1. Which of the following is not a field? Explain.
   (a) The integers $\mathbb{Z}$
   (b) The rational numbers $\mathbb{Q}$
   (c) The real numbers $\mathbb{R}$
   (d) The complex numbers $\mathbb{C}$

**Answer.** The integers $\mathbb{Z}$ are not a field since not all integers have multiplicative inverses. For example, $2 \in \mathbb{Z}$ and there is no integer $k \in \mathbb{Z}$ so that $2 \times k = 1$.

2. Which of the following is not a field? Explain.
   (a) The numbers $\{0, 1\}$ with $+$ and $\times$ defined “mod 2”.
   (b) The numbers $\{0, 1, 2\}$ with $+$ and $\times$ defined “mod 3”.
   (c) The numbers $\{0, 1, 2, 3\}$ with $+$ and $\times$ defined “mod 4”.
   (d) The numbers $\{0, 1, 2, 3, 4\}$ with $+$ and $\times$ defined “mod 5”.

**Answer.** The numbers $\{0, 1, 2, 3\}$ with $+$ and $\times$ defined “mod 4” is not a field because 2 does not have a multiplicative inverse. To see this, look at the products of 2 with every other number in the set:

\[
2 \times 0 = 0 \quad 2 \times 1 = 2 \quad 2 \times 2 = 0 \quad 2 \times 3 = 2
\]

and observe that there is no number so that when multiplied by 2 results in 1.

3. Let $\alpha \in \mathbb{C}$ be nonzero. Define the number $\frac{1}{\alpha}$ and prove that $\frac{1}{\alpha} = \alpha$.

**Answer.** For any nonzero complex number $\alpha$, the number $\frac{1}{\alpha}$ is by definition the unique complex number so that

\[
(\alpha) \left( \frac{1}{\alpha} \right) = 1.
\]

The number $\frac{1}{\alpha}$ is, by definition, the unique complex number so that when multiplied by $\frac{1}{\alpha}$ the result is 1, and that number is $\alpha$.

4. Express $\frac{1}{4 + 5i}$ in the form $a + bi$ for real numbers $a, b$. 

---

1
**Answer.** Here’s one way to do it: write $\frac{1}{4 + 5i} = a + bi$

\[
\left(\frac{1}{4 + 5i}\right)(4 + 5i) = 1 \Rightarrow (a + bi)(4 + 5i) = 1
\]

\[
\Rightarrow (4a - 5b) + (5a + 4b)i = 1 + 0i
\]

\[
\Rightarrow (4a - 5b) = 1 \text{ and } 5a + 4b = 0
\]

\[
\Rightarrow (4a - 5b) = 1 \text{ and } b = -\frac{5}{4}a
\]

\[
\Rightarrow 4a - 5\left(-\frac{5}{4}a\right) = 1 \text{ and } b = -\frac{5}{4}a
\]

\[
\Rightarrow \frac{41}{4}a = 1 \text{ and } b = -\frac{5}{41}
\]

So, the answer (which a quick computation will verify) is $\frac{1}{4 + 5i} = \frac{4}{41} - \frac{5}{41}i$

**Answer.** Here’s another way that came out of a discussion on the forum:

\[
\left(\frac{1}{4 + 5i}\right) = \left(\frac{1}{4 + 5i}\right)\left(\frac{4 - 5i}{4 - 5i}\right) = \frac{4 - 5i}{16 - (-25)} = \frac{4}{41} - \frac{5}{41}i.
\]

5. True or False:

(a) There is only one number $\alpha \in \mathbb{R}$ so that $\alpha^3 = 2$.

**Answer.** True.

(b) There is only one number $\alpha \in \mathbb{C}$ so that $\alpha^3 = 2$.

**Answer.** False. There are in fact three complex numbers whose cubes are 2. In addition to $\sqrt[3]{2}$, there are:

\[
\sqrt[3]{2}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \text{ and } \sqrt[3]{2}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)
\]

Remember, exercise 2 in section 1A of the book revealed cube roots of 1, which when multiplied by $\sqrt[3]{2}$ yields cube roots of 2.

6. True or False:

(a) There exists a number $\alpha \in \mathbb{R}$ so that $\alpha^2 = -2$.

**Answer.** False. If $x$ is real, then $x^2 \geq 0$, so there cannot be a real number whose square is $-2$.

(b) There exits a number $\alpha \in \mathbb{C}$ so that $\alpha^2 = -2$.

**Answer.** True. There are in fact: $\sqrt{2}i$ and $-\sqrt{2}i$. 

2
7. Does there exist a number \( \alpha \in \mathbb{C} \) so that \( \alpha(1 + i, 2, 2 + 2i, 3 - 2i) = (2, 2 - 2i, 4, 1 - 5i) \)?

**Answer.** Yes, a quick computation shows that \( \alpha = (1 - i) \) works:

\[
(1 - i)(1 + i, 2, 2 + 2i, 3 - 2i) = (2, 2 - 2i, 4, 1 - 5i).
\]

8. Let \( V \) be a vector space over a field \( F \). Prove that

(a) For all \( v \in V \), \( 0v = 0 \).

*Note: the zero on the left is the zero scalar in \( F \) and the zero on the right is the zero vector in \( V \).*

**Answer.** Since \( 0 = 0 + 0 \), we have \( 0v = (0+0)v \). Using the distributive property yields \( 0v = 0v + 0v \). Adding \(-0v\) to both sides gives \( 0v = 0v = 0v + 0v - 0v \). On the left, we have the zero vector and on the right, we use associativity to get \( 0 = 0v + (0v - 0v) \). Using the fact that \( 0v - 0v = 0 \) again, gives \( 0 = 0v + 0 \Rightarrow 0 = 0v \).

(b) For all \( v \in V \), \( (-1)v = -v \).

*Note: the \(-1\) on the left is a scalar in the field \( F \), the \(-v\) on the right is the additive inverse of the vector \( v \in V \).*

**Answer.** We need to show that \( (-1)v + v = 0 \). So,

\[
(-1) + v = (-1)v + 1v = (-1 + 1)v = 0v = 0.
\]

9. Using the correspondence \( a + bi \leftrightarrow (a, b) \) complex numbers can be identified with points in the Cartesian plane. The four points pictured below correspond to \( z, w, z + w, \) and \( zw \) for two complex numbers \( z, w \in \mathbb{C} \). Which are which?

**Answer.** Here’s the answer. It is also correct to swap \( z \) and \( w \).

10. Consider the vector space \( \mathbb{R}^3 \). Which of the following subsets are subspaces?

(a) \( \{(a, b, c, d) \in \mathbb{R}^3 : a + b + c = 0\} \)
Answer. Subspace.

(b) \{ (a, b, c, d) \in \mathbb{R}^3 : abc = 0 \}

Answer. Not a subspace. It’s not closed under addition: the point (1, 1, 0) and (0, 1, 1) are in the set, but the sum (1, 2, 1) is not.

(c) \{ (a, b, c, d) \in \mathbb{R}^3 : a \geq 0 \}

Answer. Not a subspace. It’s not closed under scalar multiplication: the point (1, 0, 0) is in the set but \(-3(1, 0, 0) = (-3, 0, 0)\) is not in the set.

(d) \{ (a, b, c, d) \in \mathbb{R}^3 : a = 2 \}

Answer. Not a subspace. The zero vector isn’t in the set.

(e) \{ (a, b, c, d) \in \mathbb{R}^3 : a = d \}

Answer. Subspace.

(f) \{ (a, b, c, d) \in \mathbb{R}^3 : a + b + 1 = c \}

Answer. Not a subspace. The zero vector isn’t in the set.

(g) \{ (a, b, c, d) \in \mathbb{R}^3 : a + b = 2c \}

Answer. Subspace.

11. Consider the vector space \( \mathbb{R}^\mathbb{R} \). Which of the following subsets are subspaces?

(a) \{ f : \mathbb{R} \rightarrow \mathbb{R} : f(1) = 1 \}

Answer. Not a subspace. The zero vector isn’t in the set.

(b) \{ f : \mathbb{R} \rightarrow \mathbb{R} : f(1) = 0 \}

Answer. Subspace.

(c) \{ f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ is onto} \}

Answer. Not a subspace. The zero vector isn’t in the set.

(d) \{ f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ is continuous} \}

Answer. Subspace.

(e) \{ f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ is differentiable} \}

Answer. Subspace.

(f) \{ f : \mathbb{R} \rightarrow \mathbb{R} : f''(x) = f(x) \}

Answer. Subspace.
12. Let $V = \mathbb{R}^3$. Consider the following three subspaces of $V$

$$W = \{(0,0,a) \in V : a \in \mathbb{R}\}$$
$$X = \{(a,a,a) \in V : a \in \mathbb{R}\}$$
$$Y = \{(a,b,c) \in V : a + b + c = 0\}$$
$$Z = \{(a,a,b) \in V : a,b \in \mathbb{R}\}$$

True or False:

(a) $(1,1,-2) \in W$ False
(b) $(1,1,-2) \in X$ False
(c) $(1,1,-2) \in Y$ True
(d) $(1,1,-2) \in Z$ True
(e) $W$ is a subspace of $X$ False
(f) $W$ is a subspace of $Y$ False
(g) $W$ is a subspace of $Z$ True
(h) $X$ is a subspace of $Z$ True
(i) $W$ is a subspace of $Z$ True
(j) $W \cap X = \{(0,0,0)\}$ True
(k) $X \cap Z = X$ True
(l) $Z = W + X$ True
(m) $Z = W \oplus X$ True
(n) $V = Y + Z$ True
(o) $V = Y \oplus Z$ False

Footnotes

(j) To see that $W \cap X = \{(0,0,0)\}$ note that if $(x,y,z) \in W$, then $x = y = 0$. If $(x,y,z) \in X$ then $z = x = y$. So, if $(x,y,z)$ is a vector in both $X$ and $W$ then $x = y = 0$ and $x = y = z$, which together mean that $(x,y,z) = (0,0,0)$.

(l) The statement $W + X = Z$ means that every vector in $Z$ can be expressed as a sum of a vector in $W$ and a vector in $X$. For example, $(4,4,2) \in Z$ can be written $(4,4,2) = (0,0,-2) + (4,4,4)$. To see that this is always possible, suppose $(a,a,b) \in Z$. Then we have $(a,a,a) \in X$ and $(0,0,b-a) \in W$ and $(a,a,b) = (0,0,b-a) + (a,a,a)$.

(m) To see that $Z = W \oplus X$, it suffices to know that $Z = W + X$ and that $W \cap X = \{0\}$, which are explained above. This is "Blue Box 1.45: Direct Sum of Two Subspaces".

(o) It is true that $V = Y + Z$, every vector $(a,b,c) \in V$ can be written as the sum of a vector in $Y$ and a vector in $Z$, but not uniquely, so the sum isn't a direct sum. For example $(1,2,3) = (-2,-1,3) + (3,3,0)$ and $(1,2,3) = (0,1,-1) + (1,1,4)$.