Problem 1. True or False.

(a) The list of vectors $(1, 2, 0), (0, 0, 5), (1, 0, 3)$ is a basis for $\mathbb{R}^3$.

Answer:
True. The list is independent and any list of three independent vectors in a three dimensional vector space must be a basis.

(b) The list of polynomials $1, (x - 5)^2, (x - 5)^3$ is a basis for the subspace $U$ of $\mathcal{P}_3(\mathbb{R})$ defined by $U = \{ p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0 \}$.

Answer:
True. Since $U$ is a proper subspace of a four dimensional vector space, we know $\dim(U) \leq 3$. The three given polynomials are an independent list in $U$, hence must be a basis.

(c) The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x^2, xy, y^2)$ is a linear map.

Answer:
False. $T(1, 1) = (1, 1, 1)$ and $T(2, 2) = (4, 4, 4)$, so we see that $T$ is not homogeneous.

(d) If $U$ and $W$ are two-dimensional subspaces of $\mathbb{R}^3$ then there exists a nonzero vector $v \in U \cap W$.

Answer:
True. Since for any subspaces $U$ and $W$ of $\mathbb{R}^3$ we have $\dim(U) + \dim(W) - \dim(U \cap W) = \dim(U + W) \leq \dim(\mathbb{R}^3) = 3$, if $\dim(U) = \dim(W) = 2$, $\dim(U \cap W) \geq 1$, hence must contain a nonzero vector.

(e) If $T : V \to W$ is a linear map with $\text{null}(T) = \{0\}$ then $T$ is injective.

Answer:
True. If $T(v) = T(w)$ for some $v \neq w$, we’d have $T(v - w) = T(v) - T(w) = 0$ so $v - w \in \text{null}(T)$. 
Problem 2. Let \( T : \mathbb{R}^3 \to \mathbb{R} \) be the linear map defined by \( T(x, y, z) = x + y + z \).

(a) Find a basis for \( \text{null}(T) \).

\[ \text{Answer:} \]
Note that \( \text{null}(T) = \{ (x, y, z) : x + y + z = 0 \} \) is a proper subspace of \( \mathbb{R}^3 \), which is three dimensional, and that \( (1, -1, 0), (1, 0, -1) \) are two linearly independent vectors in \( \text{null}(T) \). Therefore, \( (1, -1, 0), (1, 0, -1) \) is a basis for \( \text{null}(T) \).

(b) Find a basis for a subspace \( W \subset \mathbb{R}^3 \) so that \( \mathbb{R}^3 = \text{null}(T) \oplus W \).

\[ \text{Answer:} \]
Any single vector that is not in \( \text{null}(T) \) will be a basis for a subspace \( W \) with \( \mathbb{R}^3 = \text{null}(T) \oplus W \). For example, \( (1, 2, 3) \).
Problem 3. Let $V = \mathcal{P}_4(\mathbb{R})$ and $W = \mathbb{R}^3$ and consider $\mathcal{L}(V, W)$.

(a) Prove that the set $X = \{ T \in \mathcal{L}(V, W) : T(x^2) = 0 \}$ is a subspace of $\mathcal{L}(V, W)$.

Answer:

There are three things to check:

• First note that the zero linear map is in the set $X$.
• $X$ is closed under addition. To see this, suppose $S(x^2) = 0$ and $T(x^2) = 0$. Then $(S + T)(x^2) = S(x^2) + T(x^2) = 0 + 0 = 0$.
• $X$ is closed under scalar multiplication. To see this, suppose $T(x^2) = 0$ and that $\lambda \in \mathbb{R}$. Then $\lambda T(x^2) = \lambda(0) = 0$.

(b) Explain why the set $Y = \{ T \in \mathcal{L}(V, W) : T \text{ is injective} \}$ is not a subspace of $\mathcal{L}(V, W)$.

Answer:

Since the zero linear transformation is not injective, the set $Y$ does not contain the zero linear transformation and so cannot be a subspace.
Problem 4. Prove that there cannot be a surjective linear map \( S : \mathcal{P}_4(\mathbb{R}) \to \mathbb{R}^3 \) with

\[
\text{null}(S) = \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p(x) \, dx = 0 \right\}.
\]

Answer:
Since \( \dim(\mathcal{P}_4(\mathbb{R})) = 5 \), the fundamental theorem of linear maps says that

\[
\dim(\text{null}(S)) + \dim(\text{range}(S)) = 5.
\]

If \( S \) is surjective, then \( \dim(\text{range}(S)) = \dim(\mathbb{R}^3) = 3 \) implying that \( \dim(\text{null}(S)) = 2 \). But the dimension of the given set \( \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{0}^{1} p(x) \, dx = 0 \right\} \) is greater than two since, for example, \( x, x^2 - \frac{1}{3}, x^3 \) is a list of three independent vectors in \( \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p(x) \, dx = 0 \right\} \).