EXAM

Final Exam

Math 207: Fall 2014

December 23, 2014

ANSWERS

Problem 1. The *Hessian* matrix of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point $p \in \mathbb{R}^2$ is defined to be

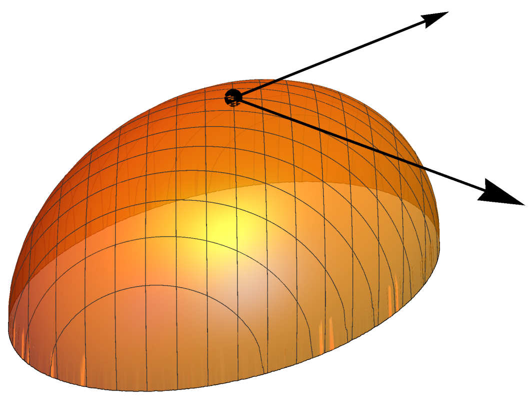
$$H_p = \begin{pmatrix} \frac{\partial^2 f(p)}{\partial x^2} & \frac{\partial^2 f(p)}{\partial y \partial x} \\ \frac{\partial^2 f(p)}{\partial x \partial y} & \frac{\partial^2 f(p)}{\partial y^2} \end{pmatrix}$$

You can think of the Hessian as a kind of second derivative of f since it's the derivative of the function $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $p \mapsto \left(\frac{\partial f(p)}{\partial x}, \frac{\partial f(p)}{\partial y}\right)$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 6\sqrt{36 - 9(x - y + 1)^2 - 4(x + y - 3)^2}.$$

There is one critical point $p \in \mathbb{R}^2$ of f. Find the eigenvectors of the Hessian at that critical point and explain why they are orthogonal.



Problem 1. Continued.

Answer:
We compute the derivative of f:

$$Df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = d\left(-18(x-y+1) - 8(x+y-3), 18(x-y+1) - 8(x+y-3)\right)$$

where $d = 3\left(36 - 9(x-y+1)^2 - 4(x+y-3)^2\right)^{-\frac{1}{2}}$. Solving

$$Df = 0 \Rightarrow \left\{ \begin{array}{c} -18(x-y+1) - 8(x+y-3) = 0 \\ 18(x-y+1) - 8(x+y-3) = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} -13x + 5y = -3 \\ 5x - 13y = -21 \end{array} \right\}$$

One finds the unique solution $(x, y) = (1, 2)$.[1] The Hessian is a little messy, but after evaluating at $p = (x, y) = (1, 2)$ we get

$$H_p = 12 \begin{pmatrix} -13 & 5 \\ 5 & -13 \end{pmatrix}$$

As a symmetric matrix, the eigenvectors will be orthogonal (and this is true in general since the equality of mixed partials imply that the Hessian will be symmetric). The characteristic polynomial is

$$t^2 + 26t + 144 = (t+18)(t+8)$$

giving eigenvalues $8, 18$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Problem 2. Find a 3×3 matrix A with nonzero integer entries so that $A^2 = I$

Answer:
The matrix I certainly satisfies $I^2 = I$ but I has many entries which are zero. To find a matrix with nonzero entries, notice that if $A^2 = I$ then $A^2 - I = 0$ which implies A satisfies the polynomial $(A - I)(A + I)$. There are several simple diagonal matrices that satisfy that equation, for example $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Any matrix similar to D will satisfy $A^2 = I$ also. So, to find the matrix A, we conjugate D by an invertible matrix P. To be sure that the result has integer entries, we choose P with integer entries and determinant 1 so that P^{-1} and hence $P^{-1}DP$, will have integer entries. Here's a good choice, I found by performing row operations on the identity, but only the row operations $row_i \rightsquigarrow row_i + row_j$ which don't change the determinant.

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and its inverse } P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Then, we obtain a solution to the problem

$$A = P^{-1}DP = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ -2 & -2 & -1 \end{pmatrix}.$$

Problem 3. Describe all the different similarity classes of matrices that have the characteristic polynomial $\chi(t) = t^5 - 2t^3 + t$? Which are invertible?

Answer:
Factoring $\chi(t)$ yields $\chi(t) = t(t-1)^2(t+1)^2$. There are several possibilities for the minimum polynomials:

- $m(t) = t(t-1)(t+1)$ corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
- $m(t) = t(t-1)^2(t+1)$ corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
- $m(t) = t(t-1)(t+1)^2$ corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
- $m(t) = t(t-1)^2(t+1)^2$corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

None are invertible since 0 is a root of $\chi(t) \Rightarrow 0$ is an eigenvalue $\Rightarrow$ the matrix has a nontrivial kernel $\Rightarrow$ the matrix is not invertible.

Problem 4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (u, v)$ where

$$\begin{aligned} u &= \sin(x + y) \\ v &= \exp(x - 2y) \end{aligned}$$

Notice that $(0, 0) \overset{f}{\mapsto} (0, 1)$. In a neighborhood of the point $(x, y) = (0, 0)$, the function f is invertible. That is, there exist an open set U containing $(u, v) = (0, 1)$ and a function $g : U \to V$ so that $g(u, v) = (x, y)$.

Use the fact that f and g are inverses to compute the total derivative Dg implicitly at the point $(u, v) = (0, 1)$.

Answer:
Differentiating $f \circ g = \mathrm{id}$ yields $Df \circ Dg = \mathrm{id}$. Therefore, Dg is the inverse of Df. We compute

$$Df = \begin{pmatrix} \cos(x + y) & \cos(x + y) \\ \exp(x - 2y) & -2\exp(x - 2y) \end{pmatrix} \text{ and at } (x, y) = (0, 0) \text{ gives } Df(0, 0) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

Inverting yields

$$Dg(0, 1) = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

Problem 5. Compute a determinant. Your choice:

$$\begin{pmatrix} \cos(1) & \cos(6) & \cos(11) & \cos(16) & \cos(21) \\ \cos(2) & \cos(7) & \cos(12) & \cos(17) & \cos(22) \\ \cos(3) & \cos(8) & \cos(13) & \cos(18) & \cos(23) \\ \cos(4) & \cos(9) & \cos(14) & \cos(19) & \cos(24) \\ \cos(5) & \cos(10) & \cos(15) & \cos(20) & \cos(25) \end{pmatrix}$$

Answer:
Replacing the first row with the first row plus the last row does not change the determinant and yields

$$\begin{pmatrix} \cos(1) + \cos(5) & \cos(6) + \cos(10) & \cos(11) + \cos(15) & \cos(16) + \cos(20) & \cos(21) + \cos(25) \end{pmatrix}$$

Using the fact that $\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, we get

$$\begin{pmatrix} 2\cos(3)\cos(2) & 2\cos(8)\cos(2) & 2\cos(13)\cos(2) & 2\cos(18)\cos(2) & 2\cos(23)\cos(2) \end{pmatrix}$$

Since this row is a multiple of the middle row (it's $2\cos(2)$ times the middle row) the determinant is zero.

Problem 5. Continued.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{9} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \end{pmatrix}$$

Answer:
Replacing row 2 by row 1 - row 2 multiplies the determinant by -1 and yields

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Replacing row 3 by row 2 - row 3 multiplies the determinant by -1 and yields

$$\begin{pmatrix} 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

Continuing in this way yields a matrix with negative the original determinant that is upper triangular with $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)}$ on the diagonals:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{72} & \frac{1}{72} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{90} \end{pmatrix}$$

So the determinant of the original is negative the determinant of the upper tirangular one (nine products by -1) which is the product of the diagonal entries. The answer is

$$\begin{aligned} &-(1)\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(\frac{1}{12}\right)\left(\frac{1}{20}\right)\left(\frac{1}{30}\right)\left(\frac{1}{42}\right)\left(\frac{1}{56}\right)\left(\frac{1}{72}\right)\left(\frac{1}{90}\right) \\ &\qquad = -\frac{1}{(2^2)(3^2)(4^2)(5^2)(6^2)(7^2)(8^2)(9^2)(10)} \end{aligned}$$