Solutions to problems on the first midterm

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1 Rewrite the equation as $-A^5+5A^4-4A^3+A^2-A=I$ and then factor out A of the left side to get $\left(-A^4+5A^3-4A^2+A-I\right)A=I$. This shows that A is invertible with the inverse $\left(-A^4+5A^3-4A^2+A-I\right)$.

2 Start with the 4×8 matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Subtract the first row from the second, then from the third and from the fourth. The result of these three row operations is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Now subtract the second row in turn from the third and the fourth to get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Then subtract the third row from the fourth to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Now divide rows by appropriate constants to get the identity matrix in the first 3×3 block.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

The inverse is the matrix consisting of the last four columns.

3 Use row operations to reduce the matrix to a triangular form. Here is what I got (yours may look somewhat different but would lead to the same answer).

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & c(c^2-2) \end{bmatrix}$$

An upper triangular matrix is invertible if and only if all of its diagonal entries are nonzero. Thus for the original matrix to be invertible $c \neq 0, \sqrt{2}, -\sqrt{2}$.

4 To verify that one matrix is the inverse of another, just multiply them to see whether the product is the identity matrix. Then to solve an equation $AX = B$ set $X = A^{-1}B$. In our case,

$$X = \frac{1}{9} \begin{bmatrix} -5 & -1 & -1 \\ 1 & -7 & 2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix}.$$

5 Rewrite the equation as $Q(A\mathbf{x}) = 0$. Since Q is invertible our equation is equivalent to $Q^{-1}Q(A\mathbf{x}) = 0$, i.e. $A\mathbf{x} = 0$

6 We use matrices and row reduction although we could work with equations themselves equally well. The augmented matrix of the system is

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -4 & 5 & 3 & b_2 \\ -2 & 1 & 1 & b_3 \end{bmatrix}.$$

We use row reduction to bring it to the form

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -3 & -1 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{bmatrix}.$$

This rewritten in terms of equations yields

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & - & x_3 & = & b_1 \\ & - & 3x_2 & - & x_3 & = & b_2 + 4b_1 \\ & & & & 0 & = & b_3 - b_2 - 2b_1. \end{array}$$

Thus $b_3 - b_2 - 2b_1 = 0$ is a necessary condition for a solution to exist. It is also sufficient since, assuming $b_3 - b_2 - 2b_1 = 0$, we can assign *any* value to x_3 and solve successively for x_2 and x_1.

7 Since $(C^T)^T = C$ for all matrices C and $(CD)^T = D^T C^T$ whenever the product CD is defined, we obtain

$$\left(BAB^T\right)^T = BA^T B^T = BAB^T.$$

The last equality follows from the fact that A is symmetric. Thus BAB^T is equal to its transpose, i.e. is symmatric.

8 Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the standard basis vectors in $\mathbb{R}^4$. Then

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \qquad T(\mathbf{e}_3) = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \qquad T(\mathbf{e}_4) = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

and the matrix A such that $T = T_A$ is

$$\begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix}.$$

Thus

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}\right) = A \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -15 \\ -2 \end{bmatrix}.$$

9 $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are respectively the first, the second, and the third columns of A. Thus

$$T(\mathbf{e}_1 + \mathbf{e}_2) = T(\mathbf{e}_1) + T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}.$$

10 A general second degree polynomial is of the form $p(x) = ax^2 + bx + c$. In our case, $p(0) = 0$, $p(-1) = 1$, and $p(1) = 2$. Thus

$$\begin{aligned} c &= 0 \\ a - b + c &= 1 \\ a + b + c &= 2. \end{aligned}$$

This system of equations is solved easily to give $a = \frac{3}{2}$, $b = \frac{1}{2}$, $c = 0$ so that $p(x) = \frac{3}{2}x^2 + \frac{1}{2}x$.