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1. Which one of the following sets of polynomials is \textit{not} a subspace of $\mathcal{P}(\mathbb{R})$?

(a) \{polynomials of degree 3\}
(b) \{polynomials $p(x)$ satisfying $p(1) = 0$ and $p'(1) = 0$\}
(c) \{even degree polynomials\}
(d) \{polynomials $p(x)$ with $\int_0^1 p(x) dx = 0$\}
(e) \{polynomials of degree $\leq 100$\}

\textbf{Answer.} (a) polynomials of degree 3 are not closed under addition.

2. Which one of the following lists of vectors is independent?

(a) $(5, 4), (7, -1), (-3, 2)$
(b) $(1, 2, 0, 4), (2, 4, 0, 8)$
(c) $(1, 1, 1), (1, 0, 0), (2, 1, 1)$
(d) $(1, 2, 0, 4), (0, 5, 0, 11), (1, 5, 9, 0)$
(e) $(1, 1, 1), (1, 2, 3), (-3, 8, 1), (3, 10, 15)$
(f) $(2, 3, 1), (0, 0, 0)$

\textbf{Answer.} (d) is independent. $(1, 2, 0, 4), (0, 5, 0, 11)$ are independent and since $(1, 5, 9, 0)$ cannot be a linear combination of $(1, 2, 0, 4), (0, 5, 0, 11)$ (since the third entry is nonzero) $(1, 2, 0, 4), (0, 5, 0, 11), (1, 5, 9, 0)$ are independent. (c) is wrong since the third vector in the list is the sum of the first two and the last is wrong since it contains the zero vector. The others can’t be independent since they are lists longer than the dimension of the vector space they live in.

3. True or False: The list of polynomials $1, (x-5)^2, (x-5)^3$ is a basis for the subspace $U$ of $\mathcal{P}_3(\mathbb{R})$ defined by $U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\}$.

\textbf{Answer.} True. Note that $U$ is a proper subspace of a four dimensional space, hence it’s dimension is three or less. Since $1, (x-5)^2, (x-5)^3$ is a list consisting of three independent polynomials in $U$, it is a basis.

4. True or False: A list of vectors $v_1, \ldots, v_n$ is a basis for a vector space $V$ if and only if every vector $v \in V$ can be expressed as a unique linear combination of the vectors $v_1, \ldots, v_n$.

\textbf{Answer.} True. Every vector can be expressed as a linear combination if and only if the list spans and the expression is unique if and only if the list is independent.

5. True or False: The linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y, x + z, 2x + y + z)$ is surjective.

\textbf{Answer.} False. Since $T(1, -1, -1) = (0, 0, 0)$, $T$ is not injective, hence by a dimension count is not surjective.

6. Which one of the following matrices is invertible?
(a) \[
\begin{pmatrix}
11 & 14 & -11 \\
5 & 36 & -5 \\
2 & 10 & -2
\end{pmatrix}
\] (c) \[
\begin{pmatrix}
58 & 14 & 2 \\
0 & 0 & 12 \\
0 & 0 & 16
\end{pmatrix}
\] (e) \[
\begin{pmatrix}
58 & 14 & 2 \\
0 & 1 & 12 \\
0 & 0 & 0
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
58 & 14 & 2 \\
0 & 36 & 12 \\
0 & 0 & 16
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
58 & 14 & 2 \\
0 & 0 & 12 \\
0 & 0 & 16
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
1 & 1 & 2
\end{pmatrix}
\]

(e) \[
\begin{pmatrix}
11 & 5 \\
22 & 10
\end{pmatrix}
\]

(f) \[
\begin{pmatrix}
11 & 5 \\
22 & 10
\end{pmatrix}
\]

**Answer.** (b) is upper triangular with nonzero entries on the diagonal, hence is invertible. The others are either upper triangular with a zero on the diagonal, or have visibly dependent columns, hence are not invertible.

7. The inverse of \[
\begin{pmatrix}
1 & 0 & 2 \\
1 & 2 & 1 \\
3 & 5 & 3
\end{pmatrix}
\]
is \[
\begin{pmatrix}
6 & 5 & -2 \\
0 & 3 & -1 \\
1 & 5 & -2
\end{pmatrix}
\]

(a) \[
\begin{pmatrix}
6 & 5 & -2 \\
0 & 3 & -1 \\
1 & 5 & -2
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
-1 & -10 & 4 \\
0 & 3 & -1 \\
1 & 5 & -2
\end{pmatrix}
\]

(e) \[
\begin{pmatrix}
0 & -10 & 4 \\
1 & 3 & -1 \\
1 & 5 & -2
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
3 & -10 & -1 \\
0 & 3 & -1 \\
-1 & 5 & 2
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
-10 & -1 & 4 \\
0 & 5 & -1 \\
1 & 3 & -2
\end{pmatrix}
\]

(f) \[
\begin{pmatrix}
-1 & -10 & 4 \\
0 & 5 & 1 \\
1 & 3 & -2
\end{pmatrix}
\]

**Answer.** (c) which can be seen by direct computation.

8. If \((x, y, z)\) is a solution to the system of equations:

\[
\begin{align*}
x + 2z &= 2 \\
x + 2y + z &= 3 \\
3x + 5y + 3z &= 4
\end{align*}
\]

then \(x = \)

(a) \(-16\)  (b) \(-9\)  (c) \(-5\)  (d) 2  (e) 6  (f) 7

**Answer.** (a). \((x, y, z)\) is a solution to the given system if and only if

\[
\begin{pmatrix}
1 & 0 & 2 \\
1 & 2 & 1 \\
3 & 5 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
2 \\
3 \\
4
\end{pmatrix}
\]

Since for an invertible matrix \(A\) we have \(Av = b \Rightarrow v = A^{-1}b\) we can solve

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
1 & -10 & 4 \\
0 & 3 & -1 \\
1 & 5 & -2
\end{pmatrix}
\begin{pmatrix}
2 \\
3 \\
4
\end{pmatrix}
= \begin{pmatrix}
-16 \\
-5 \\
9
\end{pmatrix}
\]
9. True or False: If \((x, y, z) = (0, 0, 0)\) is the only solution to the homogeneous system of linear equations

\[
\begin{align*}
a_{11}x + a_{12}y + a_{13}z &= 0 \\
a_{21}x + a_{22}y + a_{23}z &= 0 \\
a_{31}x + a_{32}y + a_{33}z &= 0
\end{align*}
\]

then the inhomogeneous system of linear equations

\[
\begin{align*}
a_{11}x + a_{12}y + a_{13}z &= b_1 \\
a_{21}x + a_{22}y + a_{23}z &= b_2 \\
a_{31}x + a_{32}y + a_{33}z &= b_3
\end{align*}
\]

has a unique solution \((x, y, z) \in \mathbb{R}^3\).

**Answer.** True. If \((x, y, z) = (0, 0, 0)\) is the only solution to the given homogeneous system, then the linear map \(T : \mathbb{R}^3 \to \mathbb{R}^3\) defined by

\[
T(x, y, z) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z)
\]

is injective, hence surjective. And \(T\) being surjective means that for any \((b_1, b_2, b_3) \in \mathbb{R}^3\) there exists an \((x, y, z)\) with \(T(x, y, z) = (b_1, b_2, b_3)\). That is, the given homogeneous solution has a solution, which must be unique since \(T\) is injective.

10. True or False: Suppose \(T : V \to W\) is a linear map and that \(v_1, \ldots, v_n\) is a list of vectors in \(V\). If \(v_1, \ldots, v_n\) is independent in \(V\) then \(Tv_1, \ldots, Tv_n\) is independent in \(W\).

**Answer.** False. For example, the zero map \(\mathbb{R}^3 \to \mathbb{R}^3\) sends the standard basis to the dependent list \((0, 0, 0), (0, 0, 0), (0, 0, 0)\).

11. Consider the linear map \(T : \mathbb{R}^3 \to \mathbb{R}^3\) given by \(T(x, y, z) = (y - z, z - x, x - y)\). Using the standard basis for \(\mathbb{R}^3\), the \((3, 2)\) entry of the matrix \(\mathcal{M}(T)\) is

\[
\begin{align*}
(a) & -2 & (b) & -1 & (c) & 0 & (d) & \frac{1}{2} & (e) & 1 & (f) & 2
\end{align*}
\]

**Answer.** (b) Since

\[
\begin{align*}
T(1, 0, 0) &= (0, -1, 1) \\
T(0, 1, 0) &= (1, 0, -1) \\
T(0, 0, 1) &= (-1, 1, 0)
\end{align*}
\]

we have

\[
\mathcal{M}(T) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}
\]
12. Let \( T : \mathbb{R}^3 \to \mathbb{R} \) be defined by \( T(x, y, z) = x + y + z \). Which one of the following statements is true?

(a) \( T \) is injective

(b) \( \text{dim}(\text{null}(T)) = 1 \)

(c) \( \text{null}(T) = \text{span}((1, 2, -3), (0, 1, -1)) \)

(d) \( \text{null}(T) = \{(x, y, z) \in \mathbb{R}^3 : x = 0 \text{ or } y = 0 \text{ or } z = 0\} \)

(e) \( (1, 1, 1) \in \text{null}(T) \)

**Answer.** (c). By dimension count \( T \) has a two dimensional nullspace. The list \((1, 2, -3), (0, 1, -1)\) consists of two independent vectors in the nullspace, hence must span the nullspace.

13. True or False: If \( A \) is an invertible matrix and \( \lambda \) is an eigenvalue for \( A \) then \( \frac{1}{\lambda} \) is an eigenvalue for \( A^{-1} \)

**Answer.** True. Suppose if \( Av = \lambda v \). If \( A \) is invertible, \( \lambda \neq 0 \) and we have \( v = \frac{1}{\lambda} Av \). Applying \( A^{-1} \) to both sides proves the result.

14. The matrix \( \begin{pmatrix} 8 & 6 \\ -3 & -1 \end{pmatrix} \) has eigenvalues

(a) 1 and 0

(b) 2 and 0

(c) 2 and 5

(d) \(-1 \) and 2

(e) 1 and \(-2\)

(f) 1 and \(-5\)

**Answer.** Note that \( \lambda \) is an eigenvalue for the given matrix if

\[
\begin{pmatrix} 8 & 6 \\ -3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 - \lambda & 6 \\ -3 & -1 - \lambda \end{pmatrix}
\]

is not invertible, which is true if and only if \((8 - \lambda)(-1 - \lambda) + 18 = 0\) which is true if and only if \( \lambda = 2 \) or \( \lambda = 5 \).

15. Let \( v_1, \ldots, v_n \) be a list of vectors in a vector space \( V \) and define a linear map \( T : \mathbb{R}^n \to V \) by

\[ T(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n. \]

Which one of the following statements about \( T \) is false?

(a) \( T \) is injective if and only if \( v_1, \ldots, v_n \) is independent.
(b) $T$ is surjective if and only if $v_1, \ldots, v_n$ spans $V$.

(c) $T$ is an isomorphism if and only if $v_1, \ldots, v_n$ is a basis for $V$.

(d) $T$ is an isomorphism if and only if $\dim(V) = n$.

(e) $T$ is invertible if and only if $v_1, \ldots, v_n$ is a basis for $V$.

**Answer.** (d) is false. $\dim(V) = n$ is not sufficient to guarantee that $T$ is an isomorphism. For example, if $v_1 = (1,0)$ and $v_2 = (2,0)$, then the given map $\mathbb{R}^2 \to \mathbb{R}^2$ is the map $(a_1, a_2) \mapsto a_1 v_1 + a_2 v_2 = (a_1 + a_2, 0)$ is not an isomorphism.

16. Which one of the following statements about $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (2x + 10y - 4z, x - y, 3x - z)$$

is false?

(a) $\lambda = 0$ is an eigenvalue for $T$

(b) $(1, 1, 3)$ is an eigenvector for $T$

(c) $(0, 2, 5)$ is an eigenvector for $T$

(d) $(2, 1, 3)$ is an eigenvector for $T$

(e) $T$ is an isomorphism

(f) $T$ is diagonalizable

**Answer.** (e) is false. Note that $T(1, 1, 3) = (0, 0, 0)$ so $T$ is not injective, hence not an isomorphism.

17. Let $S = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 3 & 5 & 3 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 10 & -4 \\ 1 & -1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$. Then the $(1, 1)$ entry of $S^{-1}TS$ is

(a) $-3$

(b) $-2$

(c) $-1$

(d) $0$

(e) $1$

(f) $2$

**Answer.** (d). The columns of $S$ are eigenvectors for $T$, so the matrix $S^{-1}TS$ is diagonal with the eigenvalues along the diagonal. So the $(1, 1)$ entry is the eigenvalue corresponding to the eigenvector $(1, 1, 3)$ which is $\lambda = 0$.

18. The map $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (-y, x, -3z)$. Which one of the following statements about $T$ is true
(a) There are three distinct one dimensional invariant subspaces for $T$.
(b) $\mathbb{R}^3$ has a basis of eigenvectors for $T$.
(c) $T$ has three distinct eigenvalues.
(d) There exists a basis for $\mathbb{R}^3$ so that with respect to that basis the matrix $M(T)$ is diagonal.
(e) The subspace $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ is a two-dimensional invariant subspace.

**Answer.** (e). $T$ acts as a rotation by 90 degrees in the $xy$-plane. However, there are no proper invariant subspaces of this invariant subspace.

19. [2 points]. Choose one of the problems above and give a complete justification of the answer. Write your explanation clearly and concisely on the back of the answer sheet. Here’s some guidance about what constitutes complete justification:

- If you choose a True/False problem that is True, prove it.
- If you choose a True/False problem that is False, give a counterexample.
- If you choose a multiple choice problem, explain why the correct response is correct and why each of the others is wrong.