EXAM

Linear Algebra Exam

Math 207: Fall 2014

November 25, 2014

ANSWERS

Problem 1. Find a 3×3 matrix A with nonzero integer entries so that $A^3 = A$.

Answer:
If $A^3 = A$ then $A^3 - A = 0$ which implies A satisfies the polynomial $A(A-I)(A+I)$. There are several simple diagonal matrices that satisfy that equation, for example $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.
Any matrix similar to D will satisfy $A^3 = A$ also. So, to find the matrix A, we conjugate D by an invertible matrix P. To be sure that the result has integer entries, we choose P with integer entries and determinant 1 so that P^{-1} will have integer entries also. Here's a good choice, I found by performing row operations on the identity, but only the row operations $row_i \rightsquigarrow row_i + row_j$ which don't change the determinant.

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and its inverse } P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

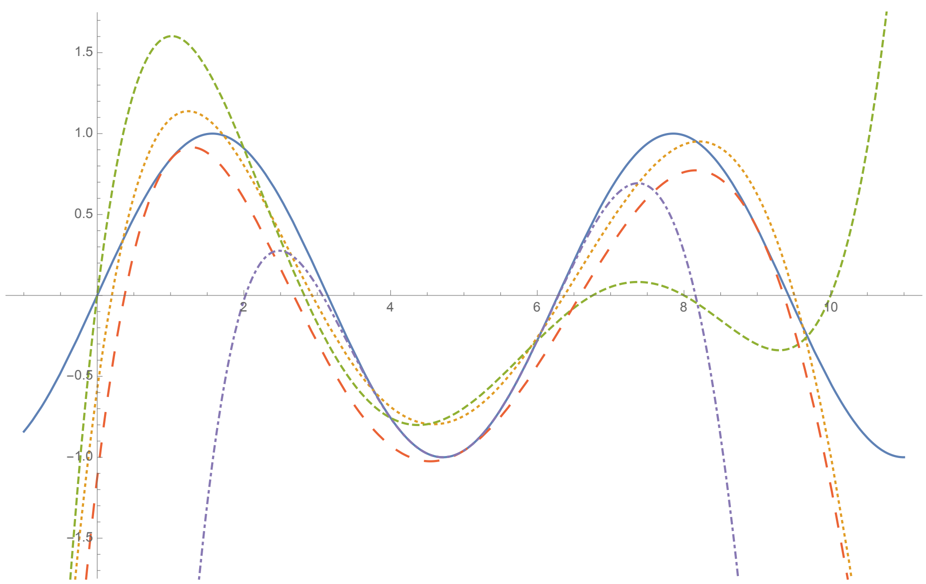
Then, we find

$$A = P^{-1}DP = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}.$$

Problem 2. Let V be the vector space of five-times differentiable real-valued functions and let W be the subspace of polynomials of degree less than or equal to five. The following formulae define symmetric bilinear functions on V, which restrict to inner products on W.

$$\langle f, g\rangle_1 = \int_0^{10} f(t)g(t)dt$$
$$\langle f, g\rangle_2 = f(0)g(0) + f(2)g(2) + f(4)g(4) + f(6)g(6) + f(8)g(8) + f(10)g(10)$$
$$\langle f, g\rangle_3 = f(1)g(1) + f(5)g(5) + f(9)g(9) + f'(1)g'(1) + f'(5)g'(5) + f'(9)g'(9)$$
$$\langle f, g\rangle_4 = f(5)g(5) + f''(5)g''(5) + f''(5)g''(5) + f'''(5)g'''(5) + f''''(5)g''''(5) + f'''''(5)g'''''(5)$$

The following picture shows the function $f(t) = \sin(t)$ together with several degree five polynomials obtained by projecting f onto W using the given inner products. Which curves go with which inner products? Justify your answer, but don't do any computations!



Answer:
The solid one is the graph of f, the dotted one corresponds to $\langle\ ,\ \rangle_1$, the dashed one corresponds to $\langle\ ,\ \rangle_2$, the long-dashed corresponds to $\langle\ ,\ \rangle_3$, and dot-dashed corresponds to $\langle\ ,\ \rangle_4$.
To determine which are which, note that each is the element of W that minimizes the distance to $f \in V \setminus W$ where distance is determined by the norm corresponding the inner product. The dotted one minimizes $\int_0^{10}(f-p)^2$, the dashed one agrees with p at $0, 2, 4, 6, 8, 10$ hence minimizes the $\langle\ ,\ \rangle_2$ distance. The long-dashed one agrees in value and derivative at $1, 5, 9$ hence minimizes the $\langle\ ,\ \rangle_3$ distance. It's not completely obvious that the dot-dashed one agrees in value and the first five derivatives at 5, but it does give the best approximation of f in a neighborhood of 5, but it minimizes the $\langle\ ,\ \rangle_4$ distance—it's fifth order Taylor polynomial for f centered at 5.

Problem 3. Let

$$M = \begin{pmatrix} 4 & -5 & -2 & 9 \\ -5 & 7 & 5 & -10 \\ 3 & -5 & -3 & 7 \\ -4 & 5 & 3 & -8 \end{pmatrix}.$$

Prove that M is not diagonalizable over the real numbers but M is diagonalizable over the complex numbers.
Hint: Compute M^2.

Answer:
A quick check shows that $M^2 = -I \Leftrightarrow M^2 + I = 0$. Therefore, the minimum polynomial $m(t)$ of M divides $t^2 + 1$, and hence must equal $t^2 + 1$. Since $m(t)$ doesn't factor into linear factors over $\mathbb{R}$, M is not diagonalizable over $\mathbb{R}$.
Over the complex numbers, the minimum polynomial of M factors $m(t) = t^2+1 = (t-i)(t+i)$, and we find that M is diagonalizable over $\mathbb{C}$. In fact, M is similar to a diagonal matrix with either i or $-i$ on the diagonal.

Problem 4. Compute e^B where $B = \begin{pmatrix} -2\pi & -\pi \\ 5\pi & 2\pi \end{pmatrix}$.

Answer:
It's easier to work with a simpler matrix similar to B. We find $\chi_B(t) = t^2+\pi^2 = (t-i\pi)(t+i\pi)$ so we have eigenvalues $\pm i\pi$. So, if we arrange corresponding eigenvectors as columns in a matrix P, we have $P^{-1}BP = D$ where

$$D = \begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix}$$

Then

$$\begin{aligned} e^B &= e^{PDP^{-1}} \\ &= Pe^D P^{-1} \\ &= P \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix} P^{-1} \\ &= P \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} P^{-1} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The second to last equality follows from the fact that $e^{ix} = \cos(x) + i\sin(x) \Rightarrow e^{i\pi} = -1$ and $e^{-i\pi} = -1$.

Problem 5. Let V be the vector space of polynomials of degree less than or equal to two and define an inner product on V by

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

A convenient orthonormal basis for V is

$$B = \left\{ \frac{1}{2}(t-1)t, 1-t^2, \frac{1}{2}t(t+1) \right\}.$$

Consider $T : V \to V$ defined by $T(p) = p'(t)$. Let T^* denote the adjoint of T.

Find $T^*(t)$.

Answer:
First, note that $\langle p, e_1\rangle = p(-1)$, $\langle p, e_2\rangle = p(0)$, and $\langle p, e_3\rangle = p(1)$ for the given basis $B = \{e_1, e_2, e_3\}$. To determine the matrix for the operator T, we differentiate each e_i and express the result in the basis B easily by evaluating $e_i'(-1)$, $e_i'(0)$, and $e_i'(1)$. The result is the following matrix:

$$\text{Rep}_{B,B}(T) = \begin{pmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{pmatrix}$$

Then, the matrix for T^* is simply the transpose:

$$\text{Rep}_{B,B}(T^*) = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

In the basis B, the polynomial t is the vector $\text{Rep}_B(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. So, $T^*(t)$ is represented by the vector

$$\begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} == \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}.$$

So, $T^*(t) = 2e_1 - 4e_2 + 2e_3 = 6t^2 - 4$.