Math 207 Fall 2014 — Practice Exam

Problem 1. Let V be the vector space of all $n \times n$ complex matrices. Prove that $\langle A, B\rangle = \operatorname{trace}(AB^*)$ defines an inner product on V.

Problem 2. Let A be any $n \times n$ complex matrix. Prove that A and A^t are similar.

Problem 3. Prove that if $A^2 = A$ then A is diagonalizable.

Problem 4. Let V be the space of real polynomials of degree ≤ 3 and let $\sigma : V \to V$ be the differentiation operator

$$\sigma(p(t)) = \frac{dp(t)}{dt}.$$

(a) What is the adjoint of σ if we equip V with the inner product defined by

$$\langle p, q\rangle := \int_0^1 p(t)q(t)dt?$$

(b) What is the adjoint of σ if we equip V with the inner product defined by

$$\langle p, q\rangle := p(1)q(1) + p'(1)q'(1) + p''(1)q''(1) + p'''(1)q'''(1)?$$

Problem 5. Let $M = \begin{pmatrix} 15 & -42 & 12 \\ -3 & 10 & -6 \\ -10 & 30 & -11 \end{pmatrix}$. Find a matrix N so that $N^2 = M$.

Problem 6. Suppose that A is a 2×2 real matrix and let $v \in \mathbb{R}^2$ be any vector. Since $\mathbb{R}^2$ is two dimensional, the three vectors $\{v, Av, A^2v\}$ must be linearly dependent. Prove that, in fact, $A^2v = \alpha Av + \beta v$ where $\alpha = \operatorname{trace}(A)$ and $\beta = -\det(A)$.

Problem 7. Suppose V is an n dimensional vector space. An operator $T : V \to V$ is called nilpotent if $T^k = 0$ for some integer k.

(a) Prove that if T is nilpotent, then $T^n = 0$.

(b) Prove that if T is nilpotent, then there is a basis of vectors of V so that the matrix for T is upper triangular with zeroes on the diagonal.

Problem 8. Let V be the space of polynomials of degree less than or equal to 2. Let $T : V \to V$ be the homomorphism defined by

$$p(x) \stackrel{T}{\mapsto} p(1)x^2 + p'(0)x + p''(0)$$

(a) Compute the determinant and trace of T.

(b) Use the ordered basis $B = \{x^2, x, 1\}$ to express T as a 3×3 matrix $A = \operatorname{Rep}_{B,B}(T)$.

(c) Find a basis E consisting of eigenvectors for T.

(d) Let Q be the change of basis matrix $Q = \operatorname{Rep}_{B,E}(\operatorname{Id})$.

(e) Use your the ordered basis E to express T as a 3×3 matrix $B = \operatorname{Rep}_{E,E}(\operatorname{Id})$.

Problem 9. Let

$$C = \begin{pmatrix} 3 & 3 & 2 & 0 & 1 \\ 3 & -5 & -2 & -4 & 1 \\ 6 & 6 & 4 & 0 & 2 \\ 3 & 5 & 3 & 1 & 1 \end{pmatrix}$$

(a) Find a basis for the kernel of C.

(b) Find a basis for the image of C.

(c) Find a basis for the space spanned by the columns of C.

(d) Find a basis for the space spanned by the columns of C^t.

Problem 10. Let

$$A = \begin{pmatrix} -1 & 8 & 2 \\ -1 & -7 & -1 \\ 2 & 8 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 1 & 3 \\ 7 & 3 & 4 \\ -10 & -2 & -4 \end{pmatrix}$$

(a) Find the characteristic and minimum polynomials and the Jordon canonical forms for A and B.

(b) Find matrices P and Q so that $P^{-1}AP$ and $Q^{-1}BQ$ are in Jordon form.

(c) Compute e^A and e^B.

Problem 11. Solve the two second order differential equations:

$$y_2'' = y_2 - 2y_1'$$
$$y_1'' = y_1 + 2y_2'$$

subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = 1.$$

By translating them into a single system of the form $Y'(t) = AY(t)$ for a 4×4 matrix A.