Problem 1. Let V be the vector space of all $n \times n$ complex matrices. Prove that $\langle A, B\rangle = \text{trace}(AB^*)$ defines an inner product on V.

Answer. We check that the axioms of an inner product are satisfied.

- *Linearity in the first entry.* If A, A', and B are $n \times n$ matrices, then

  $$\begin{aligned}\langle A + A', B\rangle &= \text{trace}((A + A')B^*) = \text{trace}(AB^* + A'B^*) \\ &= \text{trace}(AB^*) + \text{trace}(A'B^*) = \langle A, B\rangle + \langle A', B\rangle.\end{aligned}$$

  If $\alpha \in \mathbb{C}$ then

  $$\langle \alpha A, B\rangle = \text{trace}(\alpha AB^*) = \text{trace}(\alpha AB^*) = \alpha\,\text{trace}(AB^*) = \alpha\langle A, B\rangle.$$

- *Hermetian symmetry.*

  $$\langle A, B\rangle = \text{trace}(AB^*) = \overline{\text{trace}\left((AB^*)^*\right)} = \overline{\text{trace}\left((BA^*)\right)} = \overline{\langle B, A\rangle}.$$

  The first equality follows from the fact that $\text{trace}(M^*) = \overline{\text{trace}(M)}$ since $\text{trace}(M^t) = \text{trace}(M)$ and $\text{trace}(\overline{M}) = \overline{\text{trace}(M)}$.

- *Nondegeneracy.* We have to prove that if $\langle A, B\rangle = 0$ for all matrices B, then $A = 0$.

  Let B_{ij} be the matrix whose i-j-th entry is 1 and whose other entries are 0. Then $\text{trace}(AB^*_{ij})$ is the i-j-the entry of A. So if $\text{trace}(AB^*) = 0$, for all B, then $\text{trace}(AB^*_{ij}) = 0$ for all B_{ij}, hence all the entries of A must be zero.

  To see that $\text{trace}(AB^*_{ij})$ is the i-j-the entry of A, note that the matrix B^*_{ij} consists of columns of zeroes except for the j-th column, which is the i-th standard basis vector. Then the i-th column of AB^*_{ij} is the j-th column of A, and all other entries of AB_{ij} are zero. So the trace of AB^*_{ij} is a sum of terms that are almost all zero, the only possible nonzero term being in the i-i entry, in the sum i-j-entry of A.

Problem 2. Let A be any $n \times n$ complex matrix. Prove that A and A^t are similar.

Answer. There are two steps. First, if J is the Jordan form of A then A^t is similar to J^t. Second, J^t is similar to J. So, by transitivity of similarity, we have $A \sim J \sim J^t \sim A^t$.

First step. Suppose $J = P^{-1}AP$. Then by taking transpose we have

$$J^t = (P^{-1}AP)^t = P^tA^t(P^{-1})^t = P^tA^t(P^t)^{-1}.$$

So $A^t \sim J^t$.

Second step. Note that if B is a $k \times k$ Jordon block, then for

$$Q_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

we have $Q_k^{-1}BQ_k = B^t$. Then by assembling a block-diagonal matrix Q with matrices Q_k of the correct sizes as the blocks, we will have $Q^{-1}JQ = J^t$.

Problem 3. Prove that if $A^2 = A$ then A is diagonalizable.

Answer. If $A^2 = A$ then $0 = A^2 - A = A(A - I)$. Therefore, the minimum polynomial for A is either $m_A(t) = t$, $m_A(t) = (t-1)$, or $m_A(t) = t(t-1)$. In every case, $m_A(t)$ factors into distinct linear factors, hence A is diagonalizable.

Remark. In the first case, A is similar to the zero matrix, hence is the zero matrix. In the second case, A must be similar to the identity matrix, hence is the identity. In the third case, A must be similar to a diagonal matrix with 0's and 1's on the diagonal, but this matrix might not appear to be so simple. For example,

$$A = \begin{pmatrix} -2 & \frac{1}{2} & \frac{3}{2} & -2 \\ 30 & -4 & -15 & 20 \\ 6 & -1 & -2 & 4 \\ 15 & -\frac{5}{2} & -\frac{15}{2} & 11 \end{pmatrix}$$

Problem 4. Let V be the space of real polynomials of degree ≤ 3 and let $\sigma : V \to V$ be the differentiation operator

$$\sigma(p(t)) = \frac{dp(t)}{dt}.$$

(a) What is the adjoint of σ if we equip V with the inner product defined by

$$\langle p, q \rangle := \int_0^1 p(t)q(t)dt?$$

Answer. It's convenient to work with an orthonormal basis. Performing Gram-Schmidt on the basis $\{1, t, t^2, t^3\}$ yields the orthonormal basis

$$B = \left\{1, 2\sqrt{3}t - \sqrt{3}, 6\sqrt{5}t^2 - 6\sqrt{5}t + \sqrt{5}, 20\sqrt{7}t^3 - 30\sqrt{7}t^2 + 12\sqrt{7}t - \sqrt{7}\right\}$$

The matrix for σ in this basis is

$$\text{Rep}_{B,B}(\sigma) = \begin{pmatrix} 0 & 2\sqrt{3} & 0 & 2\sqrt{7} \\ 0 & 0 & 2\sqrt{15} & 0 \\ 0 & 0 & 0 & 2\sqrt{35} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, the matrix for the adjoint σ^* in the basis B is given by

$$\text{Rep}_{B,B}(\sigma^*) = (\text{Rep}_{B,B}(\sigma))^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 2\sqrt{7} & 0 & 2\sqrt{35} & 0 \end{pmatrix}.$$

Remark. You might find it interesting to express σ^* as a matrix in the basis $E = \{1, t, t^2, t^3\}$. To do so, use $\text{Rep}_{B,E}$, which is the matrix that changes basis from B to E:

$$\text{Rep}_{B,E} = \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & 12\sqrt{7} \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \\ 0 & 0 & 0 & 20\sqrt{7} \end{pmatrix}.$$

Then

$$\begin{aligned}
\mathrm{Rep}_{E,E}(\sigma^*) &= \mathrm{Rep}_{B,E}\,\mathrm{Rep}_{B,B}\,\mathrm{Rep}_{E,B} \\
&= \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & 12\sqrt{7} \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \\ 0 & 0 & 0 & 20\sqrt{7} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 2\sqrt{7} & 0 & 2\sqrt{35} & 0 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & 12\sqrt{7} \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \\ 0 & 0 & 0 & 20\sqrt{7} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} -20 & -5 & -4 & -4 \\ 180 & 60 & 58 & 60 \\ -420 & -180 & -180 & -183 \\ 280 & 140 & 140 & 140 \end{pmatrix}
\end{aligned}$$

This implies, for example, that $\sigma^*(t^2) = -4 + 58t - 180t^2 + 140t^3$. This means that for any polynomial $p(t) \in V$, we have

$$\langle \sigma(p(t)), t^2 \rangle = \langle p(t), \sigma^*(t^2) \rangle.$$

That is,

$$\int_0^1 p'(t)t^2 dt = \int_0^1 p(t)\left(-4 + 58t - 180t^2 + 140t^3\right) dt \text{ for } \textit{every} \text{ polynomial } p(t) \in V.$$

For example, for $p(t) = \pi t^3 - \sqrt{3}t^2 - \frac{t}{7} + e$, this means that

$$\int_0^1 \left(3\pi t^2 - 2\sqrt{3}t - \frac{1}{7}\right) t^2 dt = \int_0^1 \left(\pi t^3 - \sqrt{3}t^2 - \frac{t}{7} + e\right)\left(-4 + 58t - 180t^2 + 140t^3\right) dt.$$

Both sides are the number $-\frac{1}{21} - \frac{\sqrt{3}}{2} + \frac{3\pi}{5}$.

(b) What is the adjoint of σ if we equip V with the inner product defined by

$$\langle p, q \rangle := p(1)q(1) + p'(1)q'(1) + p''(1)q''(1) + p'''(1)q'''(1)?$$

Answer. Again, it's convenient to use a basis for V that is orthonormal with respect to the given inner product:

$$C = \left\{1, (t-1), \frac{1}{2}(t-1)^2, \frac{1}{3!}(t-1)^3\right\}$$

Here, we have

$$\mathrm{Rep}_{C,C}(\sigma) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathrm{Rep}_{C,C}(\sigma^*) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Remark. Again, it might be interesting to express σ^* as a matrix using the basis $E = \{1, t, t^2, t^3\}$. The result is:

$$\mathrm{Rep}_{E,E}(\sigma^*) = \begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{2} \\ 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & \frac{1}{3} & 1 \end{pmatrix}$$

Problem 5. Let $M = \begin{pmatrix} 15 & -42 & 12 \\ -3 & 10 & -6 \\ -10 & 30 & -11 \end{pmatrix}$. Find a matrix N so that $N^2 = M$.

Answer. Note that for $Q = \begin{pmatrix} -6 & 3 & -2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ we have $Q^{-1}MQ = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$. So the matrix

$$N = Q \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} Q^{-1} = \begin{pmatrix} 3 & -6 & 0 \\ -1 & 4 & -2 \\ -2 & 6 & -1 \end{pmatrix}$$

satisfies $N^2 = M$.

Problem 6. Suppose that A is a 2×2 real matrix and let $v \in \mathbb{R}^2$ be any vector. Since $\mathbb{R}^2$ is two dimensional, the three vectors $\{v, Av, A^2v\}$ must be linearly dependent. Prove that, in fact, $A^2v = \alpha Av + \beta v$ where $\alpha = \mathrm{trace}(A)$ and $\beta = -\det(A)$.

Problem 7. Suppose V is an n dimensional vector space. An operator $T : V \to V$ is called nilpotent if $T^k = 0$ for some integer k.

(a) Prove that if T is nilpotent, then $T^n = 0$.

Answer. If $T^k = 0$, then the minimum polynomial for T divides T^k, so the minimum polynomial $m_T(t)|t^k$. This means that $m_T(t) = t^r$ for some $r \leq k$. Since every linear factor of the characteristice polynomial $\chi_T(t)$ is a factor of $m_T(t)$, and $\chi_T(t)$ is degree n, we conclude $\chi_T(t) = t^n$. The Cayley Hamilton theorem implies $\chi_T(T) = 0 \Leftrightarrow T^n = 0$.

(b) Prove that if T is nilpotent, then there is a basis of vectors of V so that the matrix for T is upper triangular with zeroes on the diagonal.

Answer. The remarks above about $m_T(t) = t^r$ implies that the Jordan form for T consists of Jordon blocks of sizes $\leq r$ with zero's on the diagonal. As we defined it, the Jordon blocks have 1's on the subdiagonal, but by re-ordering the basis vectors, we will obtain the transpose matrix, an upper triangular matrix with zeroes on the diagonal (and zeros and ones on the superdiagonal).

Problem 8. Let V be the space of polynomials of degree less than or equal to 2. Let $T : V \to V$ be the homomorphism defined by

$$p(x) \stackrel{T}{\mapsto} p(1)x^2 + p'(0)x + p''(0)$$

(a) Compute the determinant and trace of T.

Answer. First, do part (b). Then, it's straightforward to determine that $\mathrm{trace}(T) = 2$ and $\det(T) = -2$.

(b) Use the ordered basis $B = \{x^2, x, 1\}$ to express T as a 3×3 matrix $A = \mathrm{Rep}_{B,B}(T)$.

Answer.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

(c) Find a basis E consisting of eigenvectors for T.

Answer.

$$\left\{t^2 - t + 1, -2, t^2 + 2t + 2\right\}$$

(d) Let Q be the change of basis matrix $Q = \mathrm{Rep}_{B,E}(\mathrm{Id})$.

Answer.

$$Q = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

(e) Use your the ordered basis E to express T as a 3×3 matrix $B = \mathrm{Rep}_{E,E}(\mathrm{Id})$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 9. Let

$$C = \begin{pmatrix} 3 & 3 & 2 & 0 & 1 \\ 3 & -5 & -2 & -4 & 1 \\ 6 & 6 & 4 & 0 & 2 \\ 3 & 5 & 3 & 1 & 1 \end{pmatrix}$$

(a) Find a basis for the kernel of C.

(b) Find a basis for the image of C.

(c) Find a basis for the space spanned by the columns of C.

(d) Find a basis for the space spanned by the columns of C^t.

Answer. It's helpful to realize that the kernel of a matrix M is the same as the kernel of a matrix M' obtained from M by row reduction. Also, the span of the rows of a matrix M are the same as the span of the rows of a matrix M' obtained from M by row reduction.

Here are two matrices obtained from C and C^t by row reduction, which will be helpful for answering all the questions.

$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 2 & \frac{5}{4} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) From the reduced row echelon form of C we see that $\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} \in \ker(C)$ if and only if $w + \frac{1}{2}x + \frac{1}{2}y = 0$ and $u + \frac{1}{6}x - \frac{1}{2}y + \frac{1}{3}z = 0$. The solutions to these two equations is a three dimensional space spanned by

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 6 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(b) To find the image of C, note that

$$C. \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = v \begin{pmatrix} 3 \\ 3 \\ 6 \\ 3 \end{pmatrix} + w \begin{pmatrix} 3 \\ -5 \\ 6 \\ 5 \end{pmatrix} + x \begin{pmatrix} 2 \\ -2 \\ 4 \\ 3 \end{pmatrix} + y \begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

That is, the image of C is the span of the columns of C, which is the same as the span of the rows of C^t, which is the same as the span of the the matrix in reduced echelon form obtained by row reducing C^t. So, the image of C is spanned by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ \frac{5}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{4} \end{pmatrix} \right\}$$

(c) The space spanned by the columns of C is exactly the image of C. So,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ \frac{5}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{4} \end{pmatrix} \right\}$$

(d) The space spanned by the columns of C^t is the span of the rows of C, which is the same as the span of the rows of the matrix obtained by row reducing C. We can read this space off as the span of

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \right\}$$

Problem 10. Let

$$A = \begin{pmatrix} -1 & 8 & 2 \\ -1 & -7 & -1 \\ 2 & 8 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 1 & 3 \\ 7 & 3 & 4 \\ -10 & -2 & -4 \end{pmatrix}$$

(a) Find the characteristic and minimum polynomials and the Jordon canonical forms for A and B.

(b) Find matrices P and Q so that $P^{-1}AP$ and $Q^{-1}BQ$ are in Jordon form.

Answer.

$$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 1 & 3 \\ 7 & 3 & 4 \\ -10 & -2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 1 \\ -2 & -6 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 1 \\ -2 & -6 & -1 \end{pmatrix} \begin{pmatrix} -1 & 8 & 2 \\ -1 & -7 & -1 \\ 2 & 8 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 & -1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

(c) Compute e^A and e^B.

$$e^B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 1 \\ -2 & -6 & -1 \end{pmatrix} \begin{pmatrix} e^2 & e^2 & \frac{e^2}{2} \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} 2 & -3 & -1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7e^2 & e^2 & \frac{7e^2}{2} \\ 8e^2 & 2e^2 & \frac{9e^2}{2} \\ -12e^2 & -2e^2 & -6e^2 \end{pmatrix}.$$

Problem 11. Solve the two second order differential equations:

$$y_2'' = y_2 - 2y_1'$$
$$y_1'' = y_1 + 2y_2'$$

subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = 1.$$

By translating them into a single system of the form $Y'(t) = AY(t)$ for a 4×4 matrix A.

Answer. Here, we translate the system into the single, first order matrix equation

$$Y' = AY, \quad Y(0) = B$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{and } Y = \begin{pmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{pmatrix}.$$

We know the solution space for Y is the span of the columns of e^{tA} and the solution we are interested in is given by $e^{tA}B$.

In order to find e^{tA}, we study the matrix A. We compute $\chi_A(t) = t^4 + 2t^2 + 1 = (t^2+1)^2 = (t-i)^2(t+i)^2$. Note that $m_A(t) = \chi_A(t)$ does not have distinct factors, so A is not diagonalizable. However, we can change basis so as to put A in a form simple enough to compute the exponential. We have $P^{-1}AP = A'$ where

$$A' = \begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{pmatrix} \text{ and } P = \begin{pmatrix} -1 & -i & -1 & i \\ -i & 1 & i & 1 \\ -i & 0 & i & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Now, we compute $e^{tA}B = e^{tPA'P^{-1}} = Pe^{tA'}P^{-1}B$ and explicitely

$$Pe^{tA'}P^{-1}B = P \begin{pmatrix} e^{ti} & te^{it} & 0 & 0 \\ 0 & e^{ti} & 0 & 0 \\ 0 & 0 & e^{-ti} & te^{-ti} \\ 0 & 0 & 0 & e^{-ti} \end{pmatrix} P^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} (e^{it}+e^{-it}) - it(e^{it}-e^{-it}) & -i(e^{it}+e^{-it}) - t(e^{it}+e^{-it} & t(e^{it}+e^{-it}) & -it(e^{it}-e^{-it}) \\ i(e^{it}-e^{-it}) + t(e^{it}+e^{-it}) & (e^{it}+e^{-it}) - it(e^{it}-e^{-it} & i(e^{it}-e^{-it}) & t(e^{it}+e^{-it}) \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -it(e^{it}-e^{-it}) \\ t(e^{it}+e^{-it}) \\ * \\ * \end{pmatrix}$$

(We ignore the last two rows since they're just the derivatives of the first two.)

Using the basic identities, $\cos(t) = \frac{1}{2}(e^{it}+e^{-it})$ and $\sin(t) = \frac{1}{2i}(e^{-it}-e^{-it})$, we simplify:

$$y_1(t) = t\sin(t)$$
$$y_2(t) = t\cos(t).$$