1. Suppose that \( T : V \to V \) is a linear map and using a basis \( v_1, v_2 \) (for both the domain and codomain) the matrix for \( T \) is
\[
M(T) = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}.
\]
What is the matrix for \( T^{-1} \)?

**Answer.** Write \( M(T^{-1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). To solve for \( a, b, c, d \) use the fact that \( T^{-1}T = \text{id} \) and \( M(\text{id}) = M(T^{-1}T) = M(T^{-1})M(T) \) so
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} a+b & 3b \\ c+d & 3d \end{pmatrix}.
\]
Solving (by setting \( a + b = 1, 3b = 0 \), etc...) we find that
\[
M(T^{-1}) = \begin{pmatrix} 1 \\ -1/3 \\ 1/3 \end{pmatrix}.
\]

2. Consider \( T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R}) \) given by
\[
h(a + bx + cx^2) = (a - 2b - 2c) + (2a - 6b - 7c)x.
\]
Use the basis \( 1, 1+x, (1+x)^2 \) for the domain and the basis \( 1, x \) for the codomain to express the matrix \( M(T) \).

**Answer.** We compute:
\[
\begin{align*}
h(1) &= 1 + 2x \\
h(1+x) &= -1 - 4x \\
h(1+2x+x^2) &= -5 - 17x
\end{align*}
\]
so
\[
M(T) = \begin{pmatrix} 1 & -1 & -5 \\ 2 & -4 & -17 \end{pmatrix}.
\]

3. Consider the linear map \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) given by
\[
T(x, y, z) = (x - 2y - 2z, 2x - 6y - 7z, -4x + 10y + 10z).
\]
Find a basis \( w_1, w_2, w_3 \) for the codomain so that using the standard basis for the domain, \( M(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).
Answer. Let \( w_1, w_2, w_3 \) be \((1, 2, -4), (-2, -6, 10), (-2, -7, 10)\). Then
\[
\begin{align*}
I(1, 0, 0) &= (1, 2, -4) = 1w_1 + 0w_0 + 0w_3 \\
I(0, 1, 0) &= (-2, -6, 10) = 0w_1 + 1w_0 + 0w_3 \\
I(0, 0, 1) &= (-2, -7, 10) = 0w_1 + 0w_0 + 1w_3
\end{align*}
\]

4. Consider the linear map \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) given by
\[
T(x, y, z) = (x - 2y - 2z, 2x - 6y - 7z, -4x + 10y + 10z).
\]
Find a basis \( v_1, v_2, v_3 \) for the domain so that using the standard basis for the codomain \( M(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

Answer. Let \( v_1, v_2, v_3 \) be \((5, 4, -2), (0, 1, -1), (1, \frac{3}{2}, -1)\). Then
\[
\begin{align*}
I(v_1) &= I(5, 4, -2) = (1, 0, 0) \\
I(v_2) &= I(0, 1, -1) = (0, 1, 0) \\
I(v_3) &= I \left( 1, \frac{3}{2}, -1 \right) = (0, 0, 1)
\end{align*}
\]

Note: Some scratchwork was required to find \( v_1, v_2, \) and \( v_3 \). For example, to determine \( v_2 \), I solved \( T(x, y, z) = (0, 1, 0) \) for \( x, y, z \) and found the unique solution \( x = 0, y = 1, \) and \( z = -1 \) after a few steps.

5. Suppose \( V \) is a finite dimensional vector space. Show that the matrix for the identity map \( I : V \to V \) is the identity matrix
\[
M(I) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}
\]
using any single basis \( v_1, \ldots, v_n \) for both the domain and codomain of \( V \).

Answer. Let \( v_1, \ldots, v_n \) be a basis for \( V \). Then the matrix \( M(I) \) is determined by
\[
\begin{align*}
M(v_1) &= v_1 = 1v_1 + 0v_2 + \cdots 0v_n \\
M(v_2) &= v_2 = 0v_1 + 1v_2 + \cdots 0v_n \\
\vdots
M(v_n) &= v_n = 0v_1 + 0v_2 + \cdots 1v_n
\end{align*}
\]
and we see that \( M(T) \) is the matrix described.
6. Study section 3D in the book and do exercises 14 and 20. Be sure you understand invertible linear maps and know examples of linear maps that are not invertible.