HOMOTOPY DG ALGEBRAS INDUCE HOMOTOPY BV ALGEBRAS

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Abstract. Let TA denote the space underlying the tensor algebra of a vector space A. In this short note, we show that if A is a differential graded algebra, then TA is a differential Batalin-Vilkovisky algebra. Moreover, if A is an A_∞ algebra, then TA is a commutative BV_∞ algebra.

1. Main Statement

Let (A, d_A) be a complex over a commutative ring R. Our convention is that d_A is of degree $+1$. The space $TA = \bigoplus_{n\geq 0} A^{\otimes n}$ is graded by declaring monomials of homogeneous elements $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ to be of degree $|a_1| + \cdots + |a_n| + n$.

There is a shuffle product $\bullet : TA \otimes TA \to TA$ generated by

$$(a_1 \otimes \cdots \otimes a_n) \bullet (a_{n+1} \otimes \cdots \otimes a_{n+m}) := \sum_{\sigma \in S(n,m)} (-1)^\kappa \cdot a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n+m)},$$

where $S(n, m)$ is the set of all (n, m)-shuffles, *i.e.* $S(n, m)$ is the set of all permutations $\sigma \in \Sigma_{n+m}$ with $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(n + 1) < \cdots < \sigma(n + m)$, (*cf.* [L]). Here $(-1)^\kappa$ is the Koszul sign, which introduces a factor of $(|a_i| + 1)(|a_j| + 1)$ whenever the elements a_i and a_j move past one another in a shuffle. Note that for degree zero elements of A, this Koszul sign is just $sgn(\sigma)$, the sign of the permutation σ. The shuffle product makes TA into a graded commutative associative product on TA.

There is a differential $d : TA \to TA$ (of degree $+1$) given by extending the differential $d_A : A \to A$ as a coderivation of the tensor coproduct, see *e.g.* [S]:

$$d(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{|a_1|+\cdots+|a_{i-1}|+i-1} a_1 \otimes \cdots \otimes d_A(a_i) \otimes \cdots \otimes a_n$$

and together with the shuffle product, the triple $(TA, d, \bullet)$ is a differential graded commutative associative algebra.

If $\mu_A : A \otimes A \to A$ is an associative product, then there is another differential $\Delta = \tilde{\mu}_A : TA \to TA$, of degree -1, given by extending the multiplication as a coderivation,

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{|a_1|+\cdots+|a_i|+i-1} a_1 \otimes \cdots \otimes \mu_A(a_i, a_{i+1}) \otimes \cdots \otimes a_n.$$

In Section 2 we show:

Theorem 1. *If (A, d_A, μ_A) is a differential graded algebra, then $(TA, d, \Delta, \bullet)$ defines a dBV algebra. The construction is functorial: If $f : A \to B$ is a morphism* *of differential associative algebras, then the induced map from TA to TB is a morphism of dBV algebras.*

Recall that a dBV algebra $(X, d, \Delta, \bullet)$ is a differential graded commutative associative algebra $(X, d, \bullet)$, with d of degree $+1$, and differential Δ of degree -1 such that d graded commutes with Δ (so that $d\Delta + \Delta d = 0$), and finally the deviation $\{,\}$ of Δ from being a derivation of $\bullet$,

$$\{x, y\} = (-1)^{|x|}\Delta(x \bullet y) - (-1)^{|x|}\Delta(x) \bullet y - x \bullet \Delta(y)$$

satisfies,

$$\{x, y\} = -(-1)^{(|x|+1)(|y|+1)}\{y, x\} \qquad \text{(Anti-symmetry)},$$
$$\{x \bullet y, z\} = x \bullet \{y, z\} + (-1)^{|y|(|z|+1)}\{x, z\} \bullet y \quad \text{(Leibniz relation)}.$$

The Leibniz relation can be read as saying that bracketing with a fixed element (on the right) is a graded derivation of the product $\bullet$. These relations imply that bracketing with a fixed element on the left is also a graded derivation

$$\{x, y \bullet z\} = \{x, y\} \bullet z + (-1)^{(|x|+1)y} y \bullet \{x, z\}$$

and also imply that bracketing with a fixed element is a graded derivation of the bracket,

$$\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x, z\}\} \qquad \text{(Jacobi identity)}.$$

A morphism of dBV algebras X and Y is a map $f : X \to Y$ that preserves the structures d, Δ, and $\bullet$.

Remark 1. In the special case where μ_A is graded commutative, Δ becomes a derivation of $\bullet$ and, thus, the bracket $\{,\}$ is zero. This is well known in the literature, see for example [L]. We were surprised we could not find in the literature the fact that TA becomes a dBV algebra when μ_A is not necessarily commutative. There is, however, a similar "Lie" version which is well known: the symmetric algebra of a Lie algebra is a BV algebra (see [TT]).

Theorem 1 generalizes naturally. If $(A, \mu_1, \mu_2, \mu_3, \ldots)$ is an A_∞ algebra, then for each $k = 1, 2, \ldots$, the linear map $\mu_k : A^{\otimes k} \to A$ can be extended to a coderivation of degree $3 - 2k$ of the tensor coproduct $\Delta_{3-2k} : TA \to TA$. In Section 3 we show:

Theorem 2. *If $(A, \mu_1, \mu_2, \mu_3, \ldots)$ is an A_∞ algebra, then $(TA, \bullet, \Delta_1, \Delta_{-1}, \Delta_{-3}, \ldots)$ defines a commutative BV_∞ algebra.*

Remark 2. A commutative BV_∞ algebra, as defined by Kravchenko [K], is a generalization of a dBV algebra, and a special case of a BV_∞ algebra, as shown in [GTV]. See also [DCV] for an additional discussion of BV_∞ algebras. The precise definition is given in Section 3, where we show the requisite property that Δ_{3-2k} has operator-order k with respect to the shuffle product.

From a logical point of view, it is probably better to prove Theorem 2 first, from which Theorem 1 follows, see Remark 3 below. However, we prefer to give a direct proof of Theorem 1 using the traditional definition of a dBV algebra, making this an easy to read self-contained section. This also has the advantage of giving an explicit formula for the bracket $\{\ ,\ \}$, and gives the opportunity to illustrate explicitly how the signs are checked in this context.

Acknowledgements. The second author was partially supported by the Max-Planck Institute in Bonn, Germany. We would like to thank Gabriel Drummond-Cole and Bruno Vallette for useful discussions about BV_∞ algebras.

2. Proof of the Theorem 1

The identities $d^2 = 0$, $\Delta^2 = 0$, $\bullet$ being associative and graded commutative, and d being a derivation of $\bullet$ are all straightforward. The (graded) anti-symmetry of the bracket follows formally from the (graded) symmetry of $\bullet$. The functoriality statement is immediate. It remains to show that the bracket $\{,\}$ satisfies the Leibniz relation.

We abbreviate $a_{i_1} \otimes \cdots \otimes a_{i_k}$ by $a_{i_1,\dots,i_k}$, and $\sigma^{-1}(i)$ by σ_i^{-1} for a permutation $\sigma \in \Sigma_k$. First, we may calculate the bracket as

$$\{a_{1,\dots,n}, a_{n+1,\dots,n+m}\} = \sum_{\sigma\in S(n,m)} \pm\Delta(a_{\sigma_1^{-1},\dots,\sigma_{n+m}^{-1}}) - (\pm\Delta(a_{1,\dots,n}) \bullet a_{n+1,\dots,n+m}) - (\pm a_{1,\dots,n} \bullet \Delta(a_{n+1,\dots,n+m}))$$

We claim that every term in the last two expressions cancels with precisely one term in $\sum_{\sigma\in S(n,m)} \pm\Delta(a_{\sigma_1^{-1},\dots,\sigma_{n+m}^{-1}})$ so that $\{a_{1,\dots,n}, a_{n+1,\dots,n+m}\}$ equals

$$\sum_{\sigma\in S(n,m)} \sum_{j\in C_\sigma^{\{1,\dots,n\},\{n+1,\dots,n+m\}}} \pm a_{\sigma_1^{-1},\dots,\sigma_{j-1}^{-1}} \otimes \mu_A(a_{\sigma_j^{-1}}, a_{\sigma_{j+1}^{-1}}) \otimes a_{\sigma_{j+1}^{-1},\dots,\sigma_{n+m}^{-1}},$$

where the set $C_\sigma^{I,J}$ is defined, for a permutation $\sigma \in \Sigma_k$ and disjoint set of indices $I \cup J \subseteq \{1,\dots,k\}$ with $I \cap J = \varnothing$, by

$$C_\sigma^{I,J} = \{j \quad : \quad \sigma_j^{-1} \in I \text{ and } \sigma_{j+1}^{-1} \in J, \text{ or } \sigma_j^{-1} \in J \text{ and } \sigma_{j+1}^{-1} \in I\}.$$

In other words, μ_A is applied in the above sum whenever exactly one of the two elements $a_{\sigma_j^{-1}}$ and $a_{\sigma_{j+1}^{-1}}$ is taken from $a_1,\dots,a_n$, and the other element is taken from $a_{n+1},\dots,a_{n+m}$. Since the correct terms appear exactly once, the only difficulty is to check the cancellation by signs, which we do now.

If we shuffle $a_{n+1,\dots,n+j}$ past a_i, for $1 \le i \le n$ and $1 \le j \le m$, and then apply Δ, we obtain the term

$$a_{1,\dots,i-1} \otimes a_{n+1,\dots,n+j} \otimes \mu_A(a_i, a_{i+1}) \otimes a_{i+2,\dots n} \otimes a_{n+j+1,\dots,n+m}$$

with sign

$$(-1)^{(|a_i|+\cdots+|a_n|+n-i+1)(|a_{n+1}|+\cdots+|a_{n+j}|+j)+(|a_1|+\cdots+|a_{i-1}|+|a_{n+1}|+\cdots+|a_{n+j}|+|a_i|+(i-1+j))}$$

while in the other order, Δ then shuffle, we obtain the same term with sign

$$(-1)^{(|a_1|+\cdots+|a_i|+i+1)+(|\mu(a_i,a_{i+1})|+|a_{i+2}|+\cdots+|a_n|+n-i)(|a_{n+1}|+\cdots+|a_{n+j}|+j)}$$

and these agree. This special case implies the general case, for any shuffle, since a more general shuffle introduces the same additional sign in both cases.

Similarly, shuffling $a_{i+1,\dots,n}$ past a_{n+j+1} for $1 \le i < n$ and $1 \le j < m$, and then applying Δ, we obtain the term

$$a_{1,\dots,i} \otimes a_{n+1,\dots,n+j-1} \otimes \mu_A(a_{n+j}, a_{n+j+1}) \otimes a_{i+1,\dots,n} \otimes a_{n+j+2,\dots,n+m}$$

with sign

$$(-1)^{(|a_{i+1}|+\cdots+|a_n|+n-i)(|a_{n+1}|+\cdots+|a_{n+j+1}|+j+1)+(|a_1|+\cdots+|a_i|+|a_{n+1}|+\cdots+|a_{n+j}|+i+j+1)}$$

while in the other order we obtain the same term with sign

$(-1)^{(|a_{n+1}|+\cdots+|a_{n+j-1}|+j-1)+(|a_{i+1}|+\cdots+|a_n|+n-i)(|a_{n+1}|+\cdots+|a_{n+j-1}|+|\mu(a_{n+j},a_{n+j+1})|+j)}$

These differ by $(-1)^{|a_1|+\cdots+|a_n|+n}$, as expected. Again, this special cases implies the general case, as before.

Now having checked the claim, if we abbreviate the expression $a_{i_1,\ldots,i_{j-1}} \otimes \mu_A(a_{i_j}, a_{i_{j+1}}) \otimes a_{i_{j+1},\ldots,i_k}$ by $a^{(j,j+1)}_{i_1,\ldots,i_k}$, then we can write,

$$\{a_{1,\ldots,n}, a_{n+1,\ldots,n+m}\} = \sum_{\sigma\in S(n,m)} \sum_{j\in C_\sigma^{\{1,\ldots,n\},\{n+1,\ldots,n+m\}}} \pm a^{(j,j+1)}_{\sigma_1^{-1},\ldots,\sigma_{n+m}^{-1}}$$

With this, we can check that $\{a_{1,\ldots,n} \bullet a_{n+1,\ldots,n+m}, a_{n+m+1,\ldots,n+m+p}\}$ equals

$$\begin{aligned}
&= \sum_{\sigma\in S(n,m)} \pm\{a_{\sigma_1^{-1},\ldots,\sigma_{n+m}^{-1}}, a_{n+m+1,\ldots,n+m+p}\} \\
&= \sum_{\rho\in S(n,m,p)} \sum_{j\in C_\rho^{\{1,\ldots,n+m\},\{n+m+1,\ldots,n+m+p\}}} \pm a^{(j,j+1)}_{\rho_1^{-1},\ldots,\rho_{n+m+p}^{-1}} \\
&= \sum_{\rho\in S(n,m,p)} \sum_{j\in C_\rho^{\{1,\ldots,n\},\{n+m+1,\ldots,n+m+p\}}} \pm a^{(j,j+1)}_{\rho_1^{-1},\ldots,\rho_{n+m+p}^{-1}} \\
&\quad + \sum_{\rho\in S(n,m,p)} \sum_{j\in C_\rho^{\{n+1,\ldots,n+m\},\{n+m+1,\ldots,n+m+p\}}} \pm a^{(j,j+1)}_{\rho_1^{-1},\ldots,\rho_{n+m+p}^{-1}} \\
&= a_{1,\ldots,n} \bullet \{a_{n+1,\ldots,n+m}, a_{n+m+1,\ldots,n+m+p}\} \\
&\qquad\qquad \pm\{a_{1,\ldots,n}, a_{n+m+1,\ldots,n+m+p}\} \bullet a_{n+1,\ldots,n+m},
\end{aligned}$$

where $S(n,m,p) \subseteq \Sigma_{n+m+p}$ consists of those permutations $\rho \in \Sigma_{n+m+p}$ that satisfy $\rho(1) < \cdots < \rho(n), \rho(n+1) < \cdots < \rho(n+m)$, and $\rho(n+m+1) < \cdots < \rho(n+m+p)$. By a careful consideration of the signs, as done above, it follows that the Leibniz identity holds, and this completes the proof of Theorem 1.

3. Proof of Theorem 2

Let $(X, \bullet)$ be a graded commutative associative algebra. An operator $\Delta : X \to X$ has operator-order n if and only if

$$\sum (-1)^{n+1-r+\kappa} \Delta(x_{i_1} \bullet \cdots \bullet x_{i_r}) \bullet x_{i_{r+1}} \bullet \cdots \bullet x_{i_{n+1}} = 0$$

where the sum is taken over nonempty subsets $\{i_1, \ldots, i_r : i_1 < \ldots < i_r\} \subseteq \{1, \ldots, n+1\}$ and $\{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_r\}$ has been ordered $i_{r+1} < \cdots < i_{n+1}$, and κ comes from the usual Koszul sign rule.

If Δ has operator-order one, then it is a derivation of $\bullet$. If Δ has operator-order two, then its deviation from being a derivation of $\bullet$, is a derivation of $\bullet$. This means that if we define $\{\,,\,\}$ to be the deviation of Δ from being a derivation of $\bullet$, then $\{\,,\,\}$ and $\bullet$ satisfy the Leibniz relation.

Remark 3. Using this fact, one can prove Theorem 1 without reference to the bracket—here is an outline: any map $A \otimes A \to A$ becomes an order 2 operator $\Delta : TA \to TA$ with respect to the shuffle product when it is lifted as a coderivation of the tensor coproduct (as we will show in the lemma below). If (A, d_A, μ_A) is a differential graded algebra, then μ_A is associative, so that $\Delta^2 = 0$. Since μ_A has degree zero, Δ has degree -1. Since d_A is a derivation of μ_A, then $d : TA \to TA$ and $\Delta : TA \to TA$ commute. That proves that $(TA, d, \Delta, \bullet)$ is a dBV algebra.

To generalize: a Kravchenko commutative BV_∞ algebra consists of a graded commutative differential graded algebra $(X, d, \bullet)$ and a collection $\{\Delta_k : X \to X\}_{k=1,-1,-3,-5,\ldots}$ of operators satisfying

- $\Delta_1 = d$,
- each Δ_{3-2k} has degree $3-2k$ and operator-order k,
- for each n, $\sum_{j+k=n} \Delta_j \Delta_k = 0$.

We use the degree convention in [K] but note that in [GTV] the opposite convention is used (there, d has degree -1 and the higher Δ operators have positive degree). As a special case, a dBV algebra is a Kravchenko commutative BV_∞ algebra with $\Delta_{-3} = \Delta_{-5} = \cdots = 0$.

To prove Theorem 2, assume that $(A, \mu_1, \mu_2, \mu_3 \ldots)$ is an A_∞ algebra. By definition of an A_∞ algebra, each μ_k lifts to a degree $3-2k$ coderivation $\Delta_{3-2k} : TA \to TA$, with $\Delta_1 = d$ and relations $\sum_{j+k=n} \Delta_j \Delta_k = 0$. Thus it only remains to prove, that each Δ_{3-2k} has order k with respect to the shuffle product $\bullet$. This follows from the following general lemma.

Lemma. *Let $f : A^{\otimes n} \to A$ be any linear map and let $F : TA \to TA$ be the lift of f to a coderivation. Then F has order n with respect to the shuffle product.*

Proof. Let $X^1, \ldots, X^{n+1}$ be monomials in TA. So, $X^i = a_1^i \otimes \cdots \otimes a_{s_i}^i$ with each $a_\ell^i \in A$. Then $(-1)^{n+1-r+\kappa} F(X^{i_1} \bullet \cdots \bullet X^{i_r}) \bullet X^{i_{r+1}} \cdots \bullet X^{i_{n+1}}$ consists of a sum of terms of the form

$$\pm \ldots \otimes f(a_{\ell_1}^{i'_1} \otimes \cdots \otimes a_{\ell_n}^{i'_n}) \otimes \ldots \text{ (the rest of the } a_\ell^i\text{'s is outside of } f), \tag{1}$$

where f is applied to $a_{\ell_1}^{i'_1} \otimes \cdots \otimes a_{\ell_n}^{i'_n}$, and the remaining tensor products are applied outside of f. The list $\{i'_1, \ldots, i'_n\}$ may contain repetition, and we may order the list from smallest to largest without repetition as $\{i_1, \ldots, i_k\}$. Every term of the form (1) which contains only the indices $\{i_1, \ldots, i_k\}$ inside f, appears for each index set $J = \{j_1, \ldots, j_q\}$ with $\{i_1, \ldots, i_k\} \subseteq J \subseteq \{1, \ldots, n+1\}$ exactly once in the sum of $(-1)^{n+1-q+\kappa} F(X^{j_1} \bullet \cdots \bullet X^{j_q}) \bullet X^{j_{q+1}} \cdots \bullet X^{j_{n+1}}$. Now, for a fixed expression in Equation (1) induced by different index sets J, the only difference in the sign of (1) is a factor of $(-1)^q$, where $q = |J|$, and all other signs coincide for varying J. We thus need to show that summing $(-1)^{|J|}$ over all J with $\{i_1, \ldots, i_k\} \subseteq J \subseteq \{1, \ldots, n+1\}$ vanishes. Since there are exactly $n+1-k$ choose $q-k$ such subsets J with q elements, we obtain that

$$\sum_J (-1)^{|J|} = \sum_{q=k}^{n+1} \binom{n+1-k}{q-k} \cdot (-1)^q = (-1)^k \cdot \sum_{q'=0}^{n+1-k} \binom{n+1-k}{q'} \cdot (-1)^{q'}$$
$$= (-1)^k \cdot (-1+1)^{n+1-k} = 0,$$

where we used the binomial theorem in the second to last equality. This completes the proof of the lemma. □

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