EXAM

Final Exam

Math 207: Fall 2014

December 23, 2014

ANSWERS

Problem 1. The *Hessian* matrix of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point $p \in \mathbb{R}^2$ is defined to be

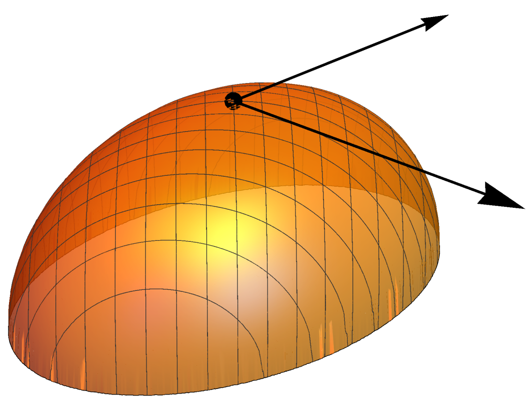
$$H_p = \begin{pmatrix} \frac{\partial^2 f(p)}{\partial x^2} & \frac{\partial^2 f(p)}{\partial y \partial x} \\ \frac{\partial^2 f(p)}{\partial x \partial y} & \frac{\partial^2 f(p)}{\partial y^2} \end{pmatrix}$$

You can think of the Hessian as a kind of second derivative of f since it's the derivative of the function $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $p \mapsto \left(\frac{\partial f(p)}{\partial x}, \frac{\partial f(p)}{\partial y}\right)$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 6\sqrt{36 - 9(x - y + 1)^2 - 4(x + y - 3)^2}.$$

There is one critical point $p \in \mathbb{R}^2$ of f. Find the eigenvectors of the Hessian at that critical point and explain why they are orthogonal.



Problem 1. ***Answer:***
We compute the derivative of f:

$$Df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = d\left(-18(x-y+1) - 8(x+y-3), 18(x-y+1) - 8(x+y-3)\right)$$

where $d = 3\left(36 - 9(x-y+1)^2 - 4(x+y-3)^2\right)^{-\frac{1}{2}}$. Solving

$$Df = 0 \Rightarrow \left\{ \begin{array}{c} -18(x-y+1) - 8(x+y-3) = 0 \\ 18(x-y+1) - 8(x+y-3) = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} -13x + 5y = -3 \\ 5x - 13y = -21 \end{array} \right\}$$

One finds the unique solution $(x, y) = (1, 2)$ (see the footnote for the computation).
The Hessian is a little messy, but if you keep your computation organized, it simplifies after evaluating at $p = (x, y) = (1, 2)$ and you find

$$H_p = \begin{pmatrix} -13 & 5 \\ 5 & -13 \end{pmatrix}$$

As a symmetric matrix, the eigenvectors will be orthogonal (and this is true in general since the equality of mixed partials imply that the Hessian of any smooth function will be symmetric). The characteristic polynomial of H_p is $t^2 + 26t + 144 = (t+18)(t+8)$ so we have eigenvalues $\lambda_1 = 8$ and $\lambda_2 = 18$ with corresponding eigenvectors

$$e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Footnote: Here's one way to find the critical point:

$$\begin{pmatrix} -13 & 5 \\ 5 & -13 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ -21 \end{pmatrix} = \frac{1}{144} \begin{pmatrix} -13 & -5 \\ -5 & -13 \end{pmatrix} \begin{pmatrix} -3 \\ -21 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

It's a coincidence of the particulars of this problem that this matrix that we invert to find the critical point happens to be the Hessian at that critical point—that's not something you would expect.

Problem 2. Find a 3×3 matrix A with nonzero integer entries so that $A^2 = I$

Answer:
The matrix I certainly satisfies $I^2 = I$ but I has many entries which are zero. To find a matrix with nonzero entries, notice that if $A^2 = I$ then $A^2 - I = 0$ which implies A satisfies the polynomial $(A - I)(A + I)$. There are several simple diagonal matrices that satisfy that equation, for example

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Any matrix similar to D will satisfy $A^2 = I$ also. So, to find the matrix A, we conjugate D by an invertible matrix P. To be sure that the result has integer entries, we choose P with integer entries and determinant 1 so that P^{-1} and hence $P^{-1}DP$, will have integer entries. Here's a good choice, I found by performing row operations on the identity, but only the row operations $row_i \rightsquigarrow row_i + row_j$ which don't change the determinant.

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and its inverse } P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Then, we obtain a solution to the problem

$$A = P^{-1}DP = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ -2 & -2 & -1 \end{pmatrix}.$$

Problem 3. Describe all the different similarity classes of matrices that have the characteristic polynomial $\chi(t) = t^5 - 2t^3 + t$? Which are invertible?

Answer:
Factoring $\chi(t)$ yields $\chi(t) = t(t-1)^2(t+1)^2$. There are several possibilities for the minimum polynomials:

- $m(t) = t(t-1)(t+1)$ corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
- $m(t) = t(t-1)^2(t+1)$ corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
- $m(t) = t(t-1)(t+1)^2$ corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
- $m(t) = t(t-1)^2(t+1)^2$corresponding to matrices similar to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

None are invertible—lots of ways to see that. For one, they all have determinant 0. For another, 0 is a root of $\chi(t) \Rightarrow 0$ is an eigenvalue $\Rightarrow$ the matrix has a nontrivial kernel $\Rightarrow$ the matrix is not invertible.

Problem 4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (u, v)$ where

$$\begin{aligned} u &= \sin(x + y) \\ v &= \exp(x - 2y) \end{aligned}$$

Notice that $(0, 0) \overset{f}{\mapsto} (0, 1)$. In a neighborhood of the point $(x, y) = (0, 0)$, the function f is invertible. That is, there exist an open set U containing $(u, v) = (0, 1)$ and a function $g : U \to V$ so that $g(u, v) = (x, y)$.

Use the fact that f and g are inverses to compute the total derivative Dg implicitly at the point $(u, v) = (0, 1)$.

Answer:
Differentiating $f \circ g = \mathrm{id}$ yields $Df \circ Dg = \mathrm{id}$. Therefore, Dg is the inverse of Df. We compute

$$Df = \begin{pmatrix} \cos(x+y) & \cos(x+y) \\ \exp(x-2y) & -2\exp(x-2y) \end{pmatrix} \text{ and at } (x,y) = (0,0) \text{ gives } Df(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

Inverting yields

$$Dg(0,1) = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

Problem 5. Compute a determinant. Your choice:

$$\begin{pmatrix} \cos(1) & \cos(6) & \cos(11) & \cos(16) & \cos(21) \\ \cos(2) & \cos(7) & \cos(12) & \cos(17) & \cos(22) \\ \cos(3) & \cos(8) & \cos(13) & \cos(18) & \cos(23) \\ \cos(4) & \cos(9) & \cos(14) & \cos(19) & \cos(24) \\ \cos(5) & \cos(10) & \cos(15) & \cos(20) & \cos(25) \end{pmatrix}$$

Answer:
This was inspried by a 2009 Putnam exam question. Replacing the first row with the first row plus the last row does not change the determinant and yields

$$\begin{pmatrix} \cos(1)+\cos(5) & \cos(6)+\cos(10) & \cos(11)+\cos(15) & \cos(16)+\cos(20) & \cos(21)+\cos(25) \end{pmatrix}$$

Using the fact that $\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, we get

$$\begin{pmatrix} 2\cos(3)\cos(2) & 2\cos(8)\cos(2) & 2\cos(13)\cos(2) & 2\cos(18)\cos(2) & 2\cos(23)\cos(2) \end{pmatrix}$$

Since this row is a multiple of the middle row (it's $2\cos(2)$ times the middle row) the determinant is zero.

Problem 5. Continued.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{9} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10}
\end{pmatrix}$$

Answer:
This was inspired by a 2014 putnam exam question. Replacing row 2 by row 1 - row 2 multiplies the determinant by -1 and yields

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Replacing row 3 by row 2 - row 3 multiplies the determinant by -1 and yields

$$\begin{pmatrix} 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

Continuing in this way yields a matrix with negative the original determinant that is upper triangular with $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)}$ on the diagonals:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\vdots & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{72} & \frac{1}{72} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{90}
\end{pmatrix}$$

So the determinant of the original is negative the determinant of the upper tirangular one (nine products by -1) which is the product of the diagonal entries. The answer is

$$-(1)\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(\frac{1}{12}\right)\left(\frac{1}{20}\right)\left(\frac{1}{30}\right)\left(\frac{1}{42}\right)\left(\frac{1}{56}\right)\left(\frac{1}{72}\right)\left(\frac{1}{90}\right) = -\frac{1}{(2^2)(3^2)(4^2)(5^2)(6^2)(7^2)(8^2)(9^2)(10)}$$