1. If x, y, z are elements of a group G and $xy = xz$ then $y = z$.

Answer. True.

$$xy = xz \Rightarrow x^{-1}(xy) = x^{-1}(xz) \Rightarrow (x^{-1}x)y = (x^{-1}x)z \Rightarrow ey = ez \Rightarrow y = z.$$

2. If f, g, h are functions from a set X to itself and $fg = fh$ then $g = h$.

Answer. False. For example, consider functions $\mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2$, $g(n) = -n$, and $h(n) = n$. Then $fg = fh$ but $g \neq h$.

3. The function $f : (\mathbb{Z}/7\mathbb{Z})^\times \to (\mathbb{Z}/11\mathbb{Z})^\times$ defined by $f(x) = 5x \mod 11$ is a left-invertible function.

Answer. True. It suffices to compute $f(x)$ for all $x \in \mathbb{Z}/7\mathbb{Z}$ and observe that f is one to one: $f(1) = 5$, $f(2) = 10$, $f(3) = 4$, $f(4) = 9$, $f(5) = 3$, $f(6) = 8$.

4. The function $f : (\mathbb{Z}/7\mathbb{Z})^\times \to (\mathbb{Z}/11\mathbb{Z})^\times$ defined by $f(x) = 5x \mod 11$ is a group homomorphism.

Answer. False. If f were a homomorphism, then $f(1) = 1$, but $f(1) = 5$.

5. $10x \equiv 1 \mod 21$ has a solution $x \in \mathbb{Z}/21\mathbb{Z}$.

Answer. True. It suffices to realize that $\gcd(10, 21) = 1$. Another way to see this is to simply observe that $x = 19$ is a solution.

6. There exists a homomorphism $g : D_8 \to S_4$ with $g(R_{90}) = (1234)$ and $g(H) = (12)$.

Answer. False. In D_8 we have $R_{90}H = HR_{90}^{-1}$ so if g were a homomorphism, we'd need $(1234)(12) = (12)(1234)^{-1}$, but $(1234)(12) = (134)$ and $(12)(1234)^{-1} = (12)(4321) = (143)$.

7. $\mathrm{GL}(2, 3)$ has order 48.

Answer. True. $\mathrm{GL}(2, 3)$ consists of 2×2 invertible matrices with entries in $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. There are eight possibilities for the first row of an invertible 2×2 matrix (anything but $\begin{bmatrix} 0 & 0 \end{bmatrix}$). Then, there are six possibilities for the second row (anything except for 0, 1, or 2 times the first row).

8. True or False: The function $\phi : \mathrm{GL}(2,7) \to (\mathbb{Z}/7\mathbb{Z})^\times$ defined by $\phi(M) = \det(M)$ is a homomorphism.

Answer. True. $\phi(AB) = \det(AB) = \det(A)\det(B) = \phi(A)\phi(B)$.

9. For any group G and any element $x \in Z(G)$ we have $C_G(x) = G$.

Answer. True. If $x \in Z(G)$, the center of G, then $xy = yx$ for all $y \in G$. By definition, the centralizer $C_G(x) = \{y \in G : xy = yx\}$.

10. For any homomorphism $\phi : G \to H$, the set $K = \{g \in G : \phi(g) = e\}$ is a subgroup of G.

Answer. True. We know $\phi(e) = e$, so $e \in K$. Also, if $x, y \in K$, then $xy \in K$ since $\phi(xy) = \phi(x)\phi(y) = (e)(e) = e$. Finally, if $x \in K$, then $x^{-1} \in K$ since $\phi(x^{-1}) = (\phi(x))^{-1} = e^{-1} = e$.

11. If G is a group with the property that $(ab)^2 = a^2b^2$ for any $a, b \in G$, then G is abelian.

Answer. True. To see this, muliply the equation $(ab)^2 = a^2b^2$ on the left by a^{-1} and the right by b^{-1}:

$$(ab)^2 = a^2b^2 \Rightarrow abab = aabb \Rightarrow bab = abb \Rightarrow ba = ab.$$

12. The permutation (653124)(5421) has order two in S_6.

Answer. True. Compute $(653124)(5421) = (13)(56)$ and observe that $((13)(56))^2 = e$

13. $\mathbb{Z}/2\mathbb{Z} \times Z/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has twelve subgroups.

Answer. False, there are sixteen. There is one trivial subgroup $\{(0,0,0)\}$. There are seven subgroups of order two generated by single nonzero elements.

$$\{(0,0,0),(1,0,0)\}$$
$$\{(0,0,0),(0,1,0)\}$$
$$\{(0,0,0),(0,0,1)\}$$
$$\{(0,0,0),(1,1,0)\}$$
$$\{(0,0,0),(1,0,1)\}$$
$$\{(0,0,0),(0,1,1)\}$$
$$\{(0,0,0),(1,1,1)\}$$

Choosing two nonzero elements will generate a subgroup of order 4. There are seven of these.

$$\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}$$
$$\{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}$$
$$\{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\}$$
$$\{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$$
$$\{(0,0,0),(1,0,1),(1,1,1),(0,1,0)\}$$
$$\{(0,0,0),(0,1,1),(1,1,1),(1,0,0)\}$$
$$\{(0,0,0),(1,1,0),(1,1,1),(0,0,1)\}$$

And there's the whole group.

14. The quaternions Q and the dihedral group D_8 are isomorphic.

Answer. False. -1 is the only element of Q of order two, but D_8 has several elements of order two, all of the reflections $H, V, \ldots$

15. $\mathrm{GL}(2,2)$ and S_3 are isomorphic.

Answer. True. Recall the presentation $S_3 = \left\langle f, g | f^3 = g^2 = e, gf = f^2 g \right\rangle$.. The map $f : S_3 \to \mathrm{GL}(2,2)$ defined by

$$(1,2,3) \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defines an isomorphism.

16. $\mathbb{Z}/6\mathbb{Z}$ are and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ are isomorphic.

Answer. True. Both are cyclic groups of order six, mapping a generator to a generator defines an isomorphism. For example, the map $1 \mapsto (1,1)$ defines an isomorphism.

17. $(Z/7\mathbb{Z})^\times$ and $\mathbb{Z}/6\mathbb{Z}$ are isomorphic.

Answer. True. Both are cyclic groups of order six, mapping a generator to a generator defines an isomorphism. For example, the map $3 \mapsto 1$ defines an isomorphism.

Part II: Short Answer. 3 points

18. Choose one of the True / False problems and write a complete justification of your answer.