THURSTON'S BOUNDARY FOR TEICHMÜLLER SPACES OF INFINITE SURFACES: THE GEODESIC CURRENTS AND THE LENGTH SPECTRUM

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Abstract. Let X be an infinite area hyperbolic surface. We introduce the Thurston's boundary to the Teichmüller space $T(X)$ of the surface X using two invariants of the hyperbolic metric: the Liouville currents and the length spectrum. The Thurston's boundary to $T(X)$ using Liouville (geodesic) currents is identified with the space $PML_{bdd}(X)$ of projective bounded measured laminations on X which naturally extends Thurston's result for closed surfaces (cf. [9], [5]). The Thurston's boundary using the length spectrum of X is a "closure" of $PML_{bdd}(X)$, and it coincides with $PML_{bdd}(X)$ when X can be decomposed into a countable union of geodesic pairs of pants whose boundary geodesics $\{\alpha_n\}_{n\in\mathbb{N}}$ have pinched lengths. When the lengths of the boundary curves of the geodesic pairs of pants $\{\alpha_n\}_n$ are only bounded from the above and the lengths of a subsequence $\{\alpha_{n_k}\}$ go to zero, the Thurston's boundary using the length spectrum is strictly larger than $PML_{bdd}(X)$.

1. Introduction

Fix a complete, borderless, infinite area hyperbolic surface X. The space of all quasiconformal deformations of X modulo isometries and isotopies is an infinite-dimensional manifold called the Teichmüller space $T(X)$ of X. We study the limiting behavior of the quasiconformal deformations of X when the dilatations of the quasiconfomal maps increase without bound. This study uses two different invariants of the hyperbolic metric: the Liouville currents and the length spectrum. Thurston [22], [9] used the length spectrum to compactify the Teichmüller space of a closed surface by adding to it the space of projective measured laminations of the surface. Bonahon [5] used Liouville currents to give an alternative description of the Thurston's boundary for the Teichmüller space of a closed surface.

In [18], the Hölder topology on the space of geodesic currents of an infinite area hyperbolic surface X is introduced in order to give a natural definition of the Thurston's boundary to the Teichmüller space $T(X)$ of an infinite area hyperbolic surface X. The Thurston's boundary is identified with the space $PML_{bdd}(X)$ of projective bounded measured laminations on X analogous to the case of closed surfaces. Our first contribution is an improvement in the choice of the topology on the geodesic currents. Namely, we adopt the uniform weak* topology (cf. [14]) to the space of the geodesic currents and prove that the Thurston's boundary to $T(X)$ is identified with $PML_{bdd}(X)$ as before (cf. [18]).

In the second part, we use the length spectrum of infinite hyperbolic surfaces in order to give an alternative definition of the Thurston's boundary for $T(X)$. Since

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we consider the length spectrum, it is natural to restrict our attention to complete, borderless, infinite hyperbolic surfaces that are obtained by gluing countably many geodesic pairs of pants (cf. [21], [3], [1]). It turns out that the Thurston's boundary using the length spectrum is a closure (in an appropriate sense) of $PML_{bdd}(X)$ and it is strictly larger than $PML_{bdd}(X)$ for certain classes of surfaces X. When the lengths of the geodesic boundaries of the pairs of pants are pinched between two non-zero constants, then the Thurston's boundary to $T(X)$ is equal to $PML_{bdd}(X)$. However, when the lengths of the geodesic boundaries of the pairs of pants are bounded from the above and a sequence of lengths goes to zero, then the Thurston's boundary of $T(X)$ is larger than $PML_{bdd}(X)$. More details follow.

Let X be a complete, borderless hyperbolic surface of infinite area (e.g. the hyperbolic plane $\mathbb{H}$, the complement of a Cantor set in the Riemann sphere, a topologically finite hyperbolic surface with funnel ends, and infinite genus surface). The universal covering $\tilde{X}$ is isometrically identified with the hyperbolic plane $\mathbb{H}$ and the isometry continuously extends to an identification of the boundary at infinity $\partial_\infty\tilde{X}$ with the unit circle S^1. The space $G(\tilde{X})$ of oriented geodesics of $\tilde{X}$ is identified with $(\partial_\infty\tilde{X}\times\partial_\infty\tilde{X})-diag\equiv(S^1\times S^1)-diag$ by assigning to each geodesic the pair of its endpoints, where $diag$ is the diagonal of $S^1\times S^1$.

The set $[a,b]\times[c,d]\subset(S^1\times S^1)-diag$ is called a *box of geodesics*, where $[a,b],[c,d]\subset S^1$ are disjoint closed arcs. The *Liouville measure* of the box of geodesic $[a,b]\times[c,d]$ is (cf. [5])

$$L([a,b]\times[c,d])=\log\frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

If $A\subset(S^1\times S^1)-diag$ is a Borel set, then its Liouville measure is given by

$$L(A)=\int_A\frac{|dx|\cdot|dy|}{|x-y|^2}.$$

The identification of $G(\tilde{X})$ with $(S^1\times S^1)-diag$ induces a full support, $\pi_1(X)$-invariant Borel measure on $G(\tilde{X})$ via the pull-back of the Liouville measure on $(S^1\times S^1)-diag$.

Two different hyperbolic metrics on X induce different identifications of $G(\tilde{X})$ and $(S^1\times S^1)-diag$ which in turn induce different measures on the space of geodesics $G(\tilde{X})$ via pull-backs of their Liouville measures. Denote by $\mathcal{M}(G(\tilde{X}))$ the space of all positive Borel measures (called *geodesic currents*) on $G(\tilde{X})$. The *Teichmüller space* $T(X)$ consists of all marked hyperbolic metrics on X modulo isometries homotopic to the identity via bounded homotopies. The *Liouville map*

$$\mathcal{L}:T(X)\to\mathcal{M}(G(\tilde{X}))$$

is defined by assigning to each marked hyperbolic metric the pull-back of the Liouville measure under the identification of $\tilde{X}$ and $\mathbb{H}^2$ induced by the hyperbolic metric (cf. Bonahon [5]).

When X is a finite closed surface of genus at least two, Bonahon [5] proved that the Liouville map is a homeomorphism onto its image when $\mathcal{M}(G(\tilde{X}))$ is equipped with the weak* topology. Moreover, the projectivization $P(\mathcal{L}(T(X)))$ of the image $\mathcal{L}(T(X))$ under the Liouville map remains a homeomorphism onto its image in the space of projective geodesic currents $P(\mathcal{M}(G(\tilde{X})))$. Bonahon [5] proved that the boundary of $P(\mathcal{L}(T(X)))$ inside $P(\mathcal{M}(G(\tilde{X})))$ consists of projective measured laminations $PML(X)$ of the closed surface X thus introducing Thurston's boundary to $T(X)$.

From now on, we assume that X is a hyperbolic surface of infinite area. A positive Borel measure m on $G(\tilde{X})$, called a geodesic current, is said to be *bounded* if

$$\sup_{[a,b]\times[c,d]} m([a,b]\times[c,d]) < \infty$$

where the supremum is over all boxes of geodesics $[a,b]\times[c,d]$ with $L([a,b]\times[c,d]) = \log 2$. Denote by $\mathcal{M}(G(\tilde{X}))$ the space of bounded geodesic currents on $G(\tilde{X})$. The Liouville map $\mathcal{L} : T(X) \to \mathcal{M}(G(\tilde{X}))$ is injective. If $\mathcal{M}(G(\tilde{X}))$ is equipped with the weak* topology then the Liouville map is not a homemorphism onto its image. In [19], a new topology on $\mathcal{M}(G(\tilde{X}))$ is introduced by embedding $\mathcal{M}(G(\tilde{X}))$ into the space of Hölder distributions on $G(\tilde{X})$ satisfying certain boundedness conditions. The Liouville map is an analytic homeomorphism onto its image in the space of the Hölder distributions (cf. Otal [15], and [19]).

The Hölder topology on $\mathcal{M}(G(\tilde{X}))$ is used to introduce Thurston's boundary to the Teichmüller space $T(X)$ when X is a hyperbolic surface of infinite area (cf. [19]). It turns out that the Thurston's boundary for $T(X)$ is the space of all projective bounded measured laminations $PML_{bdd}(X)$ of X as in the case of closed surfaces. Unlike for closed surfaces, the Thurston's bordification $T(X) \cup PML_{bdd}(X)$ is not compact, in fact it is not even locally compact.

The Hölder topology on $\mathcal{M}(G(\tilde{X}))$ is complicated for applications. The first contribution in this paper is simplifying the description of the topology on $\mathcal{M}(G(\tilde{X}))$ while obtaining the same Thurston's boundary to $T(X)$. The topology on $\mathcal{M}(G(\tilde{X}))$ that we use is called the *uniform weak* topology* and it is first introduced on the space $ML_{bdd}(\tilde{X})$ in [14] for the purposes of studying the relation between the earthquake measures and hyperbolic structures obtained by the corresponding earthquakes.

A box of geodesic $Q^* = [1,i]\times[-1,-i]$ is said to be the *standard box of geodesics*. If Q is a box of geodesics with $L(Q) = \log 2$, then there is a unique isometry γ_Q of $\mathbb{H}^2$ which maps Q onto Q^*. A sequence of measures $m_k \in \mathcal{M}(G(\tilde{X}))$ converges to $m \in \mathcal{M}(G(\tilde{X}))$ as $k \to \infty$ in the *uniform weak* topology* if for every continuous function $f : G(\tilde{X}) \to \mathbb{R}$ with its support in Q^* we have that

$$\sup_Q \int_{Q^*} f d(\gamma_Q)^*(m_k - m) \to 0$$

as $k \to \infty$, where the supremum is over all boxes Q with $L(Q) = \log 2$ (cf. [14]). In other words, all pull-backs of $m_k - m$ to the standard box Q^* must converge at the same speed to zero when integrated against a continuous function with support in Q^*. The "uniformity" comes from the fact that we consider pull-backs over all boxes of $G(\tilde{X})$ of the Liouville measure $\log 2$ in the supremum. We obtain

Theorem 1. *Let X be a complete hyperbolic surface without border with possibly infinite area. Then the Liouville map*

$$\mathcal{L} : T(X) \to \mathcal{M}(G(\tilde{X}))$$

is a homeomorphism onto its image when $\mathcal{M}(G(\tilde{X}))$ is equipped with the uniform weak topology. The image $\mathcal{L}(T(X))$ is closed and unbounded in $\mathcal{M}(G(\tilde{X}))$.*

The projectivization

$$P\mathcal{L} : T(X) \to P(\mathcal{M}(\tilde{X}))$$

of the Liouville map is a homeomorphisms and the image $P(\mathcal{L}(T(X)))$ *is not closed in* $P(\mathcal{M}(\tilde{X}))$*. The boundary of* $P(\mathcal{L}(T(X)))$ *is the space* $PML_{bdd}(X)$ *of projective bounded measured laminations-the Thurston's boundary to* $T(X)$*.*

Thurston [9] introduced boundary to the Teichmüller space of a closed surface of genus at least two using the length spectrum of the marked hyperbolic metrics on the surface. We consider an analogous construction for infinite area hyperbolic surfaces. In the context of infinite area hyperbolic surfaces, it is reasonable to restrict attention to surfaces whose hyperbolic metric is determined by its marked length spectrum. The study of the length spectrum properties for infinite surfaces is started by Shiga [21], and it was further developed by various authors(e.g. [1], [2], [3] [13], [11], [16],...). Most of the above work is done for the case of complete infinite area hyperbolic surfaces that have geodesic pants decomposition whose cuffs have length bounded from the above by a constant $M > 0$.

From now on we assume that X is a complete, borderless infinite area hyperbolic surface that has a geodesic pants decomposition. Let $\mathcal{S}$ denote the set of all simple closed geodesics on X for a fixed hyperbolic metric m_0. A choice of a hyperbolic metric m on X induces a function from $\mathcal{S}$ to $\mathbb{R}$ which assigns to each $\alpha \in \mathcal{S}$ the length of the geodesic in the metric m that is homotopic to α. Thus we have an injective map

$$\mathcal{X} : T(X) \to \mathbb{R}^{\mathcal{S}}_{\geq 0}.$$

When X is a closed hyperbolic surface then the above map is a homeomorphism onto its image if $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ is equipped with the weak* topology (cf. [9]). In the case of an infinite surface with a geodesic pants decomposition, the *length spectrum metric* is defined by (cf. [21], [2])

$$d_{ls}(m, m_1) = \sup_{\alpha \in \mathcal{S}} \Big| \log \frac{l_{m_1}(\alpha)}{l_m(\alpha)} \Big|.$$

Shiga [21] proved that the topology induced by the length spectrum metric on $T(X)$ is equal to the Teichmüller topology when the surface X has an upper and lower bounded geodesic pants decomposition. Allessandrini, Liu, Papadopoulos and Su [1] proved that the length spectrum on $T(X)$ is not even complete when X contains a sequence of simple closed geodesics whose length goes to zero. Thus the two topologies in this case are different.

We introduce a *normalized supremum norm* on $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ by

$$\|f\|_{\infty}^{norm} = \sup_{\alpha \in \mathcal{S}} \Big| \frac{f(\alpha)}{l_{m_0}(\alpha)} \Big|$$

where m_0 is a fixed hyperbolic metric on X and $f \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$. The normalized supremum norm induces the same topology on $T(X)$ as the length spectrum metric (cf. Lemma 8.1).

Analogous to the closed surface case, we projectivize $\mathcal{X}$ and obtain an injective map

$$P\mathcal{X} : T(X) \to P\mathbb{R}^{\mathcal{S}}_{\geq 0}.$$

By definition, the *length spectrum Thurston boundary* of $T(X)$ consists of the boundary points of the image $P\mathcal{X}(T(X))$ of $T(X)$, where $P\mathbb{R}^{\mathcal{S}}_{\geq 0}$ is given the quotient topology with respect to the normalized supremum norm on $\mathbb{R}^{\mathcal{S}}_{\geq 0}$.

Theorem 2. *Let X be an infinite area hyperbolic surface that has a geodesic pants decomposition $\{\alpha_n\}_{n\in\mathbb{N}}$. Then the length spectrum Thurston's boundary of $T(X)$ is the closure of the space of projective bounded measured laminations $PML_{bdd}(X)$ in $P\mathbb{R}^{\mathcal{S}}$, where $P\mathbb{R}^{\mathcal{S}}$ has the quotient topology induced by the topology on $\mathbb{R}^{\mathcal{S}}$ coming from the normalized supremum norm.*

If the lengths of $\{\alpha_n\}_{n\in\mathbb{N}}$ are pinched between two positive constants then the length spectrum Thurston's boundary is equal to $PML_{bdd}(X)$.

If the lengths of $\{\alpha_n\}_{n\in\mathbb{N}}$ are bounded from the above and there exists a subsequence $\{\alpha_{n_k}\}$ whose lengths converge to 0, then the length spectrum Thurston's boundary is strictly larger than $PML_{bdd}(X)$.

The Thurston's boundary to the Teichmüller space of a closed surface of genus at least two is the same whether we use the length spectrum or geodesic currents. However, for infinite surfaces the geodesic currents give a "canonical answer" which agrees with the finite case, while the length spectrum gives a larger set.

Let $\mathcal{P}=\{\alpha_n\}$ be a geodesic pants decomposition of X such that $\sup_n l_X(\alpha_n)<\infty$ and that there exists a subsequence $l_X(\alpha_{n_j})\to 0$. For each α_n, let γ_n be a shortest closed geodesic in X that intersects α_n in either 2 or 1 points. Define a space $ML_{\mathcal{P}}(X)$ to consists of all measured laminations μ such that

$$\mu(\alpha_n)\leq O(l_X(\alpha_n))$$

and

$$\mu(\gamma_n)\leq o(|\log l_x(\alpha_n)|).$$

Note that $ML_{bdd}(X)\subsetneq ML_{\mathcal{P}}(X)$.

Theorem 3. *Let X be a complete, borderless, infinite hyperbolic surface that has an upper bounded geodesic pants decomposition $\mathcal{P}=\{\alpha_n\}$ with a subsequence whose lengths converge to zero. Then the length spectrum Thurston's boundary to $T(X)$ is contained in $ML_{\mathcal{P}}(X)$.*

Moreover, if $\mu\in ML_{\mathcal{P}}(X)$ satisfies $\mu|_{\alpha_n}\leq o(|\log l_X(\alpha_n)|)$ and either

1. *the angles between the geodesics of the support of μ and α_n are bounded from below by a positive constant; or*
2. *$\mu(\alpha_n)\geq c l_X(\alpha_n)$ for some $c>0$ or $\mu(\alpha_n)=0$,*

then the projective class of μ is in the length spectrum Thurston's boundary of $T(X)$.

In addition, given a hyperbolic surface X whose every geodesic pants decomposition does not have an upper bound on the lengths of cuffs but that can be decomposed into bounded polygons with at most n sides (cf. Kinjo [11]) then the length spectrum Thurston's boundary equals $PML_{bdd}(X)$. Moreover, if X is the surface constructed by Shiga [21] such that the length spectrum metric is incomplete, then the length spectrum Thurston's boundary is strictly larger than $PML_{bdd}(X)$ (cf. §8.4).

Question: If a sequence of points in $T(X)$ converges to a projective bounded measured lamination in the length spectrum Thurston's boundary, is it true that it also converges in the Thurston's boundary of $T(X)$ introduced by geodesic currents?

2. Teichmüller spaces of geometrically infinite hyperbolic surfaces

Let X_0 be a complete hyperbolic surface without boundary whose area is infinite. The universal covering $\tilde{X}_0$ of the surface X_0 is isometrically identified with the hyperbolic plane $\mathbb{H}$. The boundary at infinity $\partial_\infty\tilde{X}_0$ is identified with the unit circle S^1.

The *Teichmüller space* $T(X_0)$ of the surface X_0 is the space of equivalence classes of all quasiconformal maps $f : X_0 \to X$ where X is an arbitrary complete hyperbolic surface modulo an equivalence relation. Two quasiconformal maps $f_1 : X_0 \to X_1$ and $f_2 : X_0 \to X_2$ are *equivalent* if there exists an isometry $I : X_1 \to X_2$ such that $f_2^{-1} \circ I \circ f_1$ is homotopic to the identity under a bounded homotopy. Denote by $[f]$ the equivalence class of a quasiconformal map $f : X_0 \to X$.

The *Teichmüller distance* on $T(X_0)$ is defined by

$$d_T([f_1], [f_2]) = \frac{1}{2} \log \inf_{g \simeq f_2 \circ f_1^{-1}} K(g)$$

where the infimum is taken over all quasiconformal maps g homotopic to $f_2 \circ f_1^{-1}$ and $K(g)$ is the quasiconformal constant of g. The *Teichmüller topology* on $T(X_0)$ is the topology induced by the Teichmüller distance.

Let $f : X_0 \to X$ be a quasiconformal map. Denote by $\tilde{f} : \mathbb{H} \to \mathbb{H}$ a lift of f to the universal covering. Then $\tilde{f} : \mathbb{H} \to \mathbb{H}$ extends by continuity to a quasisymmetric map $h : S^1 \to S^1$ that conjugates the covering group of X_0 onto the covering group of X. We normalize h to fix 1, i and -1 by post-composing it with an isometry of $\mathbb{H}$, if necessary.

Recall that $h : S^1 \to S^1$ is a *quasisymmetric map* if it is an orientation preserving homeomorphism and there exists $M \geq 1$ such that

$$\frac{1}{M} \leq \Big| \frac{h(e^{i(x+t)}) - h(e^{ix})}{h(e^{ix}) - h(e^{i(x-t)})} \Big| \leq M$$

for all $x, t \in \mathbb{R}$.

The Teichmüller space $T(X)$ is in a one to one correspondence with the space of quasisymmetric maps of S^1 that fix 1, i and -1, and that conjugate the covering group of X_0 onto a subgroup of the isometry group of $\mathbb{H}$. From this point on, we consider the Teichmüller space $T(X)$ to be the space of normalized quasisymmetric maps. A sequence $h_n \in T(X)$ *converges in the Teichmüller topology* to $h \in T(X)$ if

$$\sup_{x,t \in \mathbb{R}} \Big| \frac{h_n \circ h^{-1}(e^{i(x+t)}) - h_n \circ h^{-1}(e^{ix})}{h_n \circ h^{-1}(e^{ix}) - h_n \circ h^{-1}(e^{i(x-t)})} \Big| \to 0$$

as $n \to \infty$.

The *universal Teichmüller space* $T(\mathbb{H})$ is the Teichmüller space of the hyperbolic plane $\mathbb{H}$ and it consists of all normalized quasisymmetric maps of S^1 without any requirements on conjugating covering groups because $\mathbb{H}$ is simply connected. The universal Teichmüller space $T(\mathbb{H})$ contains multiple copies of Teichmüller spaces of all hyperbolic surfaces. In what follows, we mainly work with $T(\mathbb{H})$ since all the constructions, arguments and statements remain true under the conjugation requirement.

3. Measured laminations and earthquakes

A geodesic lamination on a hyperbolic surface X is a closed subset of X that is foliated by non-intersecting complete geodesics called *leaves* of the lamination. Geodesic lamination on X lifts to a geodesic lamination on $\mathbb{H}$ that is invariant under the action of the covering group of X. A *stratum* of a geodesic lamination is either a leaf of the lamination or a connected component of the complement. A connected component of the complement of a geodesic lamination in $\mathbb{H}$ is isometric to a possibly infinite sided geodesic polygon whose sides are complete geodesics and possibly arcs on S^1.

A *measured lamination* μ on X is an assignment of a positive Borel measure on each arc transverse to a geodesic lamination $|\mu|$ that is invariant under homotopies relative leaves of $|\mu|$. The geodesic lamination $|\mu|$ is called the *support* of μ. A measured lamination on X lifts to a measured lamination on $\mathbb{H}$ that is invariant under the covering group of X.

A *left earthquake* $E : X_0 \to X$ with support geodesic lamination λ is a surjective map that is isometry on each stratum of λ such that each stratum is moved to the left relative to any other stratum. An earthquake of X_0 lifts to an earthquake of $\mathbb{H}$ where the support is the lift of the support on X_0 (cf. Thurston [22]).

We give a definition of a (left) earthquake $E : \mathbb{H} \to \mathbb{H}$ with support geodesic lamination λ on $\mathbb{H}$. A *left earthquake* $E : \mathbb{H} \to \mathbb{H}$ is a bijection of $\mathbb{H}$ whose restriction to any stratum of λ is an isometry of $\mathbb{H}$; if A and B are two strata of λ then

$$E|_B \circ (E|_A)^{-1}$$

is a hyperbolic translation whose axis weakly separates A and B that moves B to the left as seen from A (cf. Thurston [22]).

An earthquake $E : \mathbb{H} \to \mathbb{H}$ induces a transverse measure μ to its support λ which defines a measured lamination μ with $|\mu| = \lambda$ (cf. [22]). An earthquake of $\mathbb{H}$ extends by continuity to a homeomorphism of S^1. Thurston's earthquake theorem states that any homeomorphism of S^1 can be obtained by continuous extension of a left earthquake (cf. Thurston [22]).

Given a measured lamination μ, there exists a map $E^\mu : \mathbb{H} \to \mathbb{H}$ whose transverse measure is μ and that satisfies all properties in the definition of an earthquake of $\mathbb{H}$ except being onto (cf. [22], [10]). E^μ is uniquely determined by μ up to post-composition by an isometry of $\mathbb{H}^2$.

We define *Thurston's norm* of a measured lamination μ as

$$\|\mu\|_{Th} = \sup_J \mu(J)$$

where the supremum is over all hyperbolic arcs J of length 1.

Since we are working with quasisymmetric maps, we consider measured laminations whose earthquakes induces quasisymmetric maps of S^1. An earthquake E^μ extends by continuity to a quasisymmetric map of S^1 if and only if $\|\mu\|_{Th} < \infty$ (cf. [22], [10], [16], [17]).

Denote by $ML_{bdd}(\mathbb{H})$ the space of all measured laminations on $\mathbb{H}$ with finite Thurston's norm. The above result gives a bijective map

$$EM : T(\mathbb{H}) \to ML_{bdd}(\mathbb{H})$$

defined by

$$EM : h \mapsto \mu$$

where μ is measured lamination induced by unique earthquake $E:\mathbb{H}\to\mathbb{H}$ whose continuous extension to S^1 equals h.

Note that $\|t\mu\|_{Th}=t\|\mu\|_{Th}$, for $t>0$. Then, for $\|\mu\|_{Th}<\infty$, we have that the earthquake path $t\mapsto E^{t\mu}|_{S^1}$, for $t>0$, defines a path of quasisymmetric maps, which is a path in $T(\mathbb{H})$ when the maps are normalized to fix 1, i and -1.

4. Liouville measure, geodesic currents and uniform weak* topology

Let $G(\mathbb{H})$ be the space of unoriented complete geodesics in the hyperbolic plane $\mathbb{H}$. Each geodesic is determined by two ideal endpoints on S^1 which gives

$$G(\mathbb{H})\cong(S^1\times S^1-diag)/\mathbb{Z}_2$$

where *diag* is the diagonal in $S^1\times S^1$ and $\mathbb{Z}_2$ interchanges elements in a pair. If $[a,b],[c,d]\subset S^1$ are disjoint closed arcs, then the set $([a,b]\times[c,d])/\mathbb{Z}_2$ is called a *box of geodesics*. We write $[a,b]\times[c,d]$ in place of $([a,b]\times[c,d])/\mathbb{Z}_2$ for short.

Liouville measure on $G(\mathbb{H})$ is given by

$$L(A)=\int_A\frac{dtds}{|e^{it}-e^{is}|^2}$$

for any Borel set $A\subset G(\mathbb{H})$. If $A=[a,b]\times[c,d]$, then we have

$$L([a,b]\times[c,d])=|\log\frac{(c-a)(d-b)}{(d-a)(c-b)}|.$$

In other words, Liouville measure of a box of geodesics is the logarithm of a cross-ratio of the four endpoints defining the box. Consequently, Liouville measure is invariant under isometrics of $\mathbb{H}$.

A *geodesic current* α is a positive Borel measure on $G(\mathbb{H})$. Define *supremum norm* of α by

$$\|\alpha\|_{\sup}=\sup_{L(Q)=\log 2}\alpha(Q)$$

The space $\mathcal{M}(G(\mathbb{H}))$ consists of all geodesic currents with finite supremum norm.

Note that measured laminations are geodesic currents whose support consist of geodesic laminations. If a measured lamination has finite Thurston's norm then it has finite supremum norm. Thus

$$ML_{bdd}(\mathcal{H})\subset\mathcal{M}(G(\mathbb{H})).$$

We define the uniform weak* topology on $\mathcal{M}(G(\mathbb{H}))$ which will be used to introduce Thurston's boundary to Teichmüller spaces of infinite surfaces. The uniform weak* topology was introduced in [14] on the space $ML_{bdd}(\mathbb{H})$.

Box $Q^*=[-i,1]\times[i,-1]$ is said to be the *standard box*. Let Q be a box of geodesic with $L(Q)=\log 2$. Then there exists an isometry γ_Q of $\mathbb{H}$ that maps Q onto the standard box Q^*. A sequence $\alpha_n\in\mathcal{M}(G(\mathbb{H}))$ converges to $\alpha\in\mathcal{M}(G(\mathbb{H}))$ in the uniform weak* topology if for any continuous $f:G(\mathbb{H})\to\mathbb{R}$ with $supp(f)\subset Q^*$,

$$\sup_Q\int_{Q^*}fd((\gamma_Q)^*(\alpha_n-\alpha))\to 0$$

as $n\to\infty$ (cf. [14]).

The uniform weak* topology was first introduced on $ML_{bdd}(\mathbb{H})$ (cf. [14]). The main result in [14] is that the earthquake measure map

$$EM:T(\mathbb{H})\to ML_{bdd}(\mathbb{H})$$

is a homeomorphism for the uniform weak* topology on $ML_{bdd}(\mathbb{H})$. In other words, the uniform weak* topology is a natural topology on measured laminations which makes correspondence between quasisymmetric maps and their earthquake measures bi-continuous.

5. Embedding of Teichmüller space into geodesic currents space

We define a map from the universal Teichmüller space $T(\mathbb{H})$ into the space of geodesic currents $\mathcal{M}(G(\mathbb{H}))$. Namely, the *Liouville map*

$$\mathcal{L} : T(\mathbb{H}) \to \mathcal{M}(G(\mathbb{H}))$$

is defined by

$$\mathcal{L}(h) = h^* L$$

where $h \in T(\mathbb{H})$.

Theorem 5.1. *The Liouville map*

$$\mathcal{L} : T(\mathbb{H}) \to \mathcal{M}(G(\mathbb{H}))$$

is a homeomorphism onto its image, where $\mathcal{M}(G(\mathbb{H}))$ is equipped with the uniform weak topology. In addition, $\mathcal{L}(T(\mathbb{H}))$ is closed and unbounded subset of $\mathcal{M}(G(\mathbb{H}))$.*

Proof. We first establish that $\mathcal{L}$ is injective. Indeed, $h \in T(\mathbb{H})$ is normalized to fix $1, i, -1 \in S^1$. For $x \in S^1 - \{1, i, -1\}$, denote by Q_x a box of geodesics whose defining intervals on S^1 have endpoints $1, i, -1$ and x. Then $L(h(Q_x))$ uniquely determines $h(x)$. Thus $\mathcal{L}$ is injective.

We prove that $\mathcal{L}$ is continuous. Consider $h_n \to h$ in $T(\mathbb{H})$. Let $f : G(\mathbb{H}) \to \mathbb{R}$ be a continuous function with $supp(f) \subset Q^*$. Define $\mathcal{L}(h_n) = \alpha_n$ and $\mathcal{L}(h) = \alpha$. Let Q be a box of geodesics with $L(Q) = \log 2$ and $\gamma_Q : Q^* \mapsto Q$ as before.

To estimate

$$\Big| \int_{Q^*} f d\Big[\Big(\gamma_Q\Big)^* \Big(\alpha_n - \alpha\Big)\Big]\Big|,$$

we divide Q^* into finitely many boxes of geodesics $\{Q_i\}_{i=1}^m$ such that

$$|\max_{Q_i} f - \min_{Q_i} f| < \epsilon_0$$

for all $1 \leq i \leq m$ and fixed ϵ_0 to be determined later.

Let

$$s = \sum_{i=1}^m (\max_{Q_i} f)\chi_{Q_i}$$

be a simple function approximating f.

Then

$$\Big| \int_{Q^*} (f - s) d\Big[\Big(\gamma_Q\Big)^* \Big(\alpha_n - \alpha\Big)\Big]\Big| \leq \epsilon_0(\alpha(Q) + \alpha_n(Q)) \leq 3\alpha(Q)\epsilon_0 \leq 3\epsilon_0 \|\alpha\|_{sup}$$

where the second inequality holds for all $n \geq n_0$ with n_0 large enough such that h_n is close enough to h in $T(\mathbb{H})$ (cf. Lemma 9.1).

By using Lemma 9.1 again,

$$\Big| \int_{Q^*} s d\Big[\Big(\gamma_Q\Big)^* \Big(\alpha_n - \alpha\Big)\Big]\Big| \leq \epsilon \max_{Q^*} |f|$$

for all $n \geq n_1$, where $n_1 = n_1(\delta, \epsilon)$ is large enough such that $h_n \in N(h, \delta, \epsilon)$ with $\delta = \min_i L(Q_i)$.

By choosing ϵ_0 and ϵ small enough, we can make $\Big|\int_{Q^*} fd\Big[\Big(\gamma_Q\Big)^*\Big(\alpha_n-\alpha\Big)\Big]\Big|$ as small as we want for all $n\geq \max\{n_0,n_1\}$, where n_0,n_1 depend on ϵ_0,ϵ. Thus $\alpha_n\to\alpha$ as $n\to\infty$ and $\mathcal{L}$ is continuous.

We prove that $\mathcal{L}^{-1}:\mathcal{L}(T(\mathbb{H}))\to T(\mathbb{H})$ is continuous. Consider $\alpha_n\to\alpha$ in $\mathcal{M}(G(\mathbb{H}))$ with $\mathcal{L}(h_n)=\alpha_n$ and $\mathcal{L}(h)=\alpha$.

First we prove that there is an upper bound on the quasisymmetric constants of $\{h_n\}$. Assume on the contrary that the quasisymmetric constants of $\{h_n\}$ go to infinity. Then there exists a sequence of boxes $\{Q_n\}$ with $L(Q_n)=\log 2$ and $\alpha_n(Q_n)\to\infty$ as $n\to\infty$. Subdivide each Q_n into two boxes of equal Liouville measure $\frac{1}{2}\log 2$ and, for each n, choose one box Q_n' of the two sub-boxes of Q_n such that $\alpha_n(Q_n')\to\infty$ as $n\to\infty$.

Let $Q_n^{\#}$ be a box such that $L(Q_n^{\#})=\log 2$, $Q_n'\Subset Q_n^{\#}$ and

$$\Big(\gamma_{Q_n^{\#}}\Big)^{-1}\Big(Q_n'\Big)\subset\Big[e^{i\frac{25\pi}{16}},e^{i\frac{31\pi}{16}}\Big]\times\Big[e^{i\frac{9\pi}{16}},e^{i\frac{15\pi}{16}}\Big].$$

Let $f:G(\mathbb{H})\to\mathbb{R}$ be a non-negative continuous function with $supp(f)\subset Q^*$ and $f|_{\Big[e^{i\frac{25\pi}{16}},e^{i\frac{31\pi}{16}}\Big]\times\Big[e^{i\frac{9\pi}{16}},e^{i\frac{15\pi}{16}}\Big]}=1$. By $\alpha_n\to\alpha$, there exists n_0 such that, for all $n\geq n_0$,

$$\int_{Q^*}fd((\gamma_{Q_n^{\#}})^*\alpha_n)\leq\int_{Q^*}fd((\gamma_{Q_n^{\#}})^*\alpha)+1\leq\sup_{L(Q)=\log 2}\int_{Q^*}fd((\gamma_Q)^*\alpha)+1.$$

On the other hand,

$$\int_{Q^*}fd((\gamma_{Q_n^{\#}})^*\alpha_n)\geq\alpha_n(Q_n')\to\infty$$

which gives a contradiction with the above inequality. Thus the quasisymmetric constants of the sequence $\{h_n\}$ are uniformly bounded.

To prove that $h_n\to h$ in $T(\mathbb{H})$, it is enough to prove that

$$\sup_{L(Q)=\log 2}|\alpha_n(Q)-\alpha(Q)|\to 0$$

as $n\to\infty$.

Let Q_δ be a sub-box of Q such that

$$\gamma_Q(Q_\delta)=[-ie^{-i\delta},e^{-i\delta}]\times[ie^{i\delta},-e^{-i\delta}].$$

Then $Q-Q_\delta$ is union of four boxes $Q_i(\delta)$, $i=1,\ldots,4$, such that $L(Q_i(\delta))\to 0$ as $\delta\to 0$ for all i. Since $\{h_n\}$ is a bounded sequence in $T(\mathbb{H})$, it follows that $\alpha_n(Q_i(\delta))\to 0$ and $\alpha(Q_i(\delta))\to 0$ as $\delta\to 0$ uniformly in n. Finally, let $f_\delta:G(\mathbb{H})\to\mathbb{R}$ be a positive continuous function with $supp(f_\delta)\subset Q^*$, $\|f_\delta\|_\infty=1$ and $f_\delta|_{[-ie^{i\delta},e^{-i\delta}]\times[ie^{i\delta},-e^{-i\delta}]}=1$.

It follows

$$\Big|\alpha_n(Q_\delta)-\alpha(Q_\delta)\Big|\leq\Big|\int_{Q^*}f_\delta d((\gamma_Q)^*(\alpha_n-\alpha))\Big|+|\alpha_n(Q-Q_\delta)|+|\alpha(Q-Q_\delta)|.$$

Since $|\alpha_n(Q-Q_\delta)|$ and $|\alpha(Q-Q_\delta)|$ are as small as we want (uniformly in n) for $\delta>0$ small enough and

$$\Big|\int_{Q^*}f_\delta d((\gamma_Q)^*(\alpha_n-\alpha))\Big|\to 0$$

as $n \to \infty$ for each f continuous with $supp(f) \subset Q^*$, it follows that

$$|\alpha_n(Q_\delta) - \alpha(Q_\delta)|$$

is small for n large. Thus

$$\sup_Q |\alpha_n(Q) - \alpha(Q)| \to 0$$

as $n \to \infty$.

We prove that $\mathcal{L}(T(\mathbb{H}))$ is closed in $\mathcal{M}(G(\mathbb{H}))$. Indeed, let $\alpha_n \to \alpha$ in $\mathcal{M}(G(\mathbb{H}))$, where $\mathcal{L}(h_n) = \alpha_n$ for $h_n \in T(\mathbb{H})$. Consequently,

$$\sup_{L(Q)=\log 2} |\int_{Q^*} f d((\gamma_Q)^*\alpha_n)| \leq C(f)$$

where $C(f)$ is independent of n.

Then $h_n = \mathcal{L}^{-1}(\alpha_n)$ is bounded in $T(\mathbb{H})$ (already proved above). It follows that there exists a subsequence h_{n_k} which pointwise converges to a quasisymmetric map h on S^1. Let $\beta = \mathcal{L}(h)$. Thus

$$\alpha_n(Q) \to \beta(Q)$$

as $n \to \infty$ for each box of geodesics Q.

This implies

$$\int_{Q^*} f d((\gamma_Q)^*\alpha_n) \to \int_{Q^*} f d((\gamma_Q)^*\beta)$$

as $n \to \infty$. Thus $\alpha = \beta$ by the uniqueness of measures.

Finally, $\mathcal{L}(T(\mathbb{H}))$ is unbounded because $\mathcal{L}^{-1}(M)$ is bounded whenever $M \subset \mathcal{M}(G(\mathbb{H}))$ is bounded by the proof above. □

6. The fundamental lemma

Lemma 6.1. *Let $\beta_n \in ML_{bdd}(\mathbb{H})$ be a bounded (in Thurston's norm) sequence that converges in the weak* topology to $\beta \in ML_{bdd}(\mathbb{H})$. Assume $Q = [a,b] \times [c,d]$ is a box of geodesics with $\beta(\partial Q) = 0$. Then, for $t_n > 0$ and $t_n \to \infty$ as $n \to \infty$,*

$$\frac{1}{t_n} L(E^{t_n\beta_n}(Q)) \to \beta(Q)$$

as $n \to \infty$.

Proof. Since $\beta_n \to \beta$ in weak* topology as $n \to \infty$ and $\beta(\partial Q) = 0$ we have $\beta_n(Q) \to \beta(Q)$ as $n \to \infty$.

Fix $\epsilon > 0$. Let $a' \in [d,a]$ be such that

$$\beta(\partial([a',a] \times [c,d])) = 0$$

and

$$\beta([a',a] \times [c,d]) < \frac{\epsilon}{2}.$$

Then there exists n_0 such that, for all $n \geq n_0$,

$$\beta_n([a',a] \times [c,d]) < \epsilon.$$

We partition measured lamination β into a finite sum of measured laminations as follows

$$
(1)\qquad
\begin{aligned}
&\beta_1^n(B) = \beta_n(B \cap Q),\\
&\beta_2^n(B) = \beta_n(B \cap \{[a,b] \times [c',c)\}),\\
&\beta_3^n(B) = \beta_n(B \cap \{[a,b] \times [b,c')\}),\\
&\beta_4^n(B) = \beta_n(B \cap \{[a',a) \times [c,d]\}),\\
&\beta_5^n(B) = \beta_n(B \cap \{[d,a') \times [c,d]\}),\\
&\beta_6^n(B) = \beta_n(B) - \textstyle\sum_{i=1}^5 \beta_i^n(B),
\end{aligned}
$$

where $B \subset G(\mathbb{H})$ is any Borel set. Note that β_i^n are defined by restricting β_n to boxes of geodesics with some of the boxes not being closed. This is done to avoid ambiguity because an intersection of two boxes along their boundaries might have non-zero β_n-mass. For example, β_2^n is defined by restricting to box $[a,b] \times [c',c)$ because $\beta_n([a,b] \times \{c\})$ might be non-zero and we defined β_n^1 by restricting to box $[a,b] \times [c,d]$. In this case the support of β_2^n might contain geodesics in $[a,b] \times \{c\}$ while $\beta_2^n([a,b] \times \{c\}) = 0$ (because the support is defined as the smallest closed set whose complement has zero mass). Similar property holds for other measures.

Let A_n be the stratum of β_n that separates the support of β_5^n from the support of β_4^n. Note that A_n could be either a hyperbolic polygon or a geodesic. In the case that A_n is a geodesic then it is in the support of both β_4^n and β_5^n. We normalize earthquakes $E^{t_n\beta_n}$ and $E^{t_n\beta_i^n}$, for $i = 1, \dots, 6$, to be identity on stratum(that contains) A_n. Let a'' be a point on the boundary of A_n in interval $[a',a]$ and let c'' be a point of A_n in interval $[c,d]$.

Then we have

$$E^{t_n\beta_n}|_{[a'',c'']} = E^{t_n\beta_4^n} \circ E^{t_n\beta_1^n} \circ E^{t_n\beta_2^n} \circ E^{t_n\beta_3^n} \circ E^{t_n\beta_6^n}$$

and

$$E^{t_n\beta_n}|_{[c'',a'']} = E^{t_n\beta_5^n} \circ E^{t_n\beta_6^n}.$$

We estimate $L(E^{t_n\beta_n}([a,b] \times [c,d]))$ from the above. The action of earthquake $E^{t_n\beta_6^n}$ fixes points b and d, and possibly moves a towards b and possibly moves c towards d because it moves all points to the left relative the stratum A_n. This decreases the Liouville measure of the box $[a,b] \times [c,d]$ and we delete $E^{t_n\beta_6^n}$ from the definition of $E^{t_n\beta_n}$.

Earthquake $E^{t_n\beta_3^n}$ moves b towards c and it can at most reach point c'. Similar, earthquake $E^{t_n\beta_5^n}$ moves d towards a and the closest it can get is a'. Therefore, it is enough to consider the action of $E^{t_n\beta_4^n} \circ E^{t_n\beta_1^n} \circ E^{t_n\beta_2^n}$ on box $[a,c'] \times [c,a']$.

Without loss of generality we assume that the support of β_i^n for $i = 4,1,2$ is finite since earthquakes with non-finite support are (pointwise) limits of earthquakes with finite support and Liouville measure of boxes is a continuos function of the vertices of boxes.

Let T be a hyperbolic translation whose repelling fixed point is in $[a,b]$ and attracting fixed point is in $[c',c]$. Then Liouville measure of $[a,T(c')] \times [c,a']$ is less than Liouville measure of $[a,T^1(c')] \times [c,a'])$, where T_1 is hyperbolic translation which shares repelling fixed point with T, whose attracting fixed point is c and that has the same translation length (cf. Lemma 9.4). In other words, we increase Liouville measure if we move the attracting fixed point to be at the starting point of the second interval and keep the same translation length.

Note that each geodesic of the support of β_2^n has one endpoint in $[a,b]$ and the other endpoint in $[c',c)$. By applying the above to the geodesics in the support of

β_2^n one at a time, we can assume that the support of β_2^n is in $[a,c']\times[c,a']$. The same reasoning (using Lemma 9.4) applies to the support of β_4^n so we can assume its support is also in $[a,c']\times[c,a']$.

From now on we assume that the support of measured lamination $\beta_4^n+\beta_1^n+\beta_2^n$ consists of finitely many geodesics inside box $[a,c']\times[c,a']$. Let T be a hyperbolic translation with repelling fixed point in $[a,c']$ and attracting fixed point in $[c,a']$. Let T_1 be a hyperbolic translation with the translation length equal to the translation length of T whose repelling fixed point is a and attracting fixed point is c. Then Lemma 9.2 gives

$$L([a,T(c')]\times[T(c),a'])\leq L([a,T_1(c')]\times[T_1(c),a']).$$

In other words, Liouville measure of the image of a box is largest when the axis of the hyperbolic translation is positioned from left endpoint of one boundary interval to right endpoint of the other boundary interval defining the box.

By applying the above reasoning to geodesics of $\beta_4^n+\beta_1^n+\beta_2^n$ one at a time, Liouville measure of $E^{t_n\beta_4^n}\circ E^{t_n\beta_1^n}\circ E^{t_n\beta_2^n}([a,c']\times[c,a'])$ is less than or equal to Liouville measure of $[a,c']\times[T(c),a']$, where T is the hyperbolic translation with repelling fixed point a, attracting fixed point c and translation length $\beta_4^n([c,d]\times[a',a])+\beta_1^n([a,b]\times[c,d])+\beta_2^n([a,b]\times[c',c])=\beta_n([a',b]\times[c',d])$.

By the choice of a' and c', we have that

$$\beta_n([a',b]\times[c',d])\leq\beta_n([a,b]\times[c,d])+2\epsilon$$

for $n\geq n_0$. Then Lemma 9.3 gives

$$L(E^{t_n\beta_n}([a,b]\times[c,d]))\leq t_n(\beta_n([a,b]\times[c,d])+2\epsilon)+L([a,b]\times[c,d]) \tag{2}$$

for $n\geq n_0(\epsilon)$. Dividing by t_n and letting $n\to\infty$, we get

$$\limsup_{n\to\infty}\frac{1}{t_n}L(E^{t_n\beta_n}([a,b]\times[c,d]))\leq\beta([a,b]\times[c,d])$$

because $\epsilon>0$ is arbitrary.

We find lower bound for $L(E^{t_n\beta_n}([a,b]\times[c,d]))$. Let $\epsilon>0$ be fixed. Since $\beta(\partial Q)=0$, it follows that there exists $b'\in[a,b]$ and $d'\in[c,d]$ such that

$$\beta_n([b',b]\times[c,d])+\beta_n([a,b]\times[d',d])\leq\epsilon$$

for $n\geq n_0(\epsilon)$, and

$$\beta([b',b]\times[c,d])+\beta([a,b]\times[d',d])\leq\epsilon.$$

Let c' be the endpoint of a geodesic in $|\beta_n|\cap([a,b']\times[c,d'])$ that is closest to c in interval $[c,d']$, and $c'=c$ if $|\beta_n|\cap([a,b']\times[c,d'])=\emptyset$. Let a' be the endpoint of a geodesic in $|\beta_n|\cap([a,b']\times[c,d'])$ that is closest to a in the interval $[a,b']$, and $a'=b'$ if $|\beta_n|\cap([a,b']\times[c,d'])=\emptyset$.

We write β_n as a finite sum of measured laminations as follows. For a Borel set $B\subset G(\mathbb{H})$, define

$$\begin{aligned}\beta_1^n(B)&=\beta_n(B\cap([a',b']\times[c',d'])),\\ \beta_2^n(B)&=\beta_n(B\cap([a,b]\times(d',d])),\\ \beta_3^n(B)&=\beta_n(B\cap((b',b]\times[c,d])),\\ \beta_4^n(B)&=\beta_n(B)-\textstyle\sum_{i=1}^3\beta_i^n(B).\end{aligned} \tag{3}$$

Normalize earthquakes $E^{t_n\beta_n}$ and $E^{t_n\beta_i^n}$, for $i=1,2,3$, to be identity on the stratum of β_n that separates $|\beta_1^n|$ and $|\beta_2^n|$. Note that the stratum might be common geodesic. Then we have

$$\begin{aligned} E^{t_n\beta_n}|_{[a',d']} &= E^{t_n\beta_1^n}\circ E^{t_n\beta_3^n}\circ E^{t_n\beta_4^n}, \\ E^{t_n\beta_n}|_{[d',a']} &= E^{t_n\beta_2^n}\circ E^{t_n\beta_4^n}. \end{aligned} \tag{4}$$

We consider the image of $[a',b]\times[c',d]$ under $E^{t_n\beta_n}$. Since $E^{t_n\beta_4^n}$ is a left earthquake, chosen normalization implies that a' and c' are fixed, and possibly b is moved towards c', and possibly d is moved towards a' for the fixed orientation on S^1. These movements increase Liouville measure and since we are looking for a lower bound, we ignore the action of $E^{t_n\beta_4^n}$. In similar fashion, earthquakes $E^{t_n\beta_2^n}$ and $E^{t_n\beta_3^n}$ can only increase Liouville measure of $[a',b]\times[c',d]$ and we ignore them.

It remains to estimate Liouville measure of $E^{t_n\beta_1^n}([a',b]\times[c',d])$. By Lemma 9.5, we have that $L(E^{t_n\beta_1^n}([a',b]\times[c',d]))$ is larger than $L([a',T(b)]\times[T(c),d])$, where T is a hyperbolic translation with the translation length $\beta_1^n([a',b]\times[c',d])$ whose repelling fixed point is b' and attracting fixed point is d'.

From above we obtain

$$L(E^{t_n\beta_n}([a,b]\times[c,d]))\geq L([b',T(b)]\times[d',d])$$

and Lemma 9.3 gives

$$L([b',T(b)]\times[d',d])\geq t_n\beta_1^n([a',b']\times[c',d'])+\log\frac{d^2}{4}$$

where d is the distance between geodesics $l(a',d)$ and $l(b,c')$. The above choice of b' and d', and the fact that $\epsilon>0$ is arbitrary gives

$$\liminf_{n\to\infty}\frac{1}{t_n}L(E^{t_n\beta_n}([a,b]\times[c,d]))\geq\beta([a,b]\times[c,d]).$$

□

7. Convergence of earthquake paths in Thurston's closure

We first prove that each box of geodesics $Q=[a,b]\times[c,d]$ is the limit (in the Hausdorf topology) of a sequence of increasing (in the sense of inclusions) boxes Q_n with $\beta(\partial Q_n)=0$. Indeed, $\partial Q=(\{a\}\times[c,d])\cup(\{b\}\times[c,d])\cup([a,b]\times\{c\})\cup([a,b]\times\{d\})$. Consider a small open interval $I_a\subset S^1$ around a. Since β is locally finite, there exists at most countably many $a'\in I_a$ such that $\beta(\{a'\}\times[c,d])>0$. Choose $a_n\in I_a$ such that $\beta(\{a_n\}\times[c,d])=0$. Similarly we choose b_n close to b such that $\beta(\{b_n\}\times[c,d])=0$. In the same fashion, we choose c_n close to c and d_n close to d such that

$$\beta(\partial([a_n,b_n]\times[c_n,d_n]))=0$$

and set $Q_n=[a_n,b_n]\times[c_n,d_n]$.

Next we prove the convergence of the earthquake paths in Thurston's boundary.

Theorem 7.1. *Let $\beta\in ML_{bdd}(\mathbb{H})$ and let $E^{t\beta}$, for $t>0$, be left earthquake with measure $t\beta$. Then*

$$\frac{1}{t}(E^{t\beta}|_{S^1})^*L\to\beta$$

as $t\to\infty$ in the uniform weak topology on $\mathcal{M}(G(\mathbb{H}))$.*

Proof. Without loss of generality we can assume that $\|\beta\|_{Th}=1$. Let $h_t=E^{t\beta}|_{S^1}$, for $t>0$, be the restriction of the earthquake path to the boundary S^1 of $\mathbb{H}$. Let

$$\alpha_t=(h_t)^*L$$

be the image of $h_t\in T(\mathbb{H})$ in $\mathcal{M}(\mathbb{H})$.

Assume on the contrary that $\frac{1}{t}\alpha_t$ does not converge to β in the uniform weak* topology as $t\to\infty$. Then there exists a continuous function $f:G(\mathbb{H})\to\mathbb{R}$ with $supp(f)\subset Q^*$, a sequence of boxes Q_n with $L(Q_n)=\log 2$ and a sequence $t_n\to\infty$ as $n\to\infty$ such that, for all $n\in\mathbb{N}$,

$$\Big|\int_{Q^*} fd\Big[\Big(\gamma_{Q_n}\Big)^*\Big(\frac{1}{t_n}\alpha_{t_n}-\beta\Big)\Big]\Big|\geq C_0>0. \tag{5}$$

Define

$$\alpha'_{t_n}=(\gamma_{Q_n})^*\alpha_{t_n}$$

and

$$\beta_n=(\gamma_{Q_n})^*\beta.$$

Note that by Lemma 9.3

$$\frac{1}{t_n}\alpha'_{t_n}(Q)\leq\beta_n(Q)+\frac{1}{t_n}L(Q)\leq\Big(\frac{L(Q)}{\log 2}+1\Big)\|\beta\|_{Th}+\frac{1}{t_n}L(Q)=C(Q)$$

for all n such that $t_n\geq 1$. Also

$$\beta_n(Q)\leq\Big(\frac{L(Q)}{\log 2}+1\Big)\|\beta\|_{Th}$$

for all n.

The above two inequalities imply that both β_n and $\frac{1}{t_n}\alpha'_{t_n}$ are uniformly bounded on each box Q. Then there exist subsequences $\frac{1}{t_{n_k}}\alpha'_{t_{n_k}}$ and β_{n_k} that converge in weak* topology on $\mathcal{M}(G(\mathbb{H}))$ to $\alpha^\#$ and $\beta^\#$, respectively, as $k\to\infty$.

Then (5) gives

$$\Big|\int_{Q^*} fd(\alpha^\#-\beta^\#)\Big|\geq C_0. \tag{6}$$

On the other hand, Lemma 6.1 implies that $\alpha^\#$ and $\beta^\#$ agree on all boxes $Q^\#$ with $\beta^\#(\partial Q^\#)=0$. These boxes are dense among all boxes in $G(\mathbb{H})$ and $\alpha^\#=\beta^\#$ contradicting (6). The contradiction proves theorem. □

The above theorem proves that Thurston's boundary contains the space of projective bounded measured laminations on X. It remains to prove the opposite.

Proposition 7.2. *A limit point of $P(\mathcal{L}(T(X)))$ in $P\mathcal{M}(G(\tilde{X}))$ is necessarily a projective bounded measured lamination.*

Proof. Let β be the limit point of a sequence $[\alpha_k]\in P(\mathcal{L}(T(X)))$, where $[\alpha_k]$ is the projective class of $\alpha_k\in\mathcal{L}(T(X))$. Then there exists $t_k\to\infty$ as $k\to\infty$ such that

$$\frac{1}{t_k}\alpha_k\to\beta$$

as $k\to\infty$ in the uniform weak* topology.

Recall that $\alpha\in\mathcal{L}(T(X))$ implies that

$$e^{-\alpha([a,b]\times[c,d])}+e^{-\alpha([b,c]\times[d,a])}=1$$

for all boxes $[a,b]\times[c,d]$ of $G(\tilde{X})$ (cf. Bonahon [5]). This implies that if $\alpha_k([a,b]\times[c,d])\to\infty$ then $\alpha_k([b,c]\times[d,a])\to 0$ as $k\to\infty$.

Assume that the support of β contains two intersecting geodesics (m,n) and (p,q). The geodesic $(m,n)\in G(\tilde{X})$ separates points p and q. Then there exists a box of geodesics $Q_{(m,n)}=[a_1,b_1]\times[c_1,d_1]$ containing (m,n) and a box of geodesics $Q_{(p,q)}=[a_2,b_2]\times[c_2,d_2]$ containing (p,q) such that $[a_1,b_1]\subset(b_2,c_2)$ and $[c_1,d_1]\subset(d_2,a_2)$.

Since $t_k\to\infty$ as $k\to\infty$ and both geodesics (m,n) and (p,q) are in the support of β, we have that $\alpha_k(Q_{(m,n)})\to\infty$ and $\alpha_k(Q_{(p,q)})\to\infty$ as $k\to\infty$. The boxes are chosen such that $Q_{(m,n)}\subset[b_2,c_2]\times[d_2,a_2]$. This implies $\alpha_k([b_2,c_2]\times[d_2,a_2])\to\infty$ as $k\to\infty$ which is in a contradiction with $\alpha_k(Q_{(p,q)})\to\infty$. Thus the geodesics of the support of β do not intersect. Therefore β is a measured lamination. Boundedness of β follows because $\mathcal{L}(G(\tilde{X}))$ consists of bounded measures. □

The proof of Theorem 1 from Introduction is now completed.

8. Thurston's boundary for Teichmüller spaces of infinite surfaces using the length spectrum

In this section we consider infinite type hyperbolic surfaces and introduce the "length spectrum" Thurston's boundary to their Teichmüller spaces. It turns out that the length spectrum Thurston's boundary differs from Thurston's boundary introduced using geodesic currents.

Let X_0 be a complete hyperbolic surface without boundary that has a geodesic pants decomposition. In other words, X_0 is formed by gluing infinitely many geodesic pairs of pants along their boundaries.

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be the family of cuffs (i.e. boundary components) of a geodesic pants decomposition of X_0 as above. Then each α_n is a simple closed geodesic. We say that X_0 has an *upper-bounded pants decomposition* if there exists $M>0$ such that, for each $n\in\mathbb{N}$,

$$l_{X_0}(\alpha_n)\leq M$$

where $l_{X_0}(\alpha_n)$ is the length of α_n for the hyperbolic metric of X_0 (cf. [1]). The surface X_0 has a *lower bounded pants decomposition* if there exists $m>0$ such that, for each $n\in\mathbb{N}$,

$$l_{X_0}(\alpha_n)\geq m.$$

8.1. **General infinite surfaces.**

Denote by $\mathcal{S}$ the set of all simple closed geodesics on a complete, borderless infinite hyperbolic surface X_0 with a geodesic pants decomposition. Let $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ be the space of non-negative functions on the set of all simple closed geodesics $\mathcal{S}$ of X_0. We define a map $\mathcal{X}$ from the Teichmüller space $T(X_0)$ into $\mathbb{R}^{\mathcal{S}}_{\geq 0}$, for $[f]\in T(X_0)$ and $\alpha\in\mathcal{S}$,

$$\mathcal{X}([f])(\alpha)=l_{f(X_0)}(f(\alpha)),$$

where $f(X_0)$ is the image hyperbolic surface under quasiconformal mapping f and $l_{f(X_0)}(f(\alpha))$ is the length of the simple closed geodesic on $f(X_0)$ that is in the homotopy class of a simple closed curve $f(\alpha)$. The map $\mathcal{X}:T(X_0)\to\mathbb{R}^{\mathcal{S}}_{\geq 0}$ is injective.

The *length spectrum metric* on $T(X_0)$ is given by

$$d_{ls}(X_1,X_2)=\sup_{\delta\in\mathcal{S}}\Big\{|\log\frac{l_{X_2}(\delta)}{l_{X_1}(\delta)}|\Big\}.$$

Shiga [21] proved that if X_0 has an upper and lower bounded geodesic pants decomposition $\{\alpha_n\}_{n\in\mathbb{N}}$ then the Teichmüller distance induces the same topology as the length-spectrum distance on $T(X_0)$.

We introduce the *normalized supremum norm* on $\mathbb{R}^{\mathcal{S}}$ by

$$\|f\|_\infty^{norm} = \sup_{\delta\in\mathcal{S}} \frac{|f(\delta)|}{l_{X_0}(\delta)}$$

for all $f\in\mathbb{R}^{\mathcal{S}}$. Note that the normalized supremum norm on $\mathbb{R}^{\mathcal{S}}$ is infinite at some points. We consider only the subset of $\mathbb{R}^{\mathcal{S}}$ where the normalized supremum is finite and, for simplicity, denote it by $\mathbb{R}^{\mathcal{S}}$.

Proposition 8.1. *The length spectrum metric on $T(X_0)$ is locally biLipschitz equivalent to the normalized supremum norm on $\mathcal{X}(T(X_0))$.*

Proof. Indeed, if

$$\sup_{\delta\in\mathcal{S}} \Big|\frac{l_{X_1}(\delta)}{l_{X_0}(\delta)} - \frac{l_{X_2}(\delta)}{l_{X_0}(\delta)}\Big| < \epsilon$$

then

$$\sup_{\delta\in\mathcal{S}} \frac{l_{X_1}(\delta)}{l_{X_0}(\delta)}\Big|1 - \frac{l_{X_2}(\delta)}{l_{X_1}(\delta)}\Big| < \epsilon.$$

Note that there exists a quasiconformal map from X_0 and X_1. Thus there exists $M>1$ such that $1/M \leq \frac{l_{X_1}(\delta)}{l_{X_0}(\delta)} \leq M$ (cf. Wolpert [24]). The above and symmetry implies

$$\Big|\frac{l_{X_2}(\delta)}{l_{X_1}(\delta)} - 1\Big|, \Big|\frac{l_{X_1}(\delta)}{l_{X_2}(\delta)} - 1\Big| \leq M\epsilon$$

for all $\delta\in\mathcal{S}$, and one direction is obtained since $|\log x| \simeq |x-1|$ for $1/2<x<2$. The other direction is obtained by reversing the order of the above inequalities and the two metrics are locally biLipschitz. □

Allessandrini, Liu, Papadopoulos and Su [2] proved that $T(X_0)$ is not complete in the length spectrum metric when there exists a sequence of simple closed geodesics on X_0 whose lengths converge to 0. Thus, $\mathcal{X}: T(X_0)\to\mathbb{R}^{\mathcal{S}}_{\geq 0}$ is not a homeomorphism onto its image for the normalized supremum norm on $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ and the Teichmüller metric on $T(X_0)$ when X_0 contains a sequence of simple closed geodesics whose lengths converge to zero.

Denote by

$$\mathcal{PX}: T(X_0)\to P\mathbb{R}^{\mathcal{S}}_{\geq 0}$$

the map from $T(X_0)$ into the projective space $P\mathbb{R}^{\mathcal{S}}_{\geq 0} = (\mathbb{R}^{\mathcal{S}}_{\geq 0} - \{\bar{0}\})/\mathbb{R}_{>0}$. The map $\mathcal{PX}$ is injective on $T(X_0)$. The *length spectrum Thurston's boundary* of $T(X_0)$ is, by the definition, the space of all limit points in $P\mathbb{R}^{\mathcal{S}}_{\geq 0}$ of the set $\mathcal{PX}(T(X_0))$ for the topology induced by the normalized supremum norm (c.f. [9] for the original Thurston's discussion on closed surfaces).

Note that a measured lamination μ on X_0 represent an element in $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ by the formula

$$\mu(\alpha) = i(\mu,\alpha)$$

for all $\alpha\in\mathcal{S}$, where $i(\mu,\alpha)$ is the intersection number.

Proposition 8.2. *Let X_0 be a complete, borderless infinite hyperbolic surface equipped with a geodesic pants decomposition. Then the length spectrum Thurston's boundary of $T(X_0)$ contains the space of projective bounded measured lamination $PML_{bdd}(X_0)$ and it equals the closure of $PML_{bdd}(X_0)$ for the topology on $P\mathbb{R}^{\mathcal{S}}_{\geq 0}$ induced by the normalized supremum norm.*

Proof. Let $\mu \in ML_{bdd}(X_0)$ be a bounded measured lamination on X_0. Denote by $E^{t\mu}$, for $t > 0$, an earthquake path with the earthquake measure $t\mu$. Then $t \mapsto E^{t\mu}(X_0)$ is an analytic path in $T(X_0)$ because $\mu \in ML_{bdd}(X_0)$ (cf. [16]). Let f_t be a quasiconformal map from X_0 to X_t which is homotopic to $E^{t\mu}$.

For a geodesic arc α, denote by $\mu(\alpha)$ the total μ-mass of the geodesics intersecting α. For $\alpha \in \mathcal{S}$, the inequality

$$l_{f_t(X_0)}(f_t(\alpha)) \leq t\mu(\alpha) + l_{X_0}(\alpha)$$

implies that

$$\frac{\frac{1}{t}\mathcal{X}([f_t])(\alpha) - \mu(\alpha)}{l_{X_0}(\alpha)} \leq \frac{1}{t} \tag{7}$$

for all $\alpha \in \mathcal{S}$ and all $t > 0$.

To obtain the opposite inequality, we choose the universal covering of X_0 such that $B(z) = e^{-l_{X_0}(\alpha)}z$ is a cover transformation corresponding to α. Let O be the stratum of the lift $\tilde{\mu}$ of μ to the universal covering $\mathbb{H}$ that contains $e^{l_{X_0}(\alpha)}i$, and let O_1 be the stratum of $\tilde{\mu}$ that contains i. Normalize the earthquake $E^{t\tilde{\mu}}$ such that $E^{t\tilde{\mu}}|_O = id$. Then

$$B^t = E^{t\tilde{\mu}}|_{O_1} \circ B$$

is a covering transformation that corresponds to the geodesic on $f_t(X_0)$ homotopic to $f_t(\alpha)$. Denote by l_t the translation length of B_t and $l = l_{X_0}(\alpha)$ the translation length of B. Let $k_1 < 0$ and $k_2 > 0$ be the endpoints of the hyperbolic translation $E^{t\tilde{\mu}}|_{O_1}$, and let m_t be its translation length. A direct computation (cf. [20]) gives

$$trace(B^t) = 2\cosh\frac{m_t - l}{2} - \frac{2k_1}{k_2 - k_1}\Big(\cosh\frac{m_t + l}{2} - \cosh\frac{m_t - l}{2}\Big)$$

Consequently

$$2\cosh\frac{l_t}{2} = trace(B^t) \geq 2\cosh\frac{m_t - l}{2}$$

which implies

$$l_t \geq m_t - l.$$

Since the translation length of a composition of two hyperbolic translations (with non-intersecting axis and translating in the same direction) is at least as large as the sum of their translation lengths (cf. [22]), it follows that

$$m_t \geq t\mu(\alpha).$$

The above two inequalities give

$$\frac{1}{t}\frac{l_t}{l} \geq \frac{\mu(\alpha)}{l} - \frac{1}{t}$$

which implies

$$\frac{1}{t}\frac{\mathcal{X}([f_t])(\alpha)}{l_{X_0}(\alpha)} - \frac{\mu(\alpha)}{l_{X_0}(\alpha)} \geq -\frac{1}{t}. \tag{8}$$

Then equations (7) and (8) give that, uniformly in $\alpha \in \mathcal{S}$,

$$\lim_{t\to\infty} \frac{1}{t}\frac{\mathcal{X}([f_t])(\alpha)}{l_{X_0}(\alpha)} = \frac{\mu(\alpha)}{l_{X_0}(\alpha)}.$$

We established that each point in $PML_{bdd}(X_0)$ is in the Thurston's boundary.

Let $\sigma \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$ be such that its projective class $[\sigma]$ is in the length spectrum Thurston's boundary. We need to establish that $[\sigma]$ is in the closure of $PML_{bdd}(X_0)$ for the normalized supremum norm.

There exists a sequence $[f_n] \in T(X_0)$ that converges to the projective class $[\sigma] \in P\mathbb{R}^{\mathcal{S}}_{\geq 0}$. Let $t_n \to \infty$ as $n \to \infty$ be such that $\frac{1}{t_n}\mathcal{X}([f_n]) \to \sigma$ as $n \to \infty$. Necessarily we have $\sup_n \|\frac{1}{t_n}\mathcal{X}([f_n])\|_{\infty}^{norm} < \infty$.

Let f_n be represented by a sequence of earthquakes $E^{t'_n\mu_n}$ with $\|\mu_n\|_{Th} = 1$ and $t'_n > 0$. Then $t'_n \to \infty$ as $n \to \infty$ and the first part of the proof gives

$$\|\frac{1}{t'_n}\mathcal{X}([f_n]) - \mu_n\|_{\infty}^{norm} < \frac{1}{t'_n}.$$

Note that if $\|\mu_n\|_{Th} = 1$ then $\|\mu_n\|_{\infty}^{norm} \leq 2$. Then the above inequality implies that $\|\frac{1}{t'_n}\mathcal{X}([f_n])\|_{\infty}^{norm} \leq 3$ for all t'_n with n large enough and the sequence $\frac{t'_n}{t_n}$ is bounded from the above and from the below by positive numbers. By choosing a subsequence, if necessary, we can assume that $\frac{t'_n}{t_n} \to c > 0$ as $n \to \infty$. It follows that, as $n \to \infty$,

$$\|\frac{1}{t_n}\mathcal{X}([f_n]) - c\mu_n\|_{\infty}^{norm} \to 0$$

which implies

$$\|c\mu_n - \sigma\|_{\infty}^{norm} \to 0$$

and the proof is completed. □

8.2. **Infinite surfaces with bounded geodesic pants decompositions.** We consider a hyperbolic surface X_0 which can be decomposed into geodesic pairs of pants with cuffs $\{\alpha_n\}_{n\in\mathbb{N}}$ such that

$$1/M \leq l_{X_0}(\alpha_n) \leq M$$

for some $M > 1$ and for all $n \in \mathbb{N}$. We say that such X_0 has a *bounded geodesic pants decomposition*. The next proposition establishes that the length spectrum Thurston's boundary coincides with Thurston's boundary for $T(X_0)$ introduced using the geodesic currents.

Proposition 8.3. *Let X_0 be a complete, borderless, infinite hyperbolic surface with bounded geodesic pants decomposition. Then the length spectrum Thurston's boundary is equal to the space of projective bounded measured laminations $PML_{bdd}(X_0)$ on X_0.*

Proof. Consider a sequence of points $[f_k] \in T(X_0)$ that converge to (the projective class of) $L^* \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$ in the length spectrum Thurston's boundary of $T(X_0)$. Let $t_k \to \infty$ as $k \to \infty$ be such that $\frac{1}{t_k}\mathcal{X}([f_k]) \to L^*$ in $\mathbb{R}^{\mathcal{S}}_{\geq 0} - \{\bar{0}\}$. Let $E^{t_k\beta_k}$ be a sequence of earthquakes of $\mathbb{H}$ such that $E^{t_k\beta_k}|_{S^1} = f_k$, where $\|\beta_k\|_{Th} < \infty$ (cf. [22]).

The proof of the above proposition gives

$$\Big|\frac{1}{t_k}\frac{\mathcal{X}(E^{t_k\beta_k})(\alpha)}{l_{X_0}(\alpha)}-\frac{\beta_k(\alpha)}{l_{X_0}(\alpha)}\Big|\leq\frac{1}{t_k} \tag{9}$$

for all $\alpha\in\mathcal{S}$.

Since $\frac{1}{t_k}\mathcal{X}([f_k])\to L^*$, the above inequality implies

$$\Big|\frac{\beta_k(\alpha)}{l_{X_0}(\alpha)}-\frac{L^*(\alpha)}{l_{X_0}(\alpha)}\Big|\rightrightarrows 0$$

as $k\to\infty$ uniformly in $\alpha\in\mathcal{S}$. Define

$$\|\beta\|_{ls}=\sup_{\alpha\in\mathcal{S}}\frac{\beta(\alpha)}{l_{X_0}(\alpha)}$$

for any $\beta\in ML_{bdd}(X_0)$. The above convergence gives

$$\sup_{k\in\mathbb{N}}\|\beta_k\|_{ls}=N<\infty.$$

We use the assumption that X_0 has a bounded geodesic pants decomposition in order to prove that $\|\beta_k\|_{Th}$ is bounded in k. Indeed, let $\{\alpha_n\}_{n\in\mathbb{N}}$ be cuffs of a geodesic pants decomposition $\mathcal{P}=\{P_i\}$ of X_0 such that there exists $M>1$ with

$$\frac{1}{M}\leq l_{X_0}(\alpha_n)\leq M$$

for all $n\in\mathbb{N}$, where $\{\alpha_n\}_n$ are cuffs of $\mathcal{P}$. Let P_i be a geodesic pair of pants in the above decomposition with the cuffs α_{i_j}, for $j=1,2,3$. Assume that α_{i_j}, for $j=1,2,3$ are different geodesics of X_0. Denote by P_j, $j=1,2,3$, adjacent pair of pants to P_i with common cuff α_{i_j}. Then there exists a simple closed geodesic $\alpha_{i_j}^*$ in $P_{i_j}\cup P_i$ that intersects α_{i_j} in two points such that $l_{X_0}(\alpha_{i_j}^*)$ is bounded from the above and from the below by positive constants depending only on $M>0$. The components of $P_i-\cup_{j=1}^3(\alpha_{i_j}\cup\alpha_{i_j}^*)$ are simply connected for each i. If two of α_{i_j}, for $j=1,2,3$ is the same geodesic then a similar construction yields $\alpha_{i_j}^*$ such that components of $P_i-\cup_{j=1}^3(\alpha_{i_j}\cup\alpha_{i_j}^*)$ are simply connected and that $l_{X_0}(\alpha_{i_j}^*)$ is bounded in terms of M.

The above convergence of β_k to L^* and boundedness of the lengths of α_{i_j} and $\alpha_{i_j}^*$ on X_0 imply that

$$\beta_k(\alpha_{i_j}),\beta_k(\alpha_{i_j}^*)<C(M)$$

for some constant $C=C(M)$ and for all $i,k\in\mathbb{N}$ and $j=1,2,3$. Since $X_0-\cup_i\cup_{j=1}^3\{\alpha_{i_j},\alpha_{i_j}^*\}$ has simply connected and uniformly bounded components (that are polygons with at most six sides) whose boundaries are subarcs of $\alpha_{i_j},\alpha_{i_j}^*$, we conclude that the supremum over all k and over all above components of the β_k-mass of the geodesics intersecting components is finite. Since each geodesic arc of length 1 on X_0 can intersect at most finitely many components of $X_0-\cup_i\cup_{j=1}^3\{\alpha_{i_j},\alpha_{i_j}^*\}$, it follows that $\sup_{k\in\mathbb{N}}\|\beta_k\|_{Th}<\infty$.

By $\sup_{k\in\mathbb{N}}\|\beta_k\|_{Th}<\infty$, there exists a subsequence β_{k_j} and $\beta^*\in ML_{bdd}(X_0)$ such that $\beta_{k_j}\to\beta^*$ as $j\to\infty$ in the weak* topology. (The weak* topology is described in terms of the lifts of the measured laminations β_k to the universal covering $\mathbb{H}$.) Then

$$L^*(\alpha)=\beta^*(\alpha)$$

for all $\alpha\in\mathcal{S}$ and

$$\|\beta^*\|_{Th}<\infty.$$

Thus any point in the length spectrum Thurston's boundary is in $PML_{bdd}(X_0)$. The above proposition gives that all points in $PML_{bdd}(X_0)$ are also in the length spectrum Thurston's boundary for $T(X_0)$. □

8.3. **Infinite hyperbolic surfaces with upper bounded geodesic pants decompositions.** Let X_0 be a complete, borderless, infinite hyperbolic surface with a geodesic pants decomposition $\mathcal{P}=\{\alpha_n\}_{n\in\mathbb{N}}$ such that

$$\sup_n l_{X_0}(\alpha_n)=M<\infty.$$

In addition, we assume that there exists a subsequence $\{\alpha_{n_j}\}_j$ with $l_{X_0}(\alpha_{n_j})\to 0$ as $j\to\infty$. Let P_n^1 and P_n^2 be the geodesic pairs of pants in $\mathcal{P}$ with a common cuff α_n (possibly $P_n^1=P_n^2$). Let γ_n be a shortest closed geodesic in $P_n^1\cup P_n^2$ that intersects α_n in either one point (when $P_n^1=P_n^2$) or in two points (when $P_n^1\neq P_n^2$). We have that (cf. [1])

$$l_{X_0}(\gamma_n)\simeq \max\{1,|\log l_{X_0}(\alpha_n)|\},$$

where for two quantities $a\simeq b$ means that a/b is between two positive constants.

We define a space $ML_{\mathcal{P}}(X_0)$ of measured laminations on X_0 adopted to the pairs of pants decomposition $\mathcal{P}$. A measured lamination μ on X_0 is in $ML_{\mathcal{P}}(X_0)$ if there exists a constant $N>0$ such that, for all cuffs α_n of $\mathcal{P}$,

$$\mu(\alpha_n)\leq Nl_{X_0}(\alpha_n),$$

and

$$\mu(\gamma_n)\leq N\max\{1,o(|\log l_{X_0}(\alpha_n)|)\},$$

where $o(|\log l_{X_0}(\alpha_n)|)/|\log l_{X_0}(\alpha_n)|\to 0$ as $|\log l_{X_0}(\alpha_n)|\to\infty$. The second condition implies that

$$\mu|_{\alpha_n}\leq N\max\{1,o(|\log l_{X_0}(\alpha_n)|)\}$$

whenever the measure μ has an atom at α_n.

Note that if the geodesic pants decomposition $\mathcal{P}$ of X_0 is bounded then $ML_{\mathcal{P}}(X_0)$ coincides with $ML_{bdd}(X_0)$.

Proposition 8.4. *Let X_0 be a complete, borderless, infinite hyperbolic surface with an upper-bounded geodesic pants decomposition $\mathcal{P}=\{\alpha_n\}_{n\in\mathbb{N}}$ such that a subsequence of cuffs has lengths going to zero. Then the length spectrum Thurston's boundary of $T(X_0)$ is contained in $PML_{\mathcal{P}}(X_0)$.*

Proof. Assume that the projective class of $\mu\in\mathbb{R}^{\mathcal{S}}_{\geq 0}$ is in the length spectrum Thurston's boundary of $T(X_0)$. Then there exists $[f_k]\in T(X_0)$ and $t_k\to\infty$ such that

$$\frac{1}{t_k}\mathcal{X}([f_k])\to\mu$$

as $t_k\to\infty$.

Let $\mu_k\in ML_{bdd}(X_0)$ such that $f_k=E^{t_k\mu_k}|_{S^1}$. By (9)

$$\Big|\frac{1}{t_k}\frac{\mathcal{X}([E^{t_k\mu_k}])(\alpha)}{l_{X_0}(\alpha)}-\frac{\mu_k(\alpha)}{l_{X_0}(\alpha)}\Big|<\frac{1}{t_k}$$

for all $\alpha\in\mathcal{S}$, and it follows that

$$\sup_{\alpha\in\mathcal{S}}\Big|\frac{\mu_k(\alpha)}{l_{X_0}(\alpha)}-\frac{\mu(\alpha)}{l_{X_0}(\alpha)}\Big|\to 0 \tag{10}$$

as $k\to\infty$.

Denote by γ_n a shortest closed geodesic that intersects α_n. Then γ_n intersects α_n in either two or one point and it is contained in the union of two pairs of pants of $\mathcal{P}$ that have α_n on their boundaries. By (10), there exists $N>0$ such that

$$\mu_k(\alpha_n)\leq Nl_{X_0}(\alpha_n)$$

and

$$\mu_k(\gamma_n)\leq Nl_{X_0}(\gamma_n)\simeq O(|\log l_{X_0}(\alpha_n)|)$$

for all $k\in\mathbb{N}$ and for all $\{\alpha_n,\gamma_n\}_{n\in\mathbb{N}}$.

Note that each pair of pants is divided into simply connected components by the union of its cuffs and three geodesics from $\{\gamma_n\}$ that intersect each cuff of the pair of pants. Therefore, any compact subset of X_0 is covered by finitely many simply connected components (that are polygons with at most six sides) whose boundary sides are arcs in $\{\alpha_n,\gamma_n\}_n$. Let $\tilde{\mu}_k$ be the lift of μ_k to the space of geodesics $G(\mathbb{H})$ of the universal covering $\mathbb{H}$. The uniformity in k of the above inequalities implies that the total mass of measured laminations $\tilde{\mu}_k$ on each compact subset K of $G(\tilde{H})$ is bounded by a constant depending on the set K and independent of k. It follows that there exists a subsequence of $\tilde{\mu}_k$ which converges in the weak* topology on $G(\tilde{H})$ to a measured lamination on $G(\mathbb{H})$. Since $\tilde{\mu}_k$ are invariant under the covering group, it follows that the weak* limit is also invariant under the covering group and it projects to a measured lamination on X_0. Therefore μ is induced by the intersection number of closed geodesic with a measured lamination, and from now on we identify μ with this measured lamination. The above inequalities give

$$\mu(\alpha_n)\leq Nl_{X_0}(\alpha_n)$$

and

$$\mu(\gamma_n)\leq Nl_{X_0}(\gamma_n)\simeq O(|\log l_{X_0}(\alpha_n)|)$$

for all $n\in\mathbb{N}$. The first inequality agrees with the inequality in the definition of $ML_{\mathcal{P}}(X_0)$, while the second inequality needs to be improved to get $o(|\log l_{X_0}(\alpha_n)|)$ on the right.

We prove $\mu(\gamma_n)\leq N\max\{1,o(|\log l_{X_0}(\alpha_n)|)\}$ for some $N>0$ by contradiction. Assume that there exist $\epsilon>0$ and a subsequence α_{n_j} such that $l_{X_0}(\alpha_{n_j})\to 0$ as $j\to\infty$ and

$$\mu(\gamma_{n_j})\geq\epsilon|\log l_{X_0}(\alpha_{n_j})|.$$

Let $P^1_{n_j}$ and $P^2_{n_j}$ be the geodesic pairs of pants in $\mathcal{P}$ that have a common cuff α_{n_j}. Let $\alpha_i^{n_j}$, for $i=1,\dots,4$, be the geodesic boundaries of $P^1_{n_j}\cup P^2_{n_j}$. Let $\bar{\mu}_k^{n_j}$ and $\bar{\mu}^{n_j}$ be geodesic laminations on $P^1_{n_j}\cup P^2_{n_j}$ whose supports are $|\mu_k|\cap(P^1_{n_j}\cup P^2_{n_j})$ and $|\mu|\cap(P^1_{n_j}\cup P^2_{n_j})$ respectively, and whose transverse measures are given by the transverse measures of μ_k and μ for geodesic arcs in $P^1_{n_j}\cup P^2_{n_j}$.

Denote by S_{n_j} a complete hyperbolic surface $P^1_{n_j}\cup P^2_{n_j}$, and let $\tilde{S}_{n_j}$ be its universal covering. Then $\tilde{S}_{n_j}$ is isometric to the hyperbolic plane minus a countable number of half-planes whose boundaries are the lifts of $\alpha_i^{n_j}$, for $i=1,\dots,4$. By definition, the boundary $\partial\tilde{S}_{n_j}$ of $\tilde{S}_{n_j}$ consists of all lifts of $\alpha_i^{n_j}$, for $i=1,\dots,4$, together with the ideal boundary points. A hyperbolic geodesic in $\tilde{S}_{n_j}$ has two endpoints on $\partial\tilde{S}_{n_j}$, where both could be ideal endpoints, or both could be finite endpoints, or one could be finite and the other ideal endpoint. Two hyperbolic geodesics are close if their endpoints are close, where the distance between endpoints is expressed in terms of their Euclidean distance when $\tilde{S}_{n_j}$ is given in the unit disk model.

Denote by $\tilde{\bar{\mu}}_k$ and $\tilde{\bar{\mu}}$ the lifts of $\bar{\mu}_k$ and $\bar{\mu}$ to the universal covering $\tilde{S}_{n_j}$. Then $\tilde{\bar{\mu}}_k$ converges to $\tilde{\bar{\mu}}$ in the weak* topology on the measures on the space of geodesics of $\tilde{S}_{n_j}$.

Note that $\mu(\gamma_{n_j}) \geq \epsilon |\log l_{X_0}(\alpha_{n_j})|$ and the convergence of μ_k to μ in the normalized supremum norm implies that, for all k large enough,

$$\mu_k(\gamma_{n_j}) \geq \frac{\epsilon}{2} |\log l_{X_0}(\alpha_{n_j})|.$$

From now, we fix one k such that the above equation holds and seek a contradiction.

A geodesic g in the support $|\bar{\mu}_k|$ of $\bar{\mu}_k$ that intersects γ_n is either α_n, or the image $\gamma_{n_j}^r$ of γ_{n_j} under r full Dehn twists around α_n for r non-zero integer, or it has both of its endpoints on the boundary geodesics of S_{n_j}. We consider three possible cases for the support of $\bar{\mu}_k$: either a subsequence of α_{n_j} is in the support of $\bar{\mu}_k$, or a subsequence of $\gamma_{n_j}^{r_j}$ is in the support of $\bar{\mu}_k$ for some non-zero r_j, or only finitely many of α_{n_j} and $\gamma_{n_j}^{r_j}$ are in the support of $\bar{\mu}_k$.

Assume first that the support of $\bar{\mu}_k$ contains a subsequence of α_{n_j}. Denote this subsequence by α_{n_j} in order to simplify the notation. In this case, any other geodesic in the support of $\bar{\mu}_k$ (different from α_{n_j}) that intersects γ_n can intersect it in at most two points and it connects two boundary geodesics of S_{n_j} (possibly the same). Since the transverse μ_k measure to the four boundary geodesics S_{n_j} is bounded from the above (independently of n_j) and each geodesic of the support $|\mu_k|$ that is different from α_{n_j} intersects γ_{n_j} in at most two points, it follows that $\sup_{n_j}[\mu_k(\gamma_{n_j}) - \mu_k|_{\alpha_{n_j}}] < \infty$. Since $\mu_k(\gamma_{n_j}) \geq \frac{\epsilon}{2}|\log l_{X_0}(\alpha_{n_j})| \to \infty$ as $j \to \infty$, it follows that $\mu_k|_{\alpha_{n_j}} \to \infty$ as $j \to \infty$. This implies $\|\mu_k\|_{Th} = \infty$ which is a contradiction.

Next assume that there exists a subsequence of $\gamma_{n_j}^{r_j}$ that is contained in the support of μ_k with non-zero integers r_j. Without loss of generality, we assume that the whole sequence $\gamma_{n_j}^{r_j}$ is in the support $|\mu_k|$. Divide the situation into two cases: either the sequence r_j is bounded or there exists a subsequence which is unbounded. If the sequence r_j is bounded, then there is an upper bound on the number of intersections between γ_{n_j} and each geodesic of $|\mu_k|$. Since $\mu_k(\gamma_{n_j}) \to \infty$ as $j \to \infty$, it follows that $\mu_k|_{\gamma_{n_j}^{r_j}} \to \infty$ as $j \to \infty$ which contradicts $\|\mu_k\|_{Th} < \infty$.

We continue with the subcase that $|r_j| \to \infty$ as $j \to \infty$. The number of intersections of $\gamma_{n_j}^{r_j}$ with γ_{n_j} is either $2|r_j|$ if S_{n_j} is of type $(0,4)$, or it is $|r_j|$ if S_{n_j} is of type $(1,1)$. Any other geodesic of $|\bar{\mu}_k|$ that intersects γ_{n_j} connects boundary geodesics of S_{n_j} and the number of intersections is either $2|r_j|$ or $|r_j|$ if S_{n_j} is of type $(0,4)$, and the number of intersections is $|r_j|$ if S_{n_j} is of type $(1,1)$. Since $\mu_k(\gamma_{n_j}) \to \infty$ as $j \to \infty$, it follows that either $|r_j|\mu_k|_{\gamma_{n_j}} \to \infty$ or $|r_j|\mu_k(\partial S_{n_j}) \to \infty$ as $j \to \infty$, where ∂S_{n_j} is the union of the four boundary geodesics β_i, for $i = 1, \ldots, 4$, of S_{n_j}. Let $g_{p,q}$ for $p, q \in \{1, \ldots, 4\}$, $p \neq q$, be the set of geodesics of $\bar{\mu}_k$ which connect β_p to β_q. Denote by $\mu_k(g_{p,q})$ the μ_k-measure of a geodesic arc transverse to the set $g_{p,q}$ that does not intersect any other geodesics of $|\bar{\mu}_k|$. If $|r_j|\mu_k(\partial S_{n_j}) \to \infty$ then there exist $p \neq q$ such that $|r_j|\mu_k(g_{p,q}) \to \infty$ as $j \to \infty$.

Let I be a hyperbolic geodesic arc of length 2 that is orthogonal to α_{n_j} with midpoint on α_{n_j}. Let r_j' be the number of intersections of $\gamma_{n_j}^{r_j}$ with I. Then each geodesic of $g_{p,q}$ intersects I in at least $r_j'-2$ points. Lemma 8.5 below implies that there exists $c>0$ such that $\frac{r_j'}{r_j}\geq c$. Thus if $|r_j|\mu_k|_{\gamma_{n_j}}\to\infty$ we also have that $r_j'\mu_k|_{\gamma_{n_j}}\to\infty$ as $j\to\infty$. This implies that the total transverse μ_k-measure of I is going to infinity as $j\to\infty$ and $\|\mu_k\|_{Th}=\infty$ which is a contradiction. Likewise, if $|r_j|\mu_k(g_{p,q})\to\infty$ then $(r_j'-2)\mu_k(g_{p,q})\to\infty$ as $j\to\infty$ and the transverse μ_k-measure of I converges to infinity as $j\to\infty$. Then $\|\mu_k\|_{Th}=\infty$ which is again a contradiction.

The last case to consider is when no geodesics α_{n_j} and no geodesic γ_{n_j} are in the support of μ_k for all $j\geq j_0>0$. Then there exist $p,q\in\{1,\dots,4\}$ such that $|r_j|\mu_k(g_{p,q})\to\infty$ as $j\to\infty$. We note that $|r_j|\to\infty$ as $j\to\infty$ because $\beta_k(g_{p,q})$ is bounded in j. Lemma 8.5 below implies that $(r_j'-2)\mu_k(g_{p,q})\to\infty$ as $j\to\infty$ which gives $\|\mu_k\|_{Th}=\infty$. This is again a contradiction.

Thus the assumption $\mu(\gamma_{n_j})\geq\epsilon|\log l_{X_0}(\alpha_{n_j})|$ is false. Therefore $\mu(\gamma_{n_j})=o(|\log l_{X_0}(\alpha_{n_j})|)$. □

Lemma 8.5. *Let S be a closed hyperbolic surface of the type $(0,4)$ or $(1,1)$ with geodesic boundaries. Let α be a simple closed geodesic in S with*

$$l_S(\alpha)\leq 1/20.$$

Let γ be a shortest simple closed geodesic which intersects α in either two or one point and let I be a closed geodesic arc of length 2 that is orthogonal to α at its midpoint. Then there exists $c>0$ such that

$$i(\delta,\gamma)\leq ci(I,\gamma)$$

for any simple geodesic δ in S which is either closed or that has its endpoints on the boundary of S with $i(\delta,\gamma)\geq 10$.

Proof. Let γ_1 be the simple closed geodesic arc connecting α to itself which is orthogonal to α at both of its endpoints and that belongs to exactly one pair of pants of S. Then

$$i(\delta,\gamma_1)\geq\frac{1}{2}i(\delta,\gamma)-2.$$

Therefore it is enough to prove that there exists $c>0$ such that

$$i(\delta,\gamma_1)\leq ci(\delta,I).$$

We lift the situation to the universal covering $\mathbb{H}$ of S. Denote by $\tilde{\alpha}$ a bi-infinite geodesic in $\mathbb{H}$ that is a component of the lift of α. The components of the lift of γ_1 are all isometric to γ_1 because γ_1 is a simple open closed arc in S. Denote by $\tilde{\gamma}_1$ a single component of the lift of γ_1 that has one endpoint C on $\tilde{\alpha}$. Then the angle between $\tilde{\gamma}_1$ and $\tilde{\alpha}$ is $\pi/2$. Let $\tilde{\delta}$ be a component of the lift of δ that intersects both $\tilde{\gamma}_1$ and $\tilde{\alpha}$ in points B and A, respectively. The hyperbolic triangle ABC has side BC on $\tilde{\gamma}_1$, side CA on $\tilde{\alpha}$ and side AB on $\tilde{\delta}$. Denote by x the length of BC, by y the length of CA and by d the length of AB. Let D be a point on CA such that the length of an arc I' orthogonal to CA at the point D until it meets the side AB is 1. Let y' be the length of DA. Let $l=l_{X_0}(\alpha)$. Then

$$i(\delta,\gamma_1)=\frac{y}{l}\pm 1$$

and

$$i(\delta,I)=\frac{y'}{l}\pm 1.$$

Therefore to give a lower estimate on $\frac{i(\delta,I)}{i(\delta,\gamma_1)}$, it is enough to estimate $\frac{y'}{y}$ when y/l is large enough. Denote by D' the endpoint of I' on AB and let d' be the length of DD'. For a fixed y, the quantity y' will be the least when x is the largest possible. Without loss of generality, we assume that x is equal to the length of γ_1 which is greater than or equal to $|\log l_S(\alpha)|$.

The right angled triangles ABC and $AD'D$ share a common vertex A. Let θ be the common angle at A. The sine rules for these two triangles are

$$\frac{\sinh x}{\sin \alpha} = \sinh d$$
$$\frac{\sinh 1}{\sin \alpha} = \sinh d'$$

which in turn gives

$$\sinh d' = \frac{\sinh 1}{\sinh x} \sinh d. \tag{11}$$

The cosine rules are

$$\cosh x \cosh y = \cosh d$$
$$\cosh 1 \cosh y' = \cosh d'$$

which in turn gives

$$\cosh y' = \frac{\cosh d'}{\cosh 1}. \tag{12}$$

From (12) we have

$$\cosh^2 y' = \frac{1 + \sinh^2 d'}{\cosh^2 1}$$

and substituting (11) and using elementary trigonometric identities, we obtain

$$\sinh y' = \tanh 1 \coth x \sinh y. \tag{13}$$

Since x is bounded from below and if $y \geq y_0 > 0$, the equation (13) implies that there exists $c_0 = c_0(y_0) < 0$ such that

$$y' \geq y + c_0$$

which implies that

$$\frac{y'}{y} = 1 + \frac{c_0}{y} \geq 1/2 > 0$$

for all $y \geq y_0'$ with $y_0' = \max\{y_0, -2c_0\}$. For $\epsilon_0 \leq y \leq y_0'$, the equation (13) implies that there exist $0 < c_1 \leq c_2$ such that $c_1 \leq y' \leq c_2$. In this case we also obtain that $\frac{y'}{y}$ is bounded away from 0 independently of x. Finally, we consider the case $0 < y < \epsilon_0$ where ϵ_0 is chosen such that $1 \geq \frac{\sinh \epsilon_0}{\epsilon_0} \geq \frac{1}{2}$. Since $0 < y' < y < \epsilon_0$ and by the equation (13), we obtain that $\frac{y'}{y}$ is bounded away from 0 independently of x and of $0 < y < \epsilon_0$. Since any y falls in one of these cases, it follows that $\frac{y'}{y}$ is bounded away from 0 independently of x and y. The lemma follows. $\square$

We fix a measured lamination $\mu \in ML_{\mathcal{P}}(X_0)$ on X_0 and consider whether it represents a point in the length spectrum Thurston's boundary of $T(X_0)$ via the intersection number with simple closed geodesics. To do so, we consider the image $X^t = E^{t\mu}(X_0)$ of X_0 under the earthquake path $E^{t\mu}$ and prove that the marked surface X^t is in the closure (for the length spectrum metric) of the Teichmüller space $T(X_0)$ under some additional conditions on μ. Since $\frac{1}{t}E^{t\mu} \to \mu$ as $t \to \infty$ in the normalized supremum norm (because (9) holds for any locally finite measured lamination), it will follow that $[\mu]$ is in the Thurston's boundary.

We use the description of the closure of $T(X_0)$ in the Fenchel-Nielsen coordinates for the pants decomposition $\mathcal{P}=\{\alpha_n\}_{n\in\mathbb{N}}$. Namely, a marked surface $f:X_0\to X$ is in $T(X_0)$ if and only if its Fenchel-Nielsen coordinates $\{(\frac{l_X(\alpha_n)}{l_{X_0}(\alpha_n)},t_X(\alpha_n))\}_{n\in\mathbb{N}}$ are uniformly bounded; $f:X_0\to X$ is in the closure of $T(X_0)$ if and only if $\{\frac{l_X(\alpha_n)}{l_{X_0}(\alpha_n)}\}_n$ is bounded and $|t_X(\alpha_n)|=o(\max\{1,|\log l_{X_0}(\alpha_n)|\})$ for all n (cf. [18]).

We first prove that the twist estimate $|t_X(\alpha_n)|=o(\max\{1,|\log l_{X_0}(\alpha_n)|\})$ holds for all $\mu\in ML_{\mathcal{P}}(X_0)$. Fix a geodesic α_n from the pants decomposition $\mathcal{P}$ and denote by P_n^i for $i=1,2$ the two pairs of pants from $\mathcal{P}$ that have α_n on their boundary with possibly $P_n^1=P_n^2$. Let γ_n^i be the shortest geodesic arc in P_n^i that starts and ends at α_n. Then necessarily γ_n^i is orthogonal to α_n at both of its endpoints. Let γ_n be the geodesic homotopic to a closed curve obtained by concatenating γ_n^1 by an arc on α_n from the endpoint of γ_n^1 to the closets endpoint of γ_n^2 in the counter clockwise direction, followed by γ_n^2, and followed by an arc of α_n from the endpoint of γ_n^2 to the closets endpoint of γ_n^1 in the clockwise direction. The total length of the two arcs on α_n is less than the length of α_n and the length of each γ_n^i is up to an additive constant equal to $2|\log l_{X_0}(\alpha_n)|$. It follows that $l_{X_0}(\gamma_n)$ is up to an additive constant equal to $4|\log l_{X_0}(\alpha_n)|$ if $P_n^1\neq P_n^2$ and that $l_{X_0}(\gamma_n)$ is up to an additive constant equal to $2|\log l_{X_0}(\alpha_n)|$ if $P_n^1=P_n^2$.

Since $\mu(\gamma_n)=o(|\log l_{X_0}(\alpha_n)|)$, it follows that $l_{X^t}(\gamma_n)\leq l_{X_0}(\gamma_n)+\mu(\gamma_n)$ which is up to additive constant equal to $4|\log l_{X_0}(\alpha_n)|+o(|\log l_{X_0}(\alpha_n)|)$ if $P_n^1\neq P_n^2$ or is equal to $2|\log l_{X_0}(\alpha_n)|+o(|\log l_{X_0}(\alpha_n)|)$ if $P_n^1=P_n^2$. On the other hand, since $\mu(\alpha_n)\leq Nl_{X_0}(\alpha_n)$ we have that $l_{X^t}(\alpha_n)\leq (N+1)l_{X_0}(\alpha_n)$.

Assume that $P_n^1=P_n^2$ and denote by γ_n^* the shortest geodesic in $P_n^1=P_n^2$ that intersects α_n in two points. Then $l_{X^t}(\gamma_n^*)=2|\log l_{X^t}(\alpha_n)|$ up to an additive constant. Since $l_{X^t}(\alpha_n)\leq (N+1)l_{X_0}(\alpha_n)$ we get that $l_{X^t}(\gamma_n^*)\geq 2|\log l_{X_0}(\alpha_n)|$ up to an additive constant. We lift the situation to the hyperbolic plane. Let $\tilde{\alpha}_n$ be a single lift of α_n and let $\tilde{\gamma}_n^*$ be a lift of γ_n^* that intersects $\tilde{\alpha}_n$. Let $\tilde{\gamma}_n$ be a single lift of γ_n that intersects $\tilde{\alpha}_n$ such that $\tilde{\gamma}_n\cap\tilde{\alpha}_n$ is closets to $\tilde{\gamma}_n^*\cap\tilde{\alpha}_n$. Let B^* and B be the minimal covering transformation whose axes are $\tilde{\gamma}_n^*$ and $\tilde{\gamma}_n$, respectively. Let A denote the hyperbolic translation with the translation length s such that $A\circ B^*=B$. Namely, γ_n is obtained by n full twists around α_n of the closed geodesic γ_n^* and $s=nl_{X^t}(\alpha_n)$. It follows that

$$\cosh l/2=\cos^2\theta\cosh\frac{l^*-s}{2}+\sin^2\theta\cosh\frac{l^*+s}{2}$$

where $l^*=l_{X^t}(\gamma_n^*)$, $l=l_{X_0}(\gamma_n)$, and $0<\theta<\pi$ is the angle between $\tilde{\alpha}_n$ and $\tilde{\gamma}_n^*$. Since $l_{X^t}(\alpha_n)$ is small and γ_n^* is the shortest geodesic in P_n^1 intersecting α_n, it follows that θ is bounded away from 0 and π. Then by dropping the first term on the right of the above equation and some elementary estimates, we get

$$l\geq l^*+s+const$$

which implies

$$s\leq l-l^*+const\leq o(|\log l_{X_0}(\alpha_n)|)+const.$$

When $s\geq 0$ the above estimate suffices. When $s\leq 0$ then we drop the second term on the right of the above equation to obtain a similar estimate. The number s is (up to a positive additive constant less than $Nl_{X_0}(\alpha_n)$) equal to the twist on α_n of the marked surface X^t and we obtained the bound on the twists on α_n for X_t of the order $o(|\log l_{X_0}(\alpha_n)|)$.

When $P_n^1 \neq P_n^2$ the above method applies as well. The only difference is that $B = A_1 \circ A_2 \circ B^*$, where A_1 and A_2 are the hyperbolic translations covering maps whose axes are two consecutive lifts of α_n that intersect $\tilde{\gamma}_n^*$. The distance between the consecutive lifts of α_n is of the order $|\log l_{X^t}(\alpha_n)|$ which is large as $l_{X_0}(\alpha_n) \to 0$. Then $A_1 \circ A_2$ is a hyperbolic translation whose axis is between the axes of A_1 and A_2, and whose translation length $s_{1,2}$ is greater than $2s$. The angle θ between the axis of $A_1 \circ A_2$ and $\tilde{\alpha}_n$ is either close to 0 or close to π which could make additive constants unbounded. However, when the twist $s > 0$, then also $s_{1,2} > 0$ and θ is close to π. This implies $\frac{\theta}{2}$ is close to $\frac{\pi}{2}$ and $\sin^2\theta$ is bounded away from 0. Then as above, when $s > 0$,

$$s \leq s_{1,2} \leq l - l^* + const \leq o(|\log l_{\alpha_n}(X_0)|) + const$$

and similar conclusion holds when $s < 0$. Thus we obtained that the bound on the twists of X^t satisfies the condition from [18] to be in the closure of $T(X_0)$.

We consider the bounds on the ratio of lengths $\frac{l_{X^t}(\alpha_n)}{l_{X_0}(\alpha_n)}$, where $X^t = E^{t\mu}(X_0)$ and $\mu \in ML_{\mathcal{P}}(X_0)$. We have the equation (cf. [20])

$$\cosh\frac{l_t'}{2} = \cos^2\frac{\theta}{2}\cosh\frac{m-l'}{2} + \sin^2\frac{\theta}{2}\cosh\frac{m+l'}{2} \tag{14}$$

where $l_t' = l_{X^t}(\alpha_n)$, $l' = l_{X_0}(\alpha_n)$, m is the translation length of the comparison map for $E^{t\tilde{\mu}}$ between a stratum O and its image $A(O)$ where A is the covering transformation for α_n and O intersects the axis of A, and $0 \leq \theta \leq \pi$ is the angle between the axis of $E^{t\tilde{\mu}}|_{A(O)}$ (when normalized such that $E^{t\tilde{\mu}}|_O = id$) and the axis of A. Since $\mu(\alpha_n) \leq N l_{X_0}(\alpha_n)$, it follows that there exists $N' = N'(N) > 0$ such that

$$m \leq N'l'.$$

When $\theta \geq \theta_0 > 0$ for each geodesic α_n, the equation (14) gives that

$$1 + c_1 l_t'^2 \geq \cosh\frac{l_t'}{2} \geq \cos^2\theta_0 + \sin^2\frac{\theta_0}{2}\cosh\frac{m+l'}{2} \geq 1 + c_0 l'^2$$

for some $c_0, c_1 > 0$ when l' is small enough; this implies $\frac{l_t'}{l'} \geq c > 0$ for all α_n and all $t > 0$. The upper bound on $\frac{l_t'}{l'}$ follows from the upper bound on m independently of the angle θ. Therefore $E^{t\mu}(X_0) = X^t$ is in the closure of $T(X_0)$ in this case.

When $\mu(\alpha_n) \geq \epsilon_0 l'$ for all α_n and some $\epsilon_0 > 0$, the equation (14) gives that, for $t > \frac{2}{\epsilon_0}$,

$$\cosh\frac{l_t'}{2} \geq \cos^2\frac{\theta}{2}\cosh\frac{l'}{2} + \sin^2\frac{\theta}{2}\cosh\frac{l'}{2} = \cosh\frac{l'}{2}$$

which implies $\frac{l_t'}{l'} \geq c > 0$ for all α_n and all $t > \frac{2}{\epsilon_0}$. Therefore $E^{t\mu}(X_0) = X^t$ is in the closure of $T(X_0)$ in this case as well.

However if $t\mu(\alpha_n) = l'$ and θ is close to zero, then the first term on the right of the equation (14) is $\cos^2\theta$ while the second term is close to $\sin^2\theta + \sin^2\theta(\frac{l'}{2})^2$. Then $\frac{l_t'}{l'}$ is small and converges to zero as the angle $\theta \to 0$. In this case X_t is not in the closure of $T(X_0)$. If we take infinite sequence $\{\alpha_n\}_n$ such that $l_{\alpha_n}(X_0) \to 0$ and that $\{\frac{1}{\mu(\alpha_n)}\}_n$ is dense in a neighborhood of ∞ with the angles $\theta_n \to 0$, we obtain that X_t is not in the closure of $T(X_0)$ for all t large.

When the measured lamination μ has a subsequence of α_n in its supports with weights $o(|\log l_{X_0}(\alpha_n)|)$ and the other part of the support of μ is Thurston bounded, then $E^{t\mu}(X_0) = X^t$ is in the closure of $T(X_0)$ (cf. [20]).

Proposition 8.6. *Let X_0 be a complete, borderless hyperbolic surface with an upper bounded geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}$. Let μ be a measured lamination in $ML_{\mathcal{P}}(X_0)$ such that one of the following holds:*

1. *μ has finite Thurston norm except that a subsequence of α_n whose lengths go to zero have weights $o(|\log l_{X_0}(\alpha_n)|)$,*
2. *the support geodesics of μ subtend angles at least $\theta_0 > 0$ with geodesic γ_n, where γ_n is a shortest geodesics that intersect α_n in one or two points with $l_{X_0}(\alpha_n) \to 0$*
3. *either $\mu(\alpha_n) \geq \epsilon_0 l_{X_0}(\alpha_n)$ for some $\epsilon_0 > 0$ or $\mu(\alpha_n) = 0$.*

Then the projective class of $\mu \in ML_{\mathcal{P}}(X_0)$ is in the length spectrum Thurston boundary for $T(X_0)$.

Theorems 2 and 3 from Introduction are established by Propositions 8.2, 8.3, 8.4 and 8.6.

8.4. **Two infinite surfaces with unbounded geodesic pants decompositions.** The first surface X_1 that we consider is introduced by Kinjo [11]. Let Γ' be the hyperbolic triangle group of signature $(2,4,8)$. Let T' be the triangle fundamental polygon for Γ' with angles $\pi/2$, $\pi/4$ and $\pi/8$. Then $\Gamma'(T')$ tiles the hyperbolic plane $\mathbb{H}$. Let T be the union of T' and $\gamma_0'(T')$, where $\gamma_0' \in \Gamma'$ is a reflection in the geodesic containing the side of T' which subtends the angles $\pi/2$ and $\pi/8$ of T'. Denote the vertices of T by a, b and c; the vertex b is where T' has angle $\pi/8$ (cf. [11, Figure 2]). We choose three points a', b' and c' close to a, b and c, respectively, in the interior of the triangle T such that b' is on the side of T' containing b. The surface X_1 is obtained by puncturing the hyperbolic plane at the points $\Gamma'\{a', b', c'\}$ (cf. [11, Figures 2,3]). Kinjo [11] proved that the Teichmüller space $T(X_1)$ is complete in the length spectrum metric.

Let $\{\gamma_i\}_{i=1,\dots,8}$ be the elements of Γ' that fix a. Let l_a be the simple closed geodesic which separates the eight points $\{\gamma_i(a)\}_{i=1,\dots,8}$ from the other punctures of X_1. We similarly define curves l_b and l_c, and then extend the definition using Γ' to all other groups of eight cusps. The lengths of all $\Gamma'(l_a)$ are the same, as well as the lengths of all $\Gamma'(l_b)$, as well as the lengths of all $\Gamma'(l_c)$.

For the triangle T, we denote by $l_{a',b'}$ the simple closed geodesic which is homotopic to a simple closed curve in T that separates a', b' from c'. We similarly extend the definition to $l_{b',c'}$ and $l_{c',a'}$, and then extend it to all triangles using the invariance under Γ'. Note that the lengths of $\Gamma'(l_{a',b'})$ are the same, as well as the lengths of all $\Gamma'(l_{b',c'})$l, and the lengths of all $\Gamma'(l_{c',a'})$.

The lengths of the family of geodesics $\Gamma'(l_a) \cup \Gamma'(l_b) \cup \Gamma'(l_c) \cup \Gamma'(l_{a',b'}) \cup \Gamma'(l_{b',c'}) \cup \Gamma'(l_{c',a'})$ are bounded from the below and from the above, and this family separates the surface X_1 into finite bounded polygons with uniformly bounded number of sides. Then the proof of Proposition 8.4 extends to show that the length spectrum Thurston's boundary coincides with $PML_{bdd}(X_1)$.

Denote by X_2 an infinite hyperbolic surface defined by Shiga [21] that has geodesic pants decomposition with cuff lengths converging to infinity. The surface X_2 contains a sequence γ_n of simple closed geodesics with $l_{X_2}(\gamma_n) \to \infty$ as $n \to \infty$

such that for each closed geodesic δ we have

$$l_{X_2}(\delta) \geq \sum_{k=1}^{\infty} k l_{X_2}(\gamma_k) i(\gamma_k, \delta), \tag{15}$$

where only finitely many terms are non-zero. Shiga [21] proved that a sequence of full Dehn twists f_n around the curve γ_n diverges in the Teichmüller metric and it converges to the identity in the length spectrum metric. Thus the two metrics produce different topologies on $T(X_2)$.

We define β_n to be a measured lamination whose support is $\{\gamma_k\}_{k=1,\dots,n}$ such that, for $k = 1, \dots, n$,

$$\beta_n|_{\gamma_k} = l_{X_2}(\gamma_k).$$

The projective class $[\beta_n]$ is in $PML_{bdd}(X_2)$. Define β_* to be a measured lamination on X_2 whose support is $\{\gamma_k\}_{k=1}^{\infty}$ such that, for all $k = 1, 2, \dots$,

$$\beta_*|_{\gamma_k} = l_{X_2}(\gamma_k).$$

It is clear that the projective class $[\beta_*]$ is not in $PML_{bdd}(X_2)$.

We prove that $[\beta_n] \to [\beta_*]$ as $n \to \infty$ in the normalized supremum norm. Indeed, let δ be a simple closed geodesic in X_2. Then

$$\frac{|\beta_n(\delta) - \beta_*(\delta)|}{l_{X_2}(\delta)} = \sum_{k=n+1}^{\infty} \frac{\beta_k(\delta)}{l_{X_2}(\delta)} = \frac{\sum_{k=n+1}^{\infty} i(\delta, \gamma_k) l_{X_2}(\gamma_k)}{\sum_{k=n+1}^{\infty} k i(\delta, \gamma_k) l_{X_2}(\gamma_k)} \leq \frac{1}{n+1}$$

and $[\beta_*]$ is in the length spectrum Thurston boundary of $T(X_2)$. Therefore the boundary is larger than $PML_{bdd}(X_2)$.

Open problem: Assume that a sequence in $T(X_0)$ converges to a bounded projective measured lamination in the length spectrum Thurston's boundary. Is it true that the sequence converges in the Thurston's boundary introduced using Liouville currents?

9. Appendix

We prove a standard lemma regarding neighborhoods in $T(\mathbb{H})$ and Liouville measure of boxes of geodesics under the maps in given neighborhoods.

Lemma 9.1. *Let $h_0 \in T(\mathbb{H})$. Given $\epsilon > 0$ and $0 < \delta < \log 2$, there exists an open neighborhood $N(h_0, \delta, \epsilon)$ of h_0 in $T(\mathbb{H})$ such that for each box of geodesics Q with*

$$\delta \leq L(Q) \leq \log 2$$

we have

$$|\alpha_0(Q) - \alpha(Q)| < \epsilon$$

where $\alpha_0 = (h_0)^* L$ and $\alpha = h^* L$, for any $h \in N(h_0, \delta, \epsilon)$.

Proof. Given a box of geodesics $Q = [a, b] \times [c, d]$, let $m(Q)$ denote the modulus of the quadrilateral with interior $\mathbb{H}$ whose a-sides are $[a, b], [c, d] \subset S^1$ and b-sides are $[b, c], [d, a] \subset S^1$. Then $m(Q)$ and $L(Q)$ are continuous functions of each other with $m(Q) = 1$ if and only if $L(Q) = \log 2$.

If $f_0 : \mathbb{H} \to \mathbb{H}$ is a K-quasiconformal continuous extension of $h_0 : S^1 \to S^1$ then

$$\frac{1}{K} m(Q) \leq m(f_0(Q)) \leq K m(Q)$$

for all quadrilaterals Q with interior $\mathbb{H}$.

If $\delta \leq L(Q) \leq \log 2$ then there exists $C = C(K,\delta) \geq 1$ such that

$$\frac{1}{C} \leq L(h_0(Q)) \leq C$$

(by the continuous dependence of $L(Q)$ on $m(Q)$).

Furthermore, there exists $C_1 = C_1(C) \geq 1$ such that

$$\frac{1}{C_1} \leq m(h_0(Q)) \leq C_1$$

for all Q with $\delta \leq L(Q) \leq \log 2$.

Let $h \in T(\mathbb{H})$ such that $h \circ h_0^{-1}$ has K_1-quasiconformal extension to $\mathbb{H}$. Then

$$|m(h_0(Q)) - m(h(Q))| \leq (K_1 - 1)m(h_0(Q)).$$

By the uniform continuity of $L(Q)$ in $m(Q)$ when $m(Q)$ is in a compact interval $[\frac{1}{C_1}, C_1]$, we obtain

$$|L(h_0(Q)) - L(h(Q))| \to 0$$

as $K_1 \to 1$.

Since $\alpha_0(Q) = L(h_0(Q))$ and $\alpha(Q) = L(h(Q))$, there exists a neighborhood $N(h_0,\delta,\epsilon)$ of $h_0 \in T(\mathbb{H})$ which satisfies the conclusions of the lemma. □

We consider the behavior of the Liouville measure of a box of geodesics under a simple (left) earthquake.

Lemma 9.2. *Fix a box of geodesics $Q = [a,b] \times [c,d]$. Let E be a simple earthquake with the support geodesic g and the earthquake measure $m \geq 0$ on g. Assume that g separates the quadruple of points $\{a,b,c,d\}$ into two non-empty sets. Then the Liouville measure of $E([a,b] \times [c,d])$ will be the largest when the geodesic g has endpoints a and c.*

Proof. Note that the Liouville measure of $[a,b] \times [c,d]$ is an increasing function of the distance between the geodesic $l(a,d)$ with endpoints a, d and the geodesic $l(b,c)$ with endpoints b, c.

Assume first that g has an endpoint in $[c,d]$ and an endpoint in (d,a). Normalize E such that it is the identity in the half-plane of the complement of g which has point b on its boundary. We have

$$E([a,b] \times [c,d]) = [a,b] \times [c,T(d)]$$

where T is the hyperbolic translation with the axis g, the translation length m and the attracting fixed point the endpoint of g in the interval $[c,d]$. An elementary observation gives that point $T(d)$ will be closets to point a for the orientation of S^1 when g has endpoints a and c. By the remark in the first paragraph the corresponding Liouville mass will be largest. We established the lemma in this case.

Assume that g has one endpoint in $[c,d]$ and the other endpoint in (b,c). We normalize E to be the identity in the half-plane of the complement of g which has point a on its boundary. Then E fixes a, b, d and moves c closer to d. In this case the Liouville measure is decreased and the lemma holds since the Liouville measure is increased when g has endpoints a and c.

Assume that g has one endpoint in $[a,b]$ and the other endpoint in $[c,d]$. We normalize earthquake E such that it is the identity on the half-plane complement of g that has interval (d,a) on its boundary. The conjugate by a hyperbolic isometry of a simple left earthquake is a simple left earthquake with the same measure and the support is the image of the support of the original earthquake. Thus we can assume that we are in the upper half-plane model $\mathbb{H}$, that g has endpoints $y > 0$ and ∞, and that $d < a = 0 \leq y \leq b < c$ on the real axis. A direct computation gives

$$L(E([a,b] \times [c,d])) = \log \frac{[e^m c + (1-e^m)y][d - e^m b + (e^m - 1)y]}{d[e^m(c-b)]}$$

which is the largest when $y = 0$. Thus if we fix one endpoint of g in the interval $[c,d]$ and vary the other endpoint in the interval $[a,b]$, the largest Liouville measure of the image box $E([a,b] \times [c,d])$ will be when g has endpoint a. Now we consider g to have one endpoint a and the other endpoint in the interval $[c,d]$. Similar as before, we can use the upper half-plane model $\mathbb{H}$ for the hyperbolic plane. We assume that $b < c = 0 \leq y \leq d$ and $a = \infty$ on the extended real axis, where g has endpoints y and $a = \infty$, and the earthquake E is normalized to be the identity for all real numbers greater than y. A direct computation gives

$$L(E([\infty,b] \times [0,d]) = \log \frac{d - e^{-m}b + (e^{-m} - 1)y}{-be^{-m}}$$

which is largest when $y = 0$. Thus the Liouville measure of the image under earthquake E of box $[a,b] \times [c,d]$ is largest when the support g has endpoint a and c. This establishes the lemma in this case.

Since the Liouville measure of $[a,b] \times [c,d]$ is equal to the Liouville measure of $[c,d] \times [a,b]$, the cases involving the position of g where we replace $[a,b]$ with $[c,d]$ are automatically proved.

It remains to consider the case when g has one endpoint in $[d,a]$ and the other endpoint in $[b,c]$. We use the upper half-plane model $\mathbb{H}$ and assume $c < d \leq y_r \leq a = 0 < b$, where g has endpoints y_r and $y_a = \infty$. The earthquake E is normalized to be the identity for real numbers less than y_r. By direct computation

$$L(E([c,d] \times [0,b])) = \log \frac{[-c - y_r(e^m - 1)][-d + e^m b - y_r(e^m - 1)]}{[-d - y_r(e^m - 1)][-c + be^m - y_r(e^m - 1)]}$$

and it is the least when $y = 0$. Thus we showed that if we fix one endpoint of g in $[b,c]$ and vary the other fixed point in $[d,a]$ (while keeping the translation length m fixed), largest Liouville measure will be when the fixed point is a. Next we consider all g with one fixed point a and the other fixed point in $[b,c]$. Earthquake E fixes a, c, d. The point $E(b)$ will be closest to c when the other endpoint of g is c by the above work. By the first paragraph in the proof, we have established that Liouville measure of $E([a,b] \times [c,d])$ is largest when the support geodesic g has endpoints a and c. This covers all the cases and the proof is finished. □

In the following lemma we establish the estimate for Liouville measure of a box of geodesics $Q = [a,b] \times [c,d]$ under simple earthquakes whose support geodesic has endpoints a and c. This is the case of the largest increase in Liouville measure as established in the previous lemma.

Lemma 9.3. *Let $Q = [a,b] \times [c,d]$ be a box of geodesics and let $d = dist(l(a,d), l(b,c))$ be the distance between the geodesic $l(a,d)$ with endpoints a, d and the geodesic $l(b,c)$ with endpoints b, c. Let E be a simple earthquake with the support $g = l(a,c)$ and* *measure* $m > 0$. *Then*

$$m + \log \frac{d^2}{4} \leq L(E([a,b] \times [c,d])) \leq m + L([a,b] \times [c,d]).$$

Proof. Normalize E to be the identity on the half-plane complement of g which contains d. We use the upper half-plane model $\mathbb{H}$ and assume that $a = 0$, $b > 0$, $c = \infty$ and $d = -1$. A direct computation yields

$$L(E([a,b] \times [c,d])) = \log(e^m b + 1)$$

which easily give estimate in the statement of the lemma. □

The following lemma establishes when the Liouville measure of the image under a simple earthquake of a box $Q = [a,b] \times [c,d]$ of geodesic will be largest given that one endpoint of the support geodesic is in $[a,b]$ and the other endpoint is in $[b,c]$.

Lemma 9.4. *Let* $Q = [a,b] \times [c,d]$ *be a box of geodesic. Let* E *be a simple earthquake whose support geodesic* g *has one endpoint in* $[a,b]$ *and the other endpoint varies in* $[b,c]$. *Then* $E(Q)$ *will have largest Liouville measure when the geodesic* g *has endpoint* c.

Proof. Let $m > 0$ be earthquake measure on g for the earthquake E. Let $y \in [b,c]$ be the variable endpoint of g. We use the upper half-plane model $\mathbb{H}$. Assume that $b \leq y \leq c = 0 < d = 1 < a$ and the other endpoint of g is ∞. A direct computation gives

$$L(E([a,b] \times [c,d])) = \log \frac{a(1 - be^{-m} - y + ye^{-m})}{(a-1)(-y + ye^{-m} - be^{-m})}$$

which is largest for $y = 0$. This establishes that $y = c$. □

In the following lemma we estimate Liouville measure of the image of a box of geodesic under a simple earthquake when the support geodesic of the simple earthquake belongs to the box.

Lemma 9.5. *Let* $Q = [a,b] \times [c,d]$ *be a box of geodesics and let* E *be a simple earthquake whose support is geodesic* $l(a',c')$ *with endpoints* $a' \in (a,b)$ *and* $c' \in (c,d)$. *If* E_1 *is a simple earthquake whose support is geodesic* $l(a'',c'')$ *with endpoints* $a'' \in [a',b)$ *and* $c'' \in [c',d)$ *whose earthquake measure* $m > 0$ *on* $l(a'',c'')$ *equals earthquake measure of* E *on* $l(a',b')$, *then*

$$L(E([a,b] \times [c,d])) \geq L(E_1([a,b] \times [c,d])).$$

Proof. We use the upper half-plane model $\mathbb{H}$ and assume that $d < a < 0 \leq a' \leq a'' \leq b = 1 < c$ and $c' = c'' = \infty$. A direct computation gives

$$L(E([a,b] \times [c,d])) = \log \frac{(e^m c + (1-e^m)a' - a)(-d + e^m + (1-e^m)a')}{(-d+a)(c-1)e^m}$$

which decreases when a' increase. Thus

$$L(E([a,b] \times [c,d])) \geq L(E_1([a,b] \times [c,d]))$$

in the case when $c' = c''$.

To finish the proof, we assume that $a' = a''$ and $c' < c''$ in interval $[c,d]$ for the orientation of S^1. This case is by symmetry reduced to the previous case and the proof is completed. □

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