VERTICAL LIMITS OF GRAPH DOMAINS

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Abstract. We consider the limiting behavior of Teichmüller geodesics in the universal Teichmüller space $T(\mathbb{H})$. Our main result states that the limits of the Teichmüller geodesics in the Thurston's boundary of $T(\mathbb{H})$ may depend on both vertical and horizontal foliation of the corresponding holomorphic quadratic differential.

1. Introduction

By Uniformization Theorem, a simply connected domain D in the complex plane $\mathbb{C}$ is conformally equivalent to the hyperbolic plane $\mathbb{H}$. The set of prime ends of D is homeomorphic to the unit circle $\mathbb{S}^1$-the ideal boundary of $\mathbb{H}$ (cf. [9]). Unless stated otherwise, we implicitly assume this identification.

The map T_ε of D that is obtained by multiplying the distances in the vertical direction by $\varepsilon > 0$ is called the *Teichmüller map*. Thus, for $\varepsilon > 0$ we have

$$T_\varepsilon(x, y) = (x, \varepsilon y).$$

The image of D under T_ε is a new simply connected domain D_ε in $\mathbb{C}$. The Teichmüller map extends by continuity to a marking homeomorphism between the space of prime ends of D and the space of prime ends of D_ε. Note that both spaces of prime ends are implicitly identified with the unit circle $\mathbb{S}^1$ (cf. [9]). We prove (cf. §5)

Theorem 1.1. *Let D be a simply connected domain under the graph of a real-valued function. Assume that Γ is the family of curves in D connecting $(a, b) \subset \mathbb{S}^1$ and $(c, d) \subset \mathbb{S}^1$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon) = \mathrm{mod}(\Gamma_v), \tag{1}$$

where $\Gamma^\varepsilon = T_\varepsilon(\Gamma)$ is the image of Γ under the Teichmuller map and Γ_v is the family, possibly empty, of vertical line segments in Γ.

In the theorem above $\mathrm{mod}(\Gamma)$ denotes the conformal modulus of a curve family Γ, see Section 3 for the definition of the modulus.

Next, we interpret Theorem 1.1 in terms of the asymptotic behavior of Teichmüller geodesics corresponding to a particular type of quadratic differentials in the universal Teichmüller space.

The universal Teichmüller space $T(\mathbb{H})$ consists of all quasisymmetric maps of $\mathbb{S}^1$ which fix $-i, 1, i \in \mathbb{S}^1$. The Teichmüller map under the identification of the prime ends of D and D^ϵ with $\mathbb{S}^1$ induces a quasisymmetric map $h_\epsilon : \mathbb{S}^1 \to \mathbb{S}^1$. Thus

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we obtain a path in the universal Teichmüller space $T(\mathbb{H})$ parameterized by $\varepsilon > 0$ which corresponds to a Teichmüller geodesic. The path $\varepsilon \mapsto h_\varepsilon$ is unbounded in the Teichmuller metric (see Section 2 for the definition) as $\varepsilon \to 0$.

We consider the question of finding the limiting behavior of the Teichmüller geodesic in the universal Teichmüller space $T(\mathbb{H})$. Masur [7] described the limiting behavior of Teichmüller geodesics for the case of compact surfaces. He showed that if the vertical foliation of the corresponding quadratic differential φ is uniquely ergodic then the limit of the Teichmüller geodesic in the Thurston's boundary is the projective class of the measured lamination which is equivalent to the vertical foliation of φ (cf. [7]). When the vertical foliation consists of finitely many cylinders, then the limit is the projective lamination with support consisting of closed geodesics homotopic to the cylinders of the vertical foliation but the weights are all equal while the cylinder heights might be different (cf. [7]). Moreover, there are examples of Teichmüller geodesics which do not have unique limiting points on Thurston's boundary (cf. [6]).

In the case of closed surfaces, the limiting behavior of Teichmüller geodesics is investigated using the lengths of simple closed geodesics. There are no closed geodesics in the hyperbolic plane $\mathbb{H}$. Thurston's boundary to the universal Teichmüller space $T(\mathbb{H})$ is identified with the space $PML_{bdd}(\mathbb{H})$ of projective bounded measured laminations on $\mathbb{H}$ (cf. [11], [12]). The bordification of $T(\mathbb{H})$ is done using geodesic currents, i.e. the space of positive Borel measures on the space of geodesics of $\mathbb{H}$ (for definition see §2 and [3]). Therefore the study of limit points involves the study of the limits of geodesic currents. Although we consider limits of Teichmüller geodesics as in the case of closed surfaces, the ideas and arguments used are somewhat more analytical in nature and are disjoint from the prior work on closed surfaces (cf. [7], [6]). We prove

Theorem 1.2. *Let $\varphi : \mathbb{H} \to \mathbb{C}$ be an integrable holomorphic quadratic differential on the hyperbolic plane $\mathbb{H}$ without zeros or poles in $\mathbb{H}$. Assume that the image in $\mathbb{C}$ of $\mathbb{H}$ in the natural parameter of φ is a domain D bounded by the graphs of two functions $f(x)$ and $g(x)$ defined on an interval I of the real line. Denote by h_ε, for $\varepsilon > 0$, the Teichmüller geodesic which scales the vertical direction of φ by $\varepsilon > 0$. Then the limit of the Teichmüller geodesic h_ε as $\varepsilon \to 0$ is equal to the projective class of the measured lamination whose support is homotopic to the vertical foliation of φ and whose transverse measure is given by*

$$\int_I \frac{1}{|f(x) - g(x)|} dx$$

where I is a horizontal arc transverse to the vertical foliation and dx is the linear measure on the horizontal interval I.

We remark that the limiting projective measured lamination, although unique, cannot be described solely in terms of the vertical foliation of the holomorphic quadratic differential φ. This is a new phenomenon which does not appear in the Teichmüller spaces of compact surfaces. To illustrate this phenomenon, assume that D is the domain under the graph of a step function. Then the transverse measure is a multiple of the linear measure by the reciprocal of the heights of the steps.

One consequence of this phenomenon is that if we consider φ on $\mathbb{H}$ and a corresponding holomorphic differential φ_1 on $f(\mathbb{H})$, where f is a marking map defining a point in $T(\mathbb{H})$, the limits of the corresponding Teichmüller geodesics in the Thurston's boundary are different even though φ and φ_1 have the same vertical foliations. In the case of closed surfaces the limits are the same.

We point out that finite area of D can be replaced by "locally finite area" condition-for each finite horizontal arc the total area of the domain formed by the vertical leaves intersecting it is finite. The convergence in Theorem 2 is in the weak* topology on the geodesic currents. It is an interesting question to determine whether the above convergence holds for the uniform weak* topology from [13]. Moreover, it would be interesting to extend Theorem 2 to the case of arbitrary finite area Jordan domain or even to arbitrary integrable holomorphic quadratic differentials.

2. Thurston's boundary of the universal Teichmüller space

Let $\mathbb{H}$ be the hyperbolic plane. The ideal boundary of $\mathbb{H}$ is homeomorphic to the unit circle $\mathbb{S}^1$ in the complex plane. A homeomorphism $h:\mathbb{S}^1\to\mathbb{S}^1$ is said to be *quasisymmetric* if there exists $M\geq 1$ such that

$$\frac{1}{M}\leq\frac{|h(I)|}{|h(J)|}\leq M$$

for all circular arcs I,J with a common boundary point and disjoint interiors such that $|I|=|J|$, where $|I|$ is the length of I. A homeomorphism is quasisymmetric if and only if it extends to a quasiconformal map of the unit disk. Since in this note we will not use quasiconformal mappings we refer to the classical lecture notes of Ahlfors [2] for background on planar quasiconformal mappings.

Definition 2.1. The universal Teichmüller space $T(\mathbb{H})$ consists of all quasisymmetric maps $h:\mathbb{S}^1\to\mathbb{S}^1$ that fix $-i,1,i\in\mathbb{S}^1$.

If $g:\mathbb{D}\to\mathbb{D}$ is a quasiconformal map, denote by $K(g)$ its quasiconformal constant. The Teichmüller metric on $T(\mathbb{H})$ is given by $d(h_1,h_2)=\inf_g\log K(g)$, where g runs over all quasiconformal extensions of the quasisymmetric map $h_1\circ h_2^{-1}$. The Teichmüller topology is induced by the Teichmüller metric.

Thurston [3],[4],[14] introduced a boundary to the Teichmüller space of a closed hyperbolic surface as follows. First, the Teichmüller space $T(S)$ of a closed surface S embeds into $\mathbb{R}^{\mathcal{S}}$, where $\mathcal{S}$ is the set of all simple closed curves of S. The embedding $T(S)\hookrightarrow\mathbb{R}^{\mathcal{S}}$ is defined by assigning to each $\alpha\in\mathcal{S}$ the length of its geodesic representative for the marked hyperbolic metric on the surface S defining the point of $T(S)$. The Teichmüller space remains embedded after projectivization $T(S)\hookrightarrow\mathbb{R}^{\mathcal{S}}\hookrightarrow P\mathbb{R}^{\mathcal{S}}$ and Thurston's boundary consists of the limit points of the image of $T(S)$. It turns out that the Thurston's boundary is identified with the space of projective measured laminations on S.

Bonahon [3] used a different approach to obtain Thurston's boundary by embedding $T(S)$ into the space of geodesic currents on S. A *geodesic current* on S is a positive Borel measure on the space of geodesics $(\mathbb{S}^1\times\mathbb{S}^1\setminus diag)/\mathbb{Z}_2$ of the universal covering $\mathbb{H}$ of S that is invariant under the action of the covering group $\pi_1(S)$. Each point in the Teichmüller space $T(S)$ is a (marked) hyperbolic metric which defines a unique (up to positive multiple) positive measure of full support, called the *Liouville measure*, on the space of geodesics of the universal covering invariant under the action of the covering group. Since the marking maps conjugate covering groups, the pull backs of the Liouville measures under the marking maps give geodesics currents on the base surface S. Then the closure of the projectivization of the embedding of $T(S)$ in the space of the geodesic currents of S gives Thurston's boundary [3].

The approach to the Thurston's boundary using geodesic currents is used in [11], [13] to introduce Thurston's boundary to the Teichmüller space of arbitrary hyperbolic Riemann surface including the universal Teichmüller space $T(\mathbb{H})$ because arbitrary Riemann surfaces might not have enough non-trivial closed curves (e.g. the hyperbolic plane $\mathbb{H}$ has no non-trivial closed curves). The space of geodesics of the hyperbolic plane $\mathbb{H}$ is identified with $\mathbb{S}^1\times\mathbb{S}^1\setminus diag$ by assigning to each geodesic the pair of its endpoints. The *Liouville measure* $\mathcal{L}$ on the space of geodesic of $\mathbb{H}$ is given by

$$\mathcal{L}(A)=\int_A\frac{d\alpha d\beta}{|e^{i\alpha}-e^{i\beta}|^2}$$

for any Borel set $A\subset\mathbb{S}^1\times\mathbb{S}^1$. If $A=[a,b]\times[c,d]$ then

$$\mathcal{L}([a,b]\times[c,d])=\log\frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

To each $h\in T(\mathbb{H})$, we assign the pull-back $h^*(\mathcal{L})$ of the Liouville measure by the quasisymmetric map $h:\mathbb{S}^1\to\mathbb{S}^1$. This assignment is a homeomorphism of $T(\mathbb{H})$ onto its image in the space of bounded geodesic currents; a geodesic current α is *bounded* if

$$\sup_{[a,b]\times[c,d]}\alpha([a,b]\times[c,d])<\infty$$

where the supremum is over all $[a,b]\times[c,d]$ with $\frac{(a-c)(b-d)}{(a-d)(b-c)}=2$. The space of bounded geodesic currents is endowed with the family of Hölder norms parametrized with the Hölder exponents $0<\nu\leq 1$ (cf. [11]). The homeomorphism of $T(\mathbb{H})$ into the space of bounded geodesic currents is differentiable with a bounded derivative given by a Hölder distribution (cf. [12]) and, in fact, Otal [8] proved that it is real-analytic. The map from $T(\mathbb{H})$ to the projective bounded geodesic currents remains a homeomorphism and the boundary points of the image of $T(\mathbb{H})$ are all projective bounded measured laminations (cf. [11]). Thus Thurston's boundary of $T(\mathbb{H})$ is the space $PML_{bdd}(\mathbb{H})$ of all projective bounded measured laminations on $\mathbb{H}$ (and an analogous statement holds for any hyperbolic Riemann surface). Alternatively, the space of geodesic currents can be endowed with the uniform weak* topology and Thurston's boundary for $T(\mathbb{H})$ is again $PML_{bdd}(\mathbb{H})$ (cf. [13]).

3. The limits of the moduli of families of curves and the Liouville measure

Let R be a simply connected region in $\mathbb{C}$ other than the complex plane. Let $f:R\to\mathbb{H}$ be the Riemann mapping, where $\mathbb{H}$ is the unit disk model of the hyperbolic plane. Then the set of prime ends of R in the sense of Caratheodory is in a one to one correspondence with the points of the unit circle $\mathbb{S}^1$ (cf. [9]). When we consider a simply connected domain we will always implicitly assume the correspondence of the prime ends with the points of the unit circle $\mathbb{S}^1$ under the Riemann mapping.

Our goal is to relate the Liouville measure associated to two closed disjoint arcs of $\mathbb{S}^1$ with the modulus of the family of curves (defined below) in $\mathbb{H}$ connecting the two closed arcs. The correspondence between the prime ends of a simply connected domain R and $\mathbb{S}^1$ directly translates to R the conclusions that we obtain for $\mathbb{S}^1$.

3.1. **Conformal modulus and its properties.** Next we define the conformal modulus of a family of curves in $\mathbb{C}$ which is the main tool in this note. Suppose Γ is a family of locally rectifiable curves in $\mathbb{C}$. A non-negative Borel measurable function $\rho : \mathbb{C} \to [0, \infty]$ is called a Γ - *addmissible metric* if for every $\gamma \in \Gamma$ we have

$$l_\rho(\gamma) = \int_\gamma \rho(z)|dz| \geq 1.$$

The quantity $l_\rho(\gamma)$ is often called the ρ-length of γ. The *conformal modulus* $\mathrm{mod}(\Gamma)$ of Γ is defined by

$$\mathrm{mod}(\Gamma) = \inf_\rho \int_{\mathbb{D}} \rho(z)^2 dxdy$$

where the infimum is over all Γ-admissible metrics ρ.

In what follows we will need some basic properties of the modulus. We refer to [5, 15] for the proofs of the properties below and for further background on conformal modulus.

We will say that Γ_1 *overflows* Γ_2 and will write $\Gamma_1 > \Gamma_2$ if every curve $\gamma_1 \in \Gamma_1$ contains some curve $\gamma_2 \in \Gamma_2$.

Lemma 3.1. *Let $\Gamma_1, \Gamma_2, \dots$ be curve families in $\mathbb{C}$. Then*

1. ***(Monotonicity)*** *If $\Gamma_1 \subset \Gamma_2$ then* $\mathrm{mod}(\Gamma_1) \leq \mathrm{mod}(\Gamma_2)$.
2. ***(Subadditivity)*** $\mathrm{mod}(\bigcup_{i=1}^\infty \Gamma_i) \leq \sum_{i=1}^\infty \mathrm{mod}(\Gamma_i)$.
3. ***(Overflowing)*** *If $\Gamma_1 < \Gamma_2$ then* $\mathrm{mod}(\Gamma_1) \geq \mathrm{mod}(\Gamma_2)$.

Another very important property of the conformal modulus is its invariance under conformal mappings of the plane.

Lemma 3.2 (Conformal invariance of modulus)**.** *Suppose Γ is a family of curves in a domain $D \subset \mathbb{C}$ and f is a conformal mapping of D onto D'. Let $f(\Gamma) \subset D'$ be the image of the family Γ, i.e. $f(\Gamma) = \{f(\gamma) : \gamma \in \Gamma\}$. Then*

$$\mathrm{mod}(f(\Gamma)) = \mathrm{mod}(\Gamma).$$

3.2. **Modulus and Liouville measure.** Let (a, b, c, d) be a quadruple of distinct points on $\mathbb{S}^1$ given in the counterclockwise order. Denote by $\Gamma_{(a,b,c,d)}$ the family of all locally rectifiable curves $\gamma \subset \mathbb{D}$ connecting (a, b) to (c, d), i.e. $(a, b) \cup \gamma \cup (c, d)$ is a connected subset of the plane.

Lemma 3.3. *Let (a, b, c, d) be a quadruple of points on $\mathbb{S}^1$ in the counterclockwise order. Let $\Gamma_{(a,b,c,d)}$ consist of all curves γ in $\mathbb{D}$ which connect $(a, b) \subset \mathbb{S}^1$ with $(c, d) \subset \mathbb{S}^1$. Then*

$$\mathrm{mod}(\Gamma_{(a,b,c,d)}) - \frac{1}{\pi}\mathcal{L}([a, b] \times [c, d]) - \frac{2}{\pi}\log 4 \to 0$$

as $\mathrm{mod}(\Gamma_{(a,b,c,d)}) \to \infty$*, where $\mathcal{L}$ is the Liouville measure.*

Remark 3.4. Note that simultaneously $\mathrm{mod}(\Gamma_{(a,b,c,d)}) \to \infty$ and $\mathcal{L}([a, b] \times [c, d]) \to \infty$.

Proof. Consider a conformal mapping of the unit disc onto the upper half plane mapping the points $a, b, c, d \in \mathbb{S}$ to $w_1, w_2, w_3, \infty \in \mathbb{R} \cup \{\infty\}$, respectively, so that $-\infty < w_1 < w_2 < w_3 < \infty$. By the conformal invariance of the modulus we have

$$\text{mod}(\Gamma_{(a,b,c,d)}) = \text{mod}(\Omega_{(w_1,w_2,w_3,\infty)}) \tag{2}$$

where $\Omega_{(w_1,w_2,w_3,\infty)}$ is the family of curves connecting the segment $[w_1, w_2]$ to the segment $[w_3, \infty]$ in in the upper half plane. Furthermore, a simple symmetry argument (cf. [5, page 81]) shows that the modulus of the family of arcs $\Omega_{(w_1,w_2,w_3,\infty)}$ connecting the segment $[w_1, w_2]$ to the segment $[w_3, \infty]$ in $\mathbb{C}$ satisfies

$$\text{mod}(\Omega_{(w_1,w_2,w_3,\infty)}) = \frac{2}{\pi}\mu(\sqrt{\frac{w_3 - w_2}{w_3 - w_1}}), \tag{3}$$

where $\mu(r)$ is the 2π-multiple of the modulus of the family of closed curves in the unit disk, which separate the unit circle $\mathbb{S}^1$ and the arc on the real axis from 0 to r with $0 < r < 1$ (cf. [5, page 53]). Careful estimates on $\mu(r)$ then yield the following asymptotics:

$$\mu(r) - \log \frac{4}{r} \to 0, \tag{4}$$

as $r \to 0$ (cf. [5, page 62, (2.11)]). Let $w_1 < w_2 < w_3$ be three real numbers.

The lemma now follows easily if we combine (2),(3) and (4) together with the fact that Liouville measure is invariant under Möbius maps and therefore $\mathcal{L}([w_1, w_2] \times [w_3, \infty]) = \log \frac{w_3 - w_1}{w_3 - w_2}$. □

4. Modulus of vertical families

Theorem 1.1 states that the limiting behaviour of moduli of certain families of curves is completely determined by the subfamily Γ_v of vertical curves in D. For this reason we start by calculating the modulus of a general family of vertical curves in $\mathbb{C}$.

Lemma 4.1. *Let $E \subset \mathbb{R}$ be a measurable set and let Γ_v be a family of vertical intervals $\{\gamma(x)\}_{x \in E}$, where $\gamma(x)$ is an interval of length $|\gamma(x)| > 0$ contained in the vertical line passing through $x \in E \subset \mathbb{R}$. Then the modulus of Γ_v can be computed using the following Lebesgue integral*

$$\text{mod}(\Gamma_v) = \int_E \frac{dx}{|\gamma(x)|}. \tag{5}$$

Proof. Define

$$\rho_0(x, y) = \begin{cases} |\gamma(x)|^{-1}, & \text{for } (x, y) \in \gamma(x), \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Since every $\gamma(x) \in \Gamma_v$ is a vertical interval, we have that $\int_{\gamma(x)} \rho_0(x, y)|dz| = |\gamma(x)|^{-1} \int_{\gamma(x)} dy = 1$. Thus ρ_0 is admissible for Γ_v and we have

$$\begin{aligned} \text{mod}\Gamma_v &\leq \iint_D \rho_0^2(x, y)dxdy = \int_E \left(\int_{\gamma(x)} |\gamma(x)|^{-2} dy \right) dx \\ &= \int_E |\gamma(x)| \cdot |\gamma(x)|^{-2} dx = \int_E \frac{dx}{|\gamma(x)|}. \end{aligned}$$

To obtain the opposite inequality we will use the following well known criterion of Beurling, cf. Theorem 4.4 in [1]. Note that in [1] the criterion is formulated for the extremal length rather than the modulus, but it is easily seen that the formulation below is equivalent to the one in [1]. Recall that a Γ-admissible metric ρ_0 is said to be extremal for the family Γ if $\mathrm{mod}(\Gamma) = \iint_D \rho_0(x,y)^2 dxdy$.

Lemma 4.2 (Beurling's criterion)**.** *The metric ρ_0 is extremal for Γ if there is a subfamily $\Gamma_0 \subset \Gamma$ such that*

- $\int_\gamma \rho_0 ds = 1, \forall \gamma \in \Gamma_0$
- *for any real valued h in D satisfying $\int_\gamma h ds \geq 0, \forall \gamma \in \Gamma_0$ the following holds*

$$\iint_D h\rho_0 dxdy \geq 0.$$

As was noted above the function $\rho_0(x,y)$ defined in (6) satisfies the first condition of the Beurling's criterion. To check that the second condition is also satisfied note that for a function h in D such that $\int_{\gamma(x)} h(x,y)dy \geq 0$ for every $x \in E$ we have by Fubini's theorem

$$\iint_D h(x,y)\rho_0(x,y)dxdy = \int_E |\gamma(x)|^{-1}\left[\int_{\gamma(x)} h(x,y)dy\right]dx \geq 0,$$

since $|\gamma(x)| > 0, \forall x \in \mathbb{R}$. Thus ρ_0 is extremal for Γ_v (in this case $\Gamma_0 = \Gamma_v$). $\square$

5. The domains under the graphs of functions

Given a function $f : (A,B) \to (0,\infty)$ it is well known that the set

$$D := \{(x,y) : x \in (A,B), 0 < y < f(x)\} \tag{7}$$

(a.k.a. hypograph of f) is open if and only if f is a lower-semicontiuous function. If this is the case we will call D the *graph domain of the function f*. In this section we prove Theorem 1.1 by first proving it in the case when f is a continuous function and then by approximating an arbitrary lower semicontinuous function by an increasing sequence of continuous ones.

5.1. **A general estimate.** To begin we establish some estimates which hold for an arbitrary curve family Γ in any planar domain D of finite area. To formulate our result we will need a notation for a subfamily of "almost vertical" curves in Γ. Namely, for $\eta > 0$ we define subfamilies $\Gamma_{<\eta}$ and $\Gamma_{\geq\eta}$ of Γ as follows

$$\begin{aligned}\Gamma_{\geq\eta} &= \{\gamma \in \Gamma : |\pi_1(\gamma)| \geq \eta\},\\ \Gamma_{<\eta} &= \{\gamma \in \Gamma : |\pi_1(\gamma)| < \eta\},\end{aligned}$$

where $|\pi_1(\gamma)|$ is the length of the the vertical projection of γ onto the real axis.

Lemma 5.1. *Let D be an arbitrary finite area domain in $\mathbb{C}$ and Γ be a family of locally rectifiable curves in D. Let $T_\varepsilon(x,y) = (x,\varepsilon y)$ and $\Gamma^\varepsilon = T_\varepsilon(\Gamma)$. Then*

$$\mathrm{mod}(\Gamma_v) \leq \limsup_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon) \leq \lim_{\eta\to 0^+} \mathrm{mod}(\Gamma_{<\eta}). \tag{8}$$

Proof. By Lemma 4.1 we have

$$\mathrm{mod}(\Gamma_v) = \int_p^q \frac{dx}{f(x)} = \varepsilon\int_p^q \frac{dx}{\varepsilon f(x)} = \varepsilon \cdot \mathrm{mod}(\Gamma_v^\varepsilon),$$

Since $\Gamma_v^\varepsilon \subset \Gamma^\varepsilon$ monotonicity of the modulus imples that $\varepsilon \cdot \mathrm{mod}(\Gamma_v^\varepsilon) \leq \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon)$. This immediately yields the first inequality in (8).

To prove the right inequality in (8) note that by the subadditivity of the modulus we have

$$\limsup_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon) \quad \leq \quad \limsup_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon_{\geq\eta}) + \limsup_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon_{<\eta}).$$

Note that $\varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon_{\geq\eta}) \to 0$ as $\varepsilon \to 0$. Indeed, since for every $\gamma \in \Gamma^\varepsilon_{\geq\eta}$ we have that the length of γ is at least η it follows that $\rho(x) = \chi_{D^\varepsilon}(x)/\eta$ is admissible for $\Gamma^\varepsilon_{\geq\eta}$ and

$$\varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon_{\geq\eta}) \leq \varepsilon \cdot \int_{D^\varepsilon} (1/\eta)^2 dxdy \leq \frac{\varepsilon}{\eta^2} \cdot A(D^\varepsilon) \leq \frac{\varepsilon^2 A(D)}{\eta^2} \xrightarrow[\varepsilon\to 0]{} 0,$$

where $A(D)$ denotes the two dimensional area of a domain $D \subset \mathbb{C}$. Thus we have that

$$\limsup_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon) \quad \leq \quad \limsup_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon_{<\eta}). \tag{9}$$

Now, since $\Gamma^\varepsilon_{<\eta} = T_\varepsilon(\Gamma_{<\eta})$ and T_ε is ε^{-1}-quasiconformal we have $\mathrm{mod}(\Gamma^\varepsilon_{<\eta}) \leq \varepsilon^{-1}\mathrm{mod}(\Gamma_{<\eta})$, and from inequality (9) it follows that

$$\limsup_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon) \leq \mathrm{mod}(\Gamma_{<\eta}).$$

Since the last inequality holds for every $\eta > 0$ and $\mathrm{mod}(\Gamma_{<\eta})$ is non-decreasing in η we obtain the right hand side inequality in (8) by taking η to 0. □

Remark 5.2. Lemma 5.1 implies that to prove Theorem 1.1 and to obtain equality (1) for a family Γ in D it is enough to show that the following inequality holds

$$\mathrm{mod}(\Gamma_v) \geq \lim_{\eta\to 0^+} \mathrm{mod}(\Gamma_{<\eta}). \tag{10}$$

Remark 5.3. Inequality (10) does not hold always. For instance let Γ be the collection of all the curves in the unit square $[0,1]^2$ connecting the horizontal sides $[0,1]\times\{0\}$ and $[0,1]\times\{1\}$, excluding the family of vertical segments $\{x\}\times[0,1], 0 \leq x \leq 1$. Then $\mathrm{mod}(\Gamma_{<\eta}) = 1$ for every $\eta > 0$ while $\mathrm{mod}(\Gamma_v) = \mathrm{mod}(\emptyset) = 0$ and the inequality (10) fails.

5.2. **Domains under graphs of continuous functions.** Let D be a finite area domain in $\mathbb{C}$ under the graph of a continuous function $f : (A, B) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, where (A, B) could be a finite or an infinite interval including $(A, B) = \mathbb{R}$. Let (a, b, c, d) be in the given cyclic order on the boundary ∂D of the domain D and let Γ be the family of curves in D connecting (a, b) to (c, d). Denote by (p, q) the intersection $(a, b) \cap (c, d)$.

Theorem 5.4. *With the notations as above the following equalities hold*

$$\lim_{\varepsilon\to 0} \varepsilon \cdot \mathrm{mod}(\Gamma^\varepsilon) = \int_p^q \frac{dx}{f(x)} = \mathrm{mod}(\Gamma_v). \tag{11}$$

Proof. The second equality in (11) holds by Lemma 4.1. By Remark 5.2 we only need to show that

$$\lim_{\eta\to 0^+} \mathrm{mod}(\Gamma_{<\eta}) \leq \int_p^q \frac{dx}{f(x)}. \tag{12}$$

Now, let $p = x_0 < x_1 < \ldots < x_n = q$ be a partition of the interval $[p, q]$. Then by subadditivity of the modulus we obtain

$$\mathrm{mod}(\Gamma_{<\eta}) \quad \leq \quad \sum_{i=0}^{n-1} \mathrm{mod}\{\gamma \in \Gamma_{<\eta} : \gamma(0) \in [x_i, x_{i+1}]\},$$

Considering the rectangles

$$R_i = [x_i - \eta, x_{i+1} + \eta] \times \min_{[x_i-\eta, x_{i+1}+\eta]} f$$

we note that every curve from the family $\{\gamma \in \Gamma_{<\eta} : \gamma(0) \in [x_i, x_{i+1}]\}$ contains a subcurve in R_i which connects the horizontal sides of the rectangle. Therefore, by the property of overflowing, we have for every $i = 0, \ldots, n-1$ the estimate

$$\mathrm{mod}\{\gamma \in \Gamma_{<\eta} : \gamma(0) \in [x_i, x_{i+1}]\} \leq \frac{x_{i+1} - x_i + 2\eta}{\min_{[x_i-\eta, x_{i+1}+\eta]} f}.$$

Summing up over i we obtain

$$\mathrm{mod}(\Gamma_{<\eta}) \leq \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i + 2\eta}{\min_{[x_i-\eta, x_{i+1}+\eta]} f}.$$

Taking η to 0 we obtain

$$\lim_{\eta \to 0^+} \mathrm{mod}(\Gamma_{<\eta}) \leq \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{\min_{[x_i, x_{i+1}]} f},$$

where the sum on the right is a Riemann sum for the integral $\int_p^q \frac{dx}{f(x)}$ and therefore can be taken to be less than $\int_p^q \frac{dx}{f(x)} + \delta$ for every $\delta > 0$. This proves (12) and thus the theorem. □

5.3. **General graph domains.** In this section, we consider a graph domain D in $\mathbb{C}$ of finite Euclidean area under the graph of an arbitrary lower semicontinuous function $f : (A, B) \to [0, \infty]$.

We consider the set of prime ends for the domain D. If D is conformally mapped onto the unit disk $\mathbb{H}$ then the set of prime ends of D is in a one to one correspondence with the unit circle $\mathbb{S}^1 = \partial B_1$ (cf. [9]). The set of prime ends inherits an orientation from $\mathbb{S}^1$. Let $\{s_i\}_{i=1}^{\infty}$ be a system of cross-cuts defining prime end a of D. A continuous ray $r : [0, 1) \to D$ is said to have an endpoint $r(1)$ equal to the prime end a if there exists $t_i < 1$ such that $r((t_i, 1))$ is contained in the component of $D_i = D \setminus s_i$ not containing $r(0)$ for all large i.

Lemma 5.5. *The imprint $I(a) = \cap_i \bar{D}_i$ of a prime end a of the domain D lies on a vertical line, where $\{D_i\}_i$ is the system of subdomains of $D \setminus s_i$ defining a.*

Proof. Assume on the contrary that $z_1, z_2 \in \cap_i \bar{D}_i$ with $Re(z_1) < Re(z_2)$. Since D is the domain under the graph of f, it follows that each D_i is a subset of the union of vertical segments between points of s_i and the graph of f, or the union of the vertical segments between points of s_i and the real axis. In either case, the vertical projection onto the x-axis of each s_i contains the interval $(Re(z_1), Re(z_2))$. Thus the length of s_i does not converge to 0. Contradiction. □

We denote by $\pi_1(a)$ the x-coordinate of the prime end a. Let us assume that the interval (a,b) is on the bottom side of D (belongs to x-axis) and the interval (c,d) is on the top side of D (lies on the part of the boundary of D above the x-axis determined by the graph of f) and $\pi_1(c) > \pi_1(d)$ (since a,b,c,d are given in a cyclic order). We consider the intersection interval $(a,b)\cap(\pi_1(d),\pi_1(c)) = (p,q)$.

Theorem 5.6. *Let D be a finite area domain in $\mathbb{C}$ under the graph of an arbitrary function $f:(A,B)\to[0,\infty]$ in the above sense. Then for any quadruple (a,b,c,d) of prime ends of D in the given cyclic order, we have*

$$\lim_{\varepsilon\to 0}\varepsilon\cdot \operatorname{mod}(\Gamma^{\varepsilon}_{(a,b,c,d)}) = \int_p^q \frac{dx}{f(x)} = \operatorname{mod}(\Gamma_v)$$

where Γ_v is the family of vertical lines that connect the interval (a,b) to the interval (c,d), and $[p,q]$ is the interval of the x-coordinates of Γ_v.

Proof. Just like in the proof of Theorem 5.4 it is enough to show that the inequality (12) holds even if f is lower semicontinuous.

We will use the well known fact that if $f:(A,B)\to[0,\infty]$ is a lower semicontinuous function then there is a sequence of continuous functions $f_n:(A,B)\to[0,\infty)$ such that $f_n(x)\le f_{n+1}(x), n=1,2,\dots$ and $f_n(x)\to f(x)$ for every $x\in(A,B)$. Next, let

$$D_n = \{(x,y): x\in(A,B), y\in(0,f_n(x))\}$$

and let $\Gamma_{<\eta,n}$ be the collection of curves γ in D_n such that $\gamma(0)\in(p,q)$, $\gamma(1)$ belongs to the graph of f_n and $|\pi_1(\gamma)|<\eta$. In other words, $\gamma\in\Gamma_{<\eta,n}$ connects the interval (p,q) to the graph of f_n and has "horizontal variation" $<\eta$.

Note that for every $\eta>0$ and every $n\in\mathbb{N}$ the family $\Gamma_{<\eta}$ overflows $\Gamma_{<\eta,n}$. Indeed, since $f_n(x)<f(x)$ a curve $\gamma\in\Gamma_{<\eta}$ would have to "hit" the graph of the continuous f_n before "reaching" the graph of f. Therefore $\operatorname{mod}(\Gamma_{<\eta})\le\operatorname{mod}(\Gamma_{<\eta,n})$ and since f_n is continuous Theorem 5.4 yields

$$\limsup_{\eta\to 0^+}\operatorname{mod}(\Gamma_{<\eta})\le\limsup_{\eta\to 0^+}\operatorname{mod}(\Gamma_{<\eta,n})\le\int_p^q\frac{dx}{f_n(x)}. \tag{13}$$

Now, by our assumption a,b,c and d are different prime ends and therefore $\operatorname{mod}(\Gamma_{(a,b,c,d)})<\infty$. Next we note that this implies that $\min_{x\in[p,q]}f(x)>0$. Indeed, since f is lower semicontinuous it attain a minimum in $[p,q]$, we will denote this minimum by m. Now, since D is a domain we have that if $m=0$ then it is attained at one (or both) of the endpoints of $[p,q]$. This would only be possible if (say) $a=d$, which is a contradiction.

Now, since there is a constant $c>0$ such that $f(x)>c$ for every $x\in[p,q]$ then it is easy to see that we can assume that $f_n(x)\ge c>0$ for $x\in(p,q)$ (just redefine the f_n to be the maximum of c and the old f_n). Hence by the dominated convergence theorem we have $\int_p^q(f_n(x))^{-1}dx\to\int_p^q(f_n(x))^{-1}dx$, which gives (12) in the case of a semi-continuous f. ∎

The above theorem immediately gives Theorem 1.1 from Introduction. Theorem 1.2 is a direct consequence of Theorem 1 and Lemma 3.3.

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