EXAM

Exam 2
Math 231: Spring 2015
Thursday, March 26, 2015

ANSWERS
Problem 1. True or False.

(a) The list of vectors \((1, 2, 0), (0, 0, 5), (1, 0, 3)\) is a basis for \(\mathbb{R}^3\).

\textbf{Answer:}

True. The list is independent and any list of three independent vectors in a three dimensional vector space must be a basis.

(b) The list of polynomials \(1, (x - 5)^2, (x - 5)^3\) is a basis for the subspace \(U\) of \(\mathcal{P}_3(\mathbb{R})\) defined by \(U = \{ p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0 \}\).

\textbf{Answer:}

True. Since \(U\) is a proper subspace of a four dimensional vector space, we know \(\dim(U) \leq 3\). The three given polynomials are an independent list in \(U\), hence must be a basis.

(c) The function \(T : \mathbb{R}^2 \to \mathbb{R}^3\) defined by \(T(x, y) = (x^2, xy, y^2)\) is a linear map.

\textbf{Answer:}

False. \(T(1, 1) = (1, 1, 1)\) and \(T(2, 2) = (4, 4, 4)\), so we see that \(T\) is not homogeneous.

(d) If \(U\) and \(W\) are two-dimensional subspaces of \(\mathbb{R}^3\) then there exists a nonzero vector \(v \in U \cap W\).

\textbf{Answer:}

True. Since for any subspaces \(U\) and \(W\) of \(\mathbb{R}^3\) we have \(\dim(U) + \dim(W) = \dim(U \cap W) + \dim(U \cap W) = \dim(\mathbb{R}^3) = 3\), if \(\dim(U) = \dim(W) = 2\), \(\dim(U \cap W) \geq 1\), hence must contain a nonzero vector.

(e) If \(T : V \to W\) is a linear map with \(\text{null}(T) = \{0\}\) then \(T\) is injective.

\textbf{Answer:}

True. If \(T(v) = T(w)\) for some \(v \neq w\), we’d have \(T(v - w) = T(v) - T(w) = 0\) so \(v - w \in \text{null}(T)\).
Problem 2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear map defined by $T(x, y, z) = x + y + z$.

(a) Find a basis for $\text{null}(T)$.

Answer:
Note that $\text{null}(T) = \{(x, y, z) : x + y + z = 0\}$ is a proper subspace of $\mathbb{R}^3$, which is three dimensional, and that $(1, -1, 0), (1, 0, -1)$ are two linearly independent vectors in $\text{null}(T)$. Therefore, $(1, -1, 0), (1, 0, -1)$ is a basis for $\text{null}(T)$.

(b) Find a basis for a subspace $W \subset \mathbb{R}^3$ so that $\mathbb{R}^3 = \text{null}(T) \oplus W$.

Answer:
Any single vector that is not in $\text{null}(T)$ will be a basis for a subspace $W$ with $\mathbb{R}^3 = \text{null}(T) \oplus W$. For example, $(1, 2, 3)$. 
Problem 3. Let $V = \mathcal{P}_4(\mathbb{R})$ and $W = \mathbb{R}^3$ and consider $\mathcal{L}(V, W)$.

(a) Prove that the set $X = \{T \in \mathcal{L}(V, W) : T(x^2) = 0\}$ is a subspace of $\mathcal{L}(V, W)$.

*Answer*:

There are three things to check:

- First note that the zero linear map is in the set $X$.
- $X$ is closed under addition. To see this, suppose $S(x^2) = 0$ and $T(x^2) = 0$. Then $(S + T)(x^2) = S(x^2) + T(x^2) = 0 + 0 = 0$.
- $X$ is closed under scalar multiplication. To see this, suppose $T(x^2) = 0$ and that $\lambda \in \mathbb{R}$. Then $\lambda T(x^2) = \lambda(0) = 0$.

(b) Explain why the set $Y = \{T \in \mathcal{L}(V, W) : T \text{ is injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

*Answer*:

Since the zero linear transformation is not injective, the set $Y$ does not contain the zero linear transformation and so cannot be a subspace.
Problem 4. Prove that there cannot be a surjective linear map \( S : \mathcal{P}_4(\mathbb{R}) \to \mathbb{R}^3 \) with

\[
\text{null}(S) = \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p(x) \, dx = 0 \right\}.
\]

Answer:
Since \( \dim(\mathcal{P}_4(\mathbb{R})) = 5 \), the fundamental theorem of linear maps says that

\[
\dim(\text{null}(S)) + \dim(\text{range}(S)) = 5.
\]

If \( S \) is surjective, then \( \dim(\text{range}(S)) = \dim(\mathbb{R}^3) = 3 \) implying that \( \dim(\text{null}(S)) = 2 \). But the dimension of the given set \( \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{0}^{1} p(x) \, dx = 0 \right\} \) is greater than two since, for example, \( x, x^2 - \frac{1}{3}, x^3 \) is a list of three independent vectors in \( \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p(x) \, dx = 0 \right\} \).