Given $z_1,z_2\in\gamma$, there exists an open nonsingular vertical strip which contains the subarc of the vertical trajectory with endpoints z_1 and z_2. Then the whole subarc from z_1 to z_2 is in γ because any γ_k with points z_1^k and z_2^k close to z_1 and z_2 is completely contained in the strip. Therefore γ is a connected subarc of a vertical trajectory of φ. If γ were not maximal trajectory then it would have endpoints z_1^* and z_2^* of which at least one is in $\mathbb{D}$. But observing the horizontal strip containing $[z_1^*,z_2^*]$ we conclude that γ can be extended beyond z_1^* or z_2^* which is a contradiction. Therefore every limit γ is a maximal vertical trajectory that necessarily belongs to $\Gamma_v([a',b'],[c',d'])$. Therefore

$$\limsup_{\epsilon\to 0^+}\epsilon\cdot mod(T_\epsilon(\Gamma)\le mod(\Gamma_v([a',b'],[c',d'])). \tag{8}$$

We choose sequences $[a_k', b_k'] \supset [a,b]$ and $[c_k', d_k'] \supset [c,d]$ on finite distance from z_0 such that $a_k' \to a$, $b_k' \to b$, $c_k' \to c$ and $d_k' \to d$ as $k \to \infty$. The inequality (**??**) holds for these sequences and we need to prove that it holds for $\Gamma_v([a,b],[c,d])$ as well. It is enough to prove that

$$mod(\Gamma_v([a_k', b_k'], [c_k', d_k']) - \Gamma_v([a,b],[c,d])) \to 0.$$

Let $\mathbb{D}_k$ be the union of trajectories in $\Gamma_v([a_k', b_k'], [c_k', d_k']) - \Gamma_v([a,b],[c,d])$. It is clear that $\cap_{k=1}^{\infty} \mathbb{D}_k = \emptyset$. Define $\rho(z) = 1/l_v(z)\sqrt{|\varphi(z)dz^2|}$ for $z \in \mathbb{D}_k$ and $\rho(z) = 0$ otherwise, where $l_v(z)$ is the length of the vertical trajectory through z with respect to the metric d^{φ}. Then ρ is allowable metric for the family $\Gamma_v([a_k', b_k'], [c_k', d_k']) - \Gamma_v([a,b],[c,d])$. Then we have

$$mod(\Gamma_v([a_k', b_k'], [c_k', d_k']) - \Gamma_v([a,b],[c,d])) \leq \iint_{\mathbb{D}_k} \frac{1}{l_v(z)^2} |\varphi(z)| dxdy.$$

We claim that $l_v(z)$ has a positive lower bound in $\mathbb{D}_k$. Indeed, since intervals $[a_k', b_k']$ and $[c_k', d_k']$ are disjoint and decreasing, their distance in d^{φ} metric is positive which implies that any vertical trajectories connecting them must have lengths bounded below by a positive constant. Thus $\frac{1}{l_v(z)^2}$ is bounded above. Then $\cap_{k=1}^{\infty} \mathbb{D}_k = \emptyset$ implies that $\iint_{\mathbb{D}_k} \frac{1}{l_v(z)^2} |\varphi(z)| dxdy \to 0$ as $k \to \infty$. The proof is finished. $\square$

Theorem 4.2. *Let Γ be the family of rectifiable arcs in $\mathbb{D}$ with one endpoint in $[a,b] \subset S^1$ and the other endpoint in $[c,d] \subset S^1$. Denote by T_ϵ the Teichmüller map of $\mathbb{D}$ that shrinks the vertical trajectories of φ by the multiplicative constant $\epsilon > 0$. Then*

$$\lim_{\epsilon \to 0^+} \epsilon \cdot mod(T_\epsilon(\Gamma)) = mod(\Gamma_v([a,b],[c,d]))$$

where $\Gamma_v([a,b],[c,d])$ is the set of vertical trajectories with one endpoint in $[a,b]$ and the other endpoint in $[c,d]$.

Proof. We keep the notation as in the proof of Proposition **??**. Since $\Gamma_v([a,b],[c,d]) \subset \Gamma$, it follows that $mod(\Gamma_v([a,b],[c,d])) \leq mod(\Gamma)$. Because $\Gamma_v([a,b],[c,d])$ consists of only vertical trajectories, it follows that

$$\epsilon \cdot mod(T_\epsilon(\Gamma_v([a,b],[c,d]))) = mod(\Gamma_v([a,b],[c,d])).$$

Thus

$$mod(\Gamma_v([a,b],[c,d])) \leq \liminf_{\epsilon \to 0} \epsilon \cdot mod(T_\epsilon(\Gamma_v([a,b],[c,d]))).$$

The opposite inequality is obtained in Proposition **??** and theorem follows. $\square$

We give an equivalent definition of $mod(\Gamma_v([a,b],[c,d]))$.

Proposition 4.3. *Let φ be an integrable holomorphic quadratic differential on the unit disk $\mathbb{D}$. Then*

$$mod(\Gamma_v([a,b],[c,d])) = \int_I \frac{1}{l(z)} |Re(\sqrt{\varphi(z)}dz)|$$

where I is at most countable set of horizontal arcs that intersects each trajectory of $\Gamma_v([a,b],[c,d])$ in one point and no other vertical trajectories up to countably many of them, and $l(z)$ is the length of the vertical trajectory through z.

Proof. Let $\rho(z) = \frac{1}{l(z)}|Re(\sqrt{\varphi(z)}dz)|$. Then $\rho(z)$ is allowable for the family $\Gamma_v([a,b],[c,d])$. It follows that $mod(\Gamma_v([a,b],[c,d])) \leq \int_I \frac{1}{l(z)}|Re(\sqrt{\varphi(z)}dz)|$.

We claim that $\rho(z)$ is extremal metric which proves that we have equality above. Using Beurlings sufficient condition for extremal metric [**?**], we need to show that if $\int_\gamma h_0(z)|dz| \geq 0$ for all $\gamma \in \Gamma_v([a,b],[c,d])$ and some $h_0 : \mathbb{D} \to \mathbb{R}$ then $\iint_{\mathbb{D}} h_0(z)\rho(z)^2 dxdy \geq 0$. By transferring the integration to the natural parameter, we get that γ are subsets of vertical lines which implies $|dz| = dy$ and $\rho(z) = 1$. Then $\int_\gamma h_0(z)|dz| = \int_I h_0(z)dy \geq 0$ and an integration in the x direction gives the desired inequality (cf. [**?**]). □

Define a measured lamination μ_φ as follows. The support of μ_φ is a geodesic lamination obtained by taking geodesics in $\mathbb{D}$ which are homotopic to the vertical trajectories of φ, i.e. a geodesic in the support has endpoints equal to a vertical trajectory of φ. For a box of geodesics $[a,b]\times[c,d]$, define

$$\mu_\varphi([a,b]\times[c,d]) = mod(\Gamma_v([a,b],[c,d])).$$

Proposition 4.4. *Let μ_φ be the measured lamination corresponding to an integrable holomorphic quadratic differential φ on $\mathbb{D}$ as above. Then*

$$\mu_\varphi(\{a\}\times[c,d]) = 0$$

for all $a \in S^1$ and $[c,d] \subset S^1$ with $a \notin [c,d]$.

Proof. We recall that $\mathbb{D}$ is covered by countably many mutually disjoint open strips $S(\beta_i)$ up to countably many vertical trajectories. Assume on the contrary that $\mu_\varphi(\{a\}\times[c,d]) > 0$. Then there exists an open strip $S(\beta_{i_0})$ such that

$$\int_{X_{i_0}} |Re(\sqrt{\varphi(z)dz^2})|) > 0,$$

where β_{i_0} is the open arc on a horizontal trajectory and $X_{i_0} = \beta_{i_0} \cap \Gamma_v(\{a\}\times[c,d])$. By the definition, $\int_{X_{i_0}} |Re(\sqrt{\varphi(z)dz^2})|$ is the μ_φ-measure of the geodesics of the support of μ_φ that are homotopic to the vertical trajectories of φ intersecting β_{i_0}.

For $z \in X_{i_0}$, let $l(z)$ be the length of the maximal vertical trajectory through z. Note that $S(\beta_{i_0})$ might not contain maximal vertical trajectory through z. Since φ is integrable, we have that

$$\int_{X_{i_0}} l(z)|Re(\sqrt{\varphi(z)dz^2})| < \infty$$

which implies that $l(z) < \infty$ for a. a. $z \in X_{i_0}$.

Let $z_1, z_2 \in X_{i_0}$ be such that there exists $z_1', z_2' \in X_{i_0}$ with $z_1 < z_1' < z_2' < z_2$ for a linear order on β_{i_0}, and $l(z_1')$ and $l(z_2')$ finite. Let $\gamma_{z_i}, \gamma_{z_i'}$ be the rays starting at z_i, z_i' respectively that have a as their endpoint. Note that vertical rays γ_{z_1} and γ_{z_2} do not intersect $\beta_{i_0'}$ except at their initial points because any two points in $\mathbb{D}$ can be joined by at most one geodesic arc in the metric $|\sqrt{\varphi(z)dz^2}|$ (cf. [**?**, Theorem 14.2.1, page 72]). Both rays have their endpoint a. Let $[z_1, z_2]$ be the subarc of the vertical trajectory between z_1 and z_2. Then $\gamma_{z_1} \cup \gamma_{z_2} \cup [z_1, z_2]$ is the boundary of a simply connected domain U inside $\mathbb{D}$.

For $z \in [z_1, z_2]$, let γ_z be the ray of the vertical trajectory with the initial point z that starts in the direction of U. Then γ_z never leaves U because it cannot intersect its boundary except at z and it necessarily accumulates at a. Moreover, the ray γ_z

cannot contain critical points of φ. Indeed, if it does contain a critical point then there exist two vertical rays starting at the critical point which make a geodesic and whose both accumulation points on S^1 are equal to a. However, a geodesic must have two different accumulation points (cf. [**?**, Theorem 19.4 and Theorem 19.6]) which gives a contradiction. Therefore every vertical trajectory in U is non-critical and its full extension accumulates at $a \in S^1$ and intersects $[z_1, z_2]$ in exactly one point. Therefore, U is foliated by γ_z for $z \in (z_1, z_2)$.

Consider the conformal mapping from U into $\mathbb{C}$ using the natural parameter $dw^2 = \varphi(z)dz^2$. Since U is simply connected and without zeroes, the natural parameter is conformal on U. It follows that w maps injectively the prime ends of U onto the prime ends of $w(U)$. The point $a \in \partial U$ is accessible by vertical rays γ_z, for all $z \in (z_1, z_2)$, and they define a prime end of U. Since $\gamma_{z_1'}$ and $\gamma_{z_2'}$ have finite lengths, it follows that the endpoints of $w(\gamma_{z_1'})$ and $w(\gamma_{z_2'})$ are different in $\partial w(U)$ and they define different prime ends. This is impossible since w^{-1} maps both points to the prime end corresponding to a. Contradiction. Thus we obtained that $\mu_\varphi(\{a\} \times [c, d]) = 0$. □

Putting the above statements together gives

Theorem 4.5. *Let φ be an integrable holomorphic quadratic differential on $\mathbb{D}$ and let T_ϵ be the Teichmüller mapping that shrinks the vertical trajectories of φ by a multiplicative constant $\epsilon > 0$. The Teichmüller ray $\epsilon \mapsto T_\epsilon$ for $\epsilon > 0$ has a unique limit point $[\mu_\varphi]$ on Thurston's boundary $PML_{bdd}(\mathbb{D})$ of $T(\mathbb{D})$ as $\epsilon \to 0^+$, where $[\mu_\varphi]$ is the projective class of a bounded measured lamination μ_φ corresponding to φ.*

Proof. The convergence $T_\epsilon \to [\mu_\varphi]$ as $\epsilon \to 0^+$ in the weak* topology on measures follows immediately from Theorem **??** and Proposition **??**. It remains to be proved that μ_φ is Thurston bounded.

Note that by the definition the measured lamination μ_φ is independent under multiplication of φ by positive constants. Let $[a, b] \times [c, d]$ be such that its Liouville measure satisfies

$$L([a, b] \times [c, d]) = \log 2.$$

Denote by $\Gamma([a, b], [c, d])$ the family of all rectifiable arcs in $\mathbb{D}$ that have one endpoint in $[a, b]$ and other endpoint in $[c, d]$. Then

$$mod(\Gamma([a, b], [c, d])) \leq const$$

for all $L([a, b] \times [c, d]) = \log 2$. Since $\Gamma_v([a, b], [c, d]) \subset \Gamma([a, b], [c, d])$, we have that

$$\mu_\varphi([a, b] \times [c, d]) = mod(\Gamma_v([a, b], [c, d])) \leq const$$

and $\|\mu_\varphi\|_{Th} < \infty$. □

5. From integrable holomorphic quadratic differentials to bounded measured laminations

Let φ be an integrable holomorphic quadratic differential on $\mathbb{D}$ (i.e. a holomorphic function $\varphi : \mathbb{D} \to \mathbb{C}$ such that $\|\varphi\|_{L^1(\mathbb{D})} = \iint_{\mathbb{D}} |\varphi(z)|dxdy < \infty$). Let $\mathcal{A}(\mathbb{D})$ be the space of all integrable holomorphic quadratic differentials on $\mathbb{D}$.

Given $\varphi \in \mathcal{A}(\mathbb{D})$, we defined a corresponding bounded measured lamination

$$\mu_\varphi([a, b] \times [c, d]) = mod(\Gamma_v([a, b], [c, d]))$$

or equivalently

$$\mu_\varphi([a,b]\times[c,d]) = \int_I \frac{1}{l(z)}|Re(\sqrt{\varphi(z)}dz)|$$

where I is transverse arc to $\Gamma_v([a,b],[c,d])$.

It follows that if $c>0$ then $\mu_{c\varphi}=\mu_\varphi$. Therefore we obtain a map from the space $P\mathcal{A}(\mathbb{D})$ of projective integrable holomorphic quadratic differentials to the space of bounded measured laminations $ML_{bdd}(\mathbb{D})$,

$$\mathcal{V}/l : P\mathcal{A}(\mathbb{D}) \to ML_{bdd}(\mathbb{D}).$$

We prove that $\mathcal{V}/l : P\mathcal{A}(\mathbb{D}) \to ML_{bdd}(\mathbb{D})$ is injective.

Theorem 5.1. *The map*

$$\mathcal{V}/l : P\mathcal{A}(\mathbb{D}) \to ML_{bdd}(\mathbb{D}).$$

defined by

$$\mathcal{V}/l(\varphi) = \mu_\varphi$$

is injective.

Proof. We assume that $\mu_\varphi = \mu_{\varphi'}$ and need to prove that $\varphi = c\varphi'$ for some $c>0$. Since $\mu_\varphi = \mu_{\varphi'}$ we have that their geodesic laminations supports $|\mu_\varphi|$ and $|\mu_{\varphi'}|$ are the same. In other words each leaf of the vertical foliation φ is homotopic to a leaf of the vertical foliation of φ' relative their two endpoints on the unit circle, and vice versa.

Additionally, assume that the corresponding leaves of the vertical foliations are not only homotopic but that they are equal to each other. In other words, the vertical foliations of φ and φ' are equal. If $z_0\in\mathbb{D}$ is a regular point of both φ and φ', denote by ζ and ζ' the corresponding natural parameters in a regular neighborhood U of z_0. Then $f=\zeta'\circ\zeta^{-1}$ is a conformal mapping from $\zeta(U)\subset\mathbb{C}$ onto $\zeta'(U)\subset\mathbb{C}$ that maps vertical lines onto vertical lines. It follows then that $f(\zeta)=a\zeta'+b$ for some $a\in\mathbb{R}$. Thus $f^*(d\zeta^2)=a^2d\zeta^2=d\zeta'^2$.

We obtained that for each regular point z_0 of φ and φ' there exist a neighborhood $U\ni z_0$ and a constant $c>0$ such that $\varphi=c\varphi'$ in U. Since the set of regular points of φ and φ' is connected and dense in $\mathbb{D}$ then $\varphi=c\varphi'$ in $\mathbb{D}$ and the proof is finished in this case.

It remains to prove that the vertical foliations of φ and φ' are the same under the assumption that $\mu_\varphi=\mu_{\varphi'}$. Let $\{S(\beta_i)\}_{i=1}^\infty$ be a family of vertical strips with transverse horizontal arcs β_i that covers $\mathbb{D}$ up to countably many singular vertical trajectories. The arcs β_i have metric $|\sqrt{\varphi(z)}dz|$ and we isometrically identify them with $[0,a_i]$, where a_i is the length of β_i. The variable in $[0,a_i]$ is x and the integration with respect dx corresponds to integration with respect $|\sqrt{\varphi(z)}dz|$ in $\mathbb{D}$. The arc $[0,a_i]$ is a horizontal arc in the natural parameter $\int\sqrt{\varphi(z)}dz$ for $\varphi(z)$.

For β_i, let $S(\beta_i,[0,x])$ be the subs trip of $S(\beta_i)$ of vertical rays going through $[0,x]\subset[0,a_i]$. The area of $S(\beta_i,[0,x])$ is

$$A^\varphi_{\beta_i}(x) = \int_{[0,x]} l^\varphi(v^\varphi_{\beta_i}(t))dt,$$

where $v^\varphi_{\beta_i}(t)$ is the vertical trajectory of φ through the point $t\in[0,x]\subset\beta_i$ and $l^\varphi(\cdot)$ is the length in the $|\sqrt{\varphi(z)}dz|$ metric. The modulus of the vertical trajectories in $S(\beta_i,[0,x])$ is

$$M^{\varphi}_{\beta_i}(x)=\int_{[0,x]}\frac{1}{l^{\varphi}(v^{\varphi}_{\beta_i}(t))}dt.$$

If necessary, we multiply φ' by a positive constant such that $\|\varphi'\|_{L^1}=\|\varphi\|_{\infty}$. Since the supports of μ_{φ} and $\mu_{\varphi'}$ are the same, to each $S(\beta_i,[0,x])$ there corresponds a vertical strip $\tilde{S}(\beta_i,[0,x])$ of vertical trajectories $v^{\varphi'}_{\beta_i}(t)$ of φ' with the same endpoints on S^1 as $v^{\varphi}_{\beta_i}(t)$. Note that $v^{\varphi'}_{\beta_i}(t)$ does not necessarily pass through $t\in\beta_i$ or even intersects β_i.

Let $A^{\varphi'}_{\beta_i}(x)$ and $M^{\varphi'}_{\beta_i}(x)$ denote the area of $\tilde{S}(\beta_i,[0,x])$ and the modulus of vertical trajectories of φ' in $\tilde{S}(\beta_i,[0,x])$. We have the following lemma.

Lemma 5.2. *Let β_i be a transverse horizontal arcs to a vertical strip $S(\beta_i)$ isometrically identified with $[0,a_i]$ in the natural parameter of φ. Then for a.e. $x\in[0,a_i]$, we have*

$$\frac{d}{dx}M^{\varphi'}_{\beta_i}(x)\leq\frac{\frac{d}{dx}A^{\varphi'}_{\beta_i}(x)}{[l^{\varphi}(v^{\varphi'}_{\beta_i}(x))]^2},$$

where $l^{\varphi}(\cdot)$ is the φ-length and $v^{\varphi'}_{\beta_i}(x)$ is the horizontal trajectory of φ' whose endpoint agree with the endpoints of $v^{\varphi}_{\beta_i}(x)$.

Proof. For $x\in[0,a_i]$ and small $\varepsilon>0$ we denote

$$L_x(\varepsilon)=\inf\{l^{\varphi}(v^{\varphi'}_{\beta_i}(t))\,:\,t\in[x,x+\varepsilon]\}.$$

Then $L_x(\varepsilon)^{-1}$ is admissible for $\tilde{S}(\beta_i,[x,x+\varepsilon])$. Since, $L_x(\varepsilon)$ is non-increasing it has a limit as $\varepsilon\to 0^+$. In fact, we have

$$L_x(\varepsilon)\xrightarrow[\varepsilon\to 0^+]{} l^{\varphi}(v^{\varphi'}_{\beta_i}(t)).$$

To see this, note first that $L_x(\varepsilon)\leq l^{\varphi}(v^{\varphi'}_{\beta_i}(x))$ and we only need to estimate the limit from below. For this, fix $\delta>0$ and choose points $\xi_0,\dots,\xi_k\in v^{\varphi'}_{\beta_i}(x)$, so that

$$\sum_{i=1}^{k}|\xi_i-\xi_{i-1}|\geq l^{\varphi}(v^{\varphi'}_{\beta_i}(x))-\frac{\delta}{2}. \tag{9}$$

We want to show that for small η the curves $v^{\varphi'}_{\beta_i}(x+\eta)$ have length at least $l^{\varphi}(v^{\varphi'}_{\beta_i}(x))-\delta$. If the set of vertical trajectories $S(\beta_i)$ foliates a neighborhood of $v^{\varphi}_{\beta_i}(x)$ (or at least a neighborhood on one side) then $\tilde{S}(\beta_i)$ must foliate a neighborhood of $v^{\varphi'}_{\beta_i}(x)$ (or at least a neighborhood on one side). Indeed, otherwise there would be a vertical trajectory in $\tilde{S}(\beta_i)$ with no corresponding trajectory in $S(\beta_i)$ and that would contradict $\mu_{\varphi}=\mu_{\varphi'}$. By choosing a small $\eta>0$, we get that a subarc of $v^{\varphi'}_{\beta_i}(x+\epsilon)$ is within small euclidean to the subarc of $v^{\varphi'}_{\beta_i}(x)$ between ξ_0 and ξ_k. Since φ is continuous, it follows that for $\eta>0$ small enough, each $v^{\varphi'}_{\beta_i}(x+\epsilon)$ for $\epsilon<\eta$ has points $\xi'_0,\dots,\xi'_k$ on the φ-distance less than $\frac{\delta}{4k}$ from $\xi_0,\dots,\xi_k$, respectively.

Therefore by (**??**) we have

$$l^{\varphi}(v^{\varphi'}_{\beta_i}(x+\eta))\geq\sum_{i=1}^{k}|\xi'_i-\xi'_{i-1}|\geq\sum_{i=1}^{k}\left(|\xi_i-\xi_{i-1}|-\frac{\delta}{2k}\right)\geq l^{\varphi}(v^{\varphi'}_{\beta_i}(x))-\delta.$$

Thus $L_x(\epsilon) \geq l^{\varphi}(v^{\varphi'}_{\beta_i}(x)) - \delta$ for all $\epsilon < \eta$ which implies that $\lim_{\epsilon\to 0^+} L_x(\epsilon) = l^{\varphi}(v^{\varphi'}_{\beta_i}(x))$ because $\delta > 0$ is arbitrary. Thus for a.e. $x \in [0, a_i]$ we have

$$\frac{d}{dx}M^{\varphi'}_{\beta_i}(x) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{mod}\tilde{S}(\beta_i, [x, x+\varepsilon])}{\varepsilon} \leq \limsup_{\varepsilon\to 0^+} \frac{A^{\varphi'}_{\beta_i}(x+\varepsilon) - A^{\varphi'}_{\beta_i}(x)}{\varepsilon L^2_x(\varepsilon)} = \frac{\frac{d}{dx}A^{\varphi'}_{\beta_i}(x)}{[l^{\varphi}(v^{\varphi'}_{\beta_i}(x))]^2},$$

and the proof is complete. □

Using the above lemma we establish the next lemma which finishes the proof.

Lemma 5.3. *If for every β_i and for every $x \in [0, a_i]$ we have $M^{\varphi}_{\beta_i}(x) = M^{\varphi'}_{\beta_i}(x)$ then the vertical foliations of φ and φ' are equal.*

Proof. By the previous lemma and absolute continuity of $M^{\varphi}_{\beta_i}(x)$ we have

$$\frac{\frac{d}{dx}A^{\varphi}_{\beta_i}(x)}{[l^{\varphi}(v^{\varphi}_{\beta_i}(x))]^2} = \frac{1}{[l^{\varphi}(v^{\varphi}_{\beta_i}(x))]^2} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_x^{x+\varepsilon} l^{\varphi}(v^{\varphi}_{\beta_i}(t))dt =$$

$$\frac{1}{l^{\varphi}(v^{\varphi}_{\beta_i}(x))} = \frac{d}{dx}M^{\varphi}_{\beta_i}(x) = \frac{d}{dx}M^{\varphi'}_{\beta_i}(x) \leq \frac{\frac{d}{dx}A^{\varphi'}_{\beta_i}(x)}{l^{\varphi}(v^{\varphi'}_{\beta_i}(x))}.$$

Since $l^{\varphi}(v^{\varphi}_{\beta_i}(x)) \leq l^{\varphi}(v^{\varphi'}_{\beta_i}(x))$ with equality implying that the two curves are the same, from the above inequality we obtain

$$\frac{d}{dx}A^{\varphi}_{\beta_i}(x) \leq \frac{d}{dx}A^{\varphi'}_{\beta_i}(x)$$

for a.e. $x \in [0, a_i]$.

Thus

$$A^{\varphi}_{\beta_i}(x) = \int_0^x \frac{d}{dt}A^{\varphi}_{\beta_i}(t)dt \leq \int_0^x \frac{d}{dt}A^{\varphi'}_{\beta_i}(t)dt \leq A^{\varphi}_{\beta_i}(x)$$

for all $x \in \beta_i$. Since

$$\|\varphi\|_{L^1} = \sum_i A^{\varphi}_{\beta_i}(a_i) \leq \sum_i A^{\varphi'}_{\beta_i}(a_i) = \|\varphi'\|_{L^1}$$

and $\|\varphi\|_{L^1} = \|\varphi'\|_{L^1}$, we necessarily have equality for each term β_i and for all $x \in [0, a_i]$.

Thus $A^{\varphi}_{\beta_i}(x) = A^{\varphi'}_{\beta_i}(x)$ for all $x \in [0, a_i]$ which implies $\frac{d}{dx}A^{\varphi}_{\beta_i}(x) = \frac{d}{dx}A^{\varphi'}_{\beta_i}(x)$. The first inequality in the proof gives $\frac{\frac{d}{dx}A^{\varphi}_{\beta_i}(x)}{[l^{\varphi}(v^{\varphi}_{\beta_i}(x))]^2} \leq \frac{\frac{d}{dx}A^{\varphi'}_{\beta_i}(x)}{[l^{\varphi}(v^{\varphi'}_{\beta_i}(x))]^2}$ which together with the above inequality gives $l^{\varphi}(v^{\varphi'}_{\beta_i}(x)) = l^{\varphi}(v^{\varphi}_{\beta_i}(x))$ for all x. By the uniqueness of geodesics in simply connected domains, we obtain that all vertical trajectories of φ and φ' are the same. □

The above lemma together with the above finishes the proof of the theorem. □

Given an integrable holomorphic quadratic differential φ on the unit disk, we denote by ν_{φ} the measured lamination whose support is homotopic to the leaves of the vertical foliation of φ and the transverse measure is given by $\int_I Re(\sqrt{\varphi(z)}dz)$, where I is an arc intersecting the leaves of the vertical foliation corresponding to the leaves of ν_{φ}. We first prove that ν_{φ} is Thurston bounded.

Proposition 5.4. *Let φ be an integrable holomorphic quadratic differential on the unit disk $\mathbb{D}$. Then the vertical foliation measure ν_φ defined above is Thurston bounded.*

Proof. Let $\mathcal{V}^{\geq 1}$ be the the set of all vertical trajectories of φ whose φ-length is ≥ 1. Let $\mathcal{V}^{<1}$ be the the set of all vertical trajectories of φ whose φ-length is < 1. Let $\mathbb{D}^{\geq 1} = \cup_{\gamma\in\mathcal{V}^{\geq 1}}\gamma$ and $\mathbb{D}^{<1} = \cup_{\gamma\in\mathcal{V}^{<1}}\gamma$.

Let $[a,b]\times[c,d]\subset(S^1\times S^1)-diag$ be a box of geodesics with $cr(a,b,c,d)=2$. This implies that $\frac{1}{C}\leq mod(\mathbb{D}(a,b,c,d))\leq C$ for some $C>1$, where $\mathbb{D}(a,b,c,d)$ is the quadrilateral with interior $\mathbb{D}$ and a-sides $[a,b]\subset S^1$ and $[c,d]\subset S^1$. Let I be a differentiable transverse arc to the leaves of the vertical foliation with one endpoint in $[a,b]$ and the other endpoint in $[c,d]$ that does not contain zeros of φ. Then we have

$$\|\varphi\|_{L^1} > \iint_{\mathbb{D}^{\geq 1}}|\varphi(z)|dA = \int_{I\cap\mathbb{D}^{\geq 1}} l^\varphi(v^\varphi(z))dx \geq \int_{I\cap\mathbb{D}^{\geq 1}}dx,$$

where x is the real part of the natural parameter along I. Moreover, we have

$$C\geq mod(\mathbb{D}(a,b,c,d))\geq mod(\mathcal{V}^{<1}) = \int_{I\cap\mathbb{D}^{<1}}\frac{1}{l^\varphi(v^\varphi(z))}dx\geq\int_{I\cap\mathbb{D}^{<1}}dx.$$

Since $\nu_\varphi([a,b]\times[c,d]) = \int_I dx\leq\|\varphi\|_{L^1}+C$, we have that $\|\nu_\varphi\|_{Th}<\infty$. ∎

Note the formula

$$\|\varphi\|_{L^1} = \int_{S^1\times S^1-diag}\frac{d\mu^\varphi}{d\nu^\varphi}d\mu^\varphi.$$

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