Problem 1. Let $V$ be the vector space of all $n \times n$ complex matrices. Prove that $\langle A, B \rangle = \text{trace}(AB^*)$ defines an inner product on $V$.

Answer. We check that the axioms of an inner product are satisfied.

- **Linearity in the first entry.** If $A, A'$, and $B$ are $n \times n$ matrices, then
  \[
  \langle A + A', B \rangle = \text{trace}((A + A')B^*) = \text{trace}(AB^* + A'B^*) = \text{trace}(AB^*) + \text{trace}(A'B^*) = \langle A, B \rangle + \langle A', B \rangle.
  \]

  If $\alpha \in \mathbb{C}$ then
  \[
  \langle \alpha A, B \rangle = \text{trace}(\alpha AB^*) = \text{trace}(\alpha AB^*) = \alpha \text{trace}(AB^*) = \langle A, B \rangle.
  \]
  
- **Hermitian symmetry.**
  \[
  \langle A, B \rangle = \text{trace}(AB^*) = \text{trace}((AB^*)^t) = \text{trace}((BA^*)^t) = \langle B, A \rangle.
  \]

  The first equality follows from the fact that $\text{trace}(M^t) = \text{trace}(M)$ since $\text{trace}(M^t) = \text{trace}(M)$ and $\text{trace}(M) = \text{trace}(M)$.

- **Nondegeneracy.** We have to prove that if $\langle A, B \rangle = 0$ for all matrices $B$, then $A = 0$.

  Let $B_{ij}$ be the matrix whose $i$-$j$-th entry is 1 and whose other entries are 0. Then $\text{trace}(AB_{ij}^*)$ is the $i$-$j$-th entry of $A$. So if $\text{trace}(AB^*) = 0$, for all $B$, then $\text{trace}(AB_{ij}^*) = 0$ for all $B_{ij}$, hence all the entries of $A$ must be zero.

  To see that $\text{trace}(AB_{ij}^*)$ is the $i$-$j$-th entry of $A$, note that the matrix $B_{ij}^*$ consists of columns of zeroes except for the $j$-th column, which is the $i$-th standard basis vector. Then the $i$-th column of $AB_{ij}^*$ is the $j$-th column of $A$, and all other entries of $AB_{ij}$ are zero. So the trace of $AB_{ij}$ is a sum of terms that are almost all zero, the only possible nonzero term being in the $i$-$i$ entry, in the sum $i$-$j$-entry of $A$.

Problem 2. Let $A$ be any $n \times n$ complex matrix. Prove that $A$ and $A^t$ are similar.

Answer. There are two steps. First, if $J$ is the Jordan form of $A$ then $A^t$ is similar to $J^t$. Second, $J^t$ is similar to $J$. So, by transitivity of similarity, we have $A \sim J \sim J^t \sim A^t$.

First step. Suppose $J = P^{-1}AP$. Then by taking transpose we have

\[
J^t = (P^{-1}AP)^t = P^tA^t(P^{-1})^t = P^tA^t(P^t)^{-1}.
\]

So $A^t \sim J^t$.

Second step. Note that if $B$ is a $k \times k$ Jordon block, then for

\[
Q_k = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

we have $Q_k^{-1}BQ_k = B^t$. Then by assembling a block-diagonal matrix $Q$ with matrices $Q_k$ of the correct sizes as the blocks, we will have $Q^{-1}JQ = J^t$. 

Problem 3. Prove that if $A^2 = A$ then $A$ is diagonalizable.

Answer. If $A^2 = A$ then $0 = A^2 - A = A(A - I)$. Therefore, the minimum polynomial for $A$ is either $m_A(t) = t$, $m_A(t) = (t - 1)$, or $m_A(t) = t(t - 1)$. In every case, $m_A(t)$ factors into distinct linear factors, hence $A$ is diagonalizable.

Remark. In the first case, $A$ is similar to the zero matrix, hence is the zero matrix. In the second case, $A$ must be similar to the identity matrix, hence is the identity. In the third case, $A$ must be similar to a diagonal matrix with 0's and 1's on the diagonal, but this matrix might not appear to be so simple. For example,

$$A = \begin{pmatrix} -2 & \frac{1}{2} & \frac{3}{2} & -2 \\ 30 & -4 & -15 & 20 \\ 6 & -1 & -2 & 4 \\ 15 & -\frac{5}{7} & -\frac{15}{7} & 11 \end{pmatrix}$$

Problem 4. Let $V$ be the space of real polynomials of degree $\leq 3$ and let $\sigma : V \rightarrow V$ be the differentiation operator

$$\sigma(p(t)) = \frac{dp(t)}{dt}.$$

(a) What is the adjoint of $\sigma$ if we equip $V$ with the inner product defined by

$$\langle p, q \rangle := \int_0^1 p(t)q(t)dt?$$

Answer. It’s convenient to work with an orthonormal basis. Performing Gram-Schmidt on the basis $\{1, t, t^2, t^3\}$ yields the orthonormal basis

$$B = \{1, 2\sqrt{3}t - \sqrt{3}, 6\sqrt{5}t^2 - 6\sqrt{5}t + \sqrt{5}, 20\sqrt{7}t^3 - 30\sqrt{7}t^2 + 12\sqrt{7}t - \sqrt{7}\}$$

The matrix for $\sigma$ in this basis is

$$\text{Rep}_{B,B}(\sigma) = \begin{pmatrix} 0 & 2\sqrt{3} & 0 & 2\sqrt{7} \\ 0 & 0 & 2\sqrt{15} & 0 \\ 0 & 0 & 0 & 2\sqrt{35} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, the matrix for the adjoint $\sigma^*$ in the basis $B$ is given by

$$\text{Rep}_{B,B}(\sigma^*) = (\text{Rep}_{B,B}(\sigma))^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 2\sqrt{7} & 0 & 2\sqrt{35} & 0 \end{pmatrix}.$$  

Remark. You might find it interesting to express $\sigma^*$ as a matrix in the basis $E = \{1, t, t^2, t^3\}$. To do so, use $\text{Rep}_{B,E}$, which is the matrix that changes basis from $B$ to $E$:

$$\text{Rep}_{B,E} = \begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{5} & -\sqrt{7} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & 12\sqrt{7} \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \\ 0 & 0 & 0 & 20\sqrt{7} \end{pmatrix}.$$
Then
\[ \text{Rep}_{E,E}(\sigma^*) = \text{Rep}_{B,E} \text{Rep}_{B,B} \text{Rep}_{E,B} = \left( \begin{array}{cccc} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & 12\sqrt{7} \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \\ 0 & 0 & 0 & 20\sqrt{7} \end{array} \right) \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \end{array} \right) \left( \begin{array}{cccc} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & 12\sqrt{7} \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \\ 0 & 0 & 0 & 20\sqrt{7} \end{array} \right) \]

This implies, for example, that \( \sigma^*(t^2) = -4 + 58t - 180t^2 + 140t^3 \). This means that for any polynomial \( p(t) \in V \), we have
\[ \langle \sigma(p(t)), t^2 \rangle = \langle p(t), \sigma^*(t^2) \rangle. \]

That is,
\[ \int_0^1 p'(t)t^2 dt = \int_0^1 p(t) \left( -4 + 58t - 180t^2 + 140t^3 \right) dt \text{ for every polynomial } p(t) \in V. \]

For example, for \( p(t) = \pi t^3 - \sqrt{3}t^2 - \frac{t}{7} + e \), this means that
\[ \int_0^1 \left( 3\pi t^2 - 2\sqrt{3}t - \frac{1}{7} \right) t^2 dt = \int_0^1 \left( \pi t^3 - \sqrt{3}t^2 - \frac{t}{7} + e \right) \left( -4 + 58t - 180t^2 + 140t^3 \right) dt. \]

Both sides are the number \( -\frac{1}{21} - \frac{\sqrt{3}}{2} + \frac{3\pi}{5} \).

(b) What is the adjoint of \( \sigma \) if we equip \( V \) with the inner product defined by
\[ \langle p, q \rangle := p(1)q(1) + p'(1)q'(1) + p''(1)q''(1) + p'''(1)q'''(1)? \]

**Answer.** Again, it’s convenient to use a basis for \( V \) that is orthonormal with respect to the given inner product:
\[ C = \left\{ 1, (t-1), \frac{1}{2}(t-1)^2, \frac{1}{3!}(t-1)^3 \right\} \]

Here, we have
\[ \text{Rep}_{C,C}(\sigma) = \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow \text{Rep}_{C,C}(\sigma^*) = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right). \]
Remark. Again, it might be interesting to express $\sigma^*$ as a matrix using the basis $E = \{1, t, t^2, t^3\}$. The result is:

$$
\text{Rep}_{E,E}(\sigma^*) = \begin{pmatrix}
-1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{2} \\
1 & 0 & 0 & 1 \\
0 & \frac{1}{2} & 0 & -\frac{3}{2} \\
0 & 0 & \frac{1}{3} & 1
\end{pmatrix}
$$

Problem 5. Let $M = \begin{pmatrix} 15 & -42 & 12 \\ -3 & 10 & -6 \\ -10 & 30 & -11 \end{pmatrix}$. Find a matrix $N$ so that $N^2 = M$.

Answer. Note that for $Q = \begin{pmatrix} -6 & 3 & -2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ we have $Q^{-1}MQ = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$. So the matrix

$$
N = Q \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} Q^{-1} = \begin{pmatrix} 3 & -6 & 0 \\ -1 & 4 & -2 \\ -2 & 6 & -1 \end{pmatrix}
$$

satisfies $N^2 = M$.

Problem 6. Suppose that $A$ is a $2 \times 2$ real matrix and let $v \in \mathbb{R}^2$ be any vector. Since $\mathbb{R}^2$ is two dimensional, the three vectors $\{v, Av, A^2v\}$ must be linearly dependent. Prove that, in fact, $A^2v = \alpha Av + \beta v$ where $\alpha = \text{trace}(A)$ and $\beta = -\det(A)$.

Problem 7. Suppose $V$ is an $n$ dimensional vector space. An operator $T : V \to V$ is called nilpotent if $T^k = 0$ for some integer $k$.

(a) Prove that if $T$ is nilpotent, then $T^n = 0$.

Answer. If $T^k = 0$, then the minimum polynomial for $T$ divides $T^k$, so the minimum polynomial $m_T(t)|t^k$. This means that $m_T(t) = t^r$ for some $r \leq k$. Since every linear factor of the characteristic polynomial $\chi_T(t)$ is a factor of $m_T(t)$, and $\chi_T(t)$ is degree $n$, we conclude $\chi_T(t) = t^n$. The Cayley Hamilton theorem implies $\chi_T(T) = 0 \iff T^n = 0$.

(b) Prove that if $T$ is nilpotent, then there is a basis of vectors of $V$ so that the matrix for $T$ is upper triangular with zeroes on the diagonal.

Answer. The remarks above about $m_T(t) = t^r$ implies that the Jordan form for $T$ consists of Jordan blocks of sizes $\leq r$ with zero’s on the diagonal. As we defined it, the Jordan blocks have 1’s on the subdiagonal, but by re-ordering the basis vectors, we will obtain the transpose matrix, an upper triangular matrix with zeroes on the diagonal (and zeros and ones on the superdiagonal).

Problem 8. Let $V$ be the space of polynomials of degree less than or equal to 2. Let $T : V \to V$ be the homomorphism defined by

$$
p(x) \mapsto p(1)x^2 + p'(0)x + p''(0)
$$
(a) Compute the determinant and trace of $T$.

Answer. First, do part (b). Then, it’s straightforward to determine that $\text{trace}(T) = 2$ and $\det(T) = -2$.

(b) Use the ordered basis $B = \{x^2, x, 1\}$ to express $T$ as a $3 \times 3$ matrix $A = \text{Rep}_{B,B}(T)$.

Answer.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

(c) Find a basis $E$ consisting of eigenvectors for $T$.

Answer.

$$\{t^2 - t + 1, -2, t^2 + 2t + 2\}$$

(d) Let $Q$ be the change of basis matrix $Q = \text{Rep}_{B,E}(\text{Id})$.

Answer.

$$Q = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

(e) Use your the ordered basis $E$ to express $T$ as a $3 \times 3$ matrix $B = \text{Rep}_{E,E}(\text{Id})$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 9. Let

$$C = \begin{pmatrix} 3 & 3 & 2 & 0 & 1 \\ 3 & -5 & -2 & -4 & 1 \\ 6 & 6 & 4 & 0 & 2 \\ 3 & 5 & 3 & 1 & 1 \end{pmatrix}$$

(a) Find a basis for the kernel of $C$.

(b) Find a basis for the image of $C$.

(c) Find a basis for the space spanned by the columns of $C$.

(d) Find a basis for the space spanned by the columns of $C^t$.

Answer. It’s helpful to realize that the kernel of a matrix $M$ is the same as the kernel of a matrix $M'$ obtained from $M$ by row reduction. Also, the span of the rows of a matrix $M$ are the same as the span of the rows of a matrix $M'$ obtained from $M$ by row reduction.
Here are two matrices obtained from $C$ and $C^t$ by row reduction, which will be helpful for answering all the questions.

\[
\begin{pmatrix}
1 & 0 & \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} \\
0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 2 & \frac{5}{4} \\
0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(a) From the reduced row echelon form of $C$ we see that
\[
\begin{pmatrix}
v \\
w \\
x \\
y \\
z
\end{pmatrix}
\in \text{ker}(C) \text{ if and only if }
\begin{pmatrix}
w + \frac{1}{2}x + \frac{1}{2}y = 0 \\
u + \frac{1}{6}x - \frac{1}{2}y + \frac{1}{3}z = 0
\end{pmatrix}
\text{.}
\]
The solutions to these two equations is a three dimensional space spanned by
\[
\left\{ \begin{pmatrix}
-1 \\
0 \\
0 \\
0 \\
3
\end{pmatrix}, \begin{pmatrix}
1 \\
-1 \\
0 \\
2 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
-3 \\
6 \\
0 \\
0
\end{pmatrix} \right\}
\]

(b) To find the image of $C$, note that
\[
C \begin{pmatrix}
v \\
w \\
x \\
y \\
z
\end{pmatrix}
= v \begin{pmatrix}
3 \\
3 \\
6 \\
6 \\
3
\end{pmatrix} + w \begin{pmatrix}
3 \\
-5 \\
-6 \\
5 \\
4
\end{pmatrix} + x \begin{pmatrix}
2 \\
-2 \\
-4 \\
3 \\
1
\end{pmatrix} + y \begin{pmatrix}
0 \\
-4 \\
0 \\
0 \\
1
\end{pmatrix} + z \begin{pmatrix}
1 \\
1 \\
2 \\
0 \\
1
\end{pmatrix}
\]
That is, the image of $C$ is the span of the columns of $C$, which is the same as the span of the rows of $C^t$, which is the same as the span of the the matrix in reduced echelon form obtained by row reducing $C^t$. So, the image of $C$ is spanned by
\[
\left\{ \begin{pmatrix}
1 \\
0 \\
2 \\
\frac{5}{4}
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0 \\
-\frac{1}{4}
\end{pmatrix} \right\}
\]

(c) The space spanned by the columns of $C$ is exactly the image of $C$. So,
\[
\left\{ \begin{pmatrix}
1 \\
0 \\
2 \\
\frac{5}{4}
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0 \\
-\frac{1}{4}
\end{pmatrix} \right\}
\]
(d) The space spanned by the columns of $C^t$ is the span of the rows of $C$, which is the same as the span of the rows of the matrix obtained by row reducing $C$. We can read this space off as the span of
\[
\begin{pmatrix}
1 \\
0 \\
\frac{1}{6} \\
-\frac{1}{2} \\
\frac{1}{3}
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
1 \\
\frac{1}{2} \\
\frac{1}{2} \\
0
\end{pmatrix}
\]

Problem 10. Let
\[
A = \begin{pmatrix}
-1 & 8 & 2 \\
-1 & -7 & -1 \\
2 & 8 & -1
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
7 & 1 & 3 \\
7 & 3 & 4 \\
-10 & -2 & -4
\end{pmatrix}
\]

(a) Find the characteristic and minimum polynomials and the Jordon canonical forms for $A$ and $B$.

(b) Find matrices $P$ and $Q$ so that $P^{-1}AP$ and $Q^{-1}BQ$ are in Jordan form.

Answer.
\[
\begin{pmatrix}
2 & -3 & -1 \\
-1 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}\begin{pmatrix}
7 & 1 & 3 \\
7 & 3 & 4 \\
-10 & -2 & -4
\end{pmatrix}\begin{pmatrix}
1 & 3 & 1 \\
1 & 4 & 1 \\
-2 & -6 & -1
\end{pmatrix} = \begin{pmatrix}
2 & 1 & 0 \\
0 & 2 & 1 \\
0 & 0 & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 3 & 1 \\
1 & 4 & 1 \\
-2 & -6 & -1
\end{pmatrix}\begin{pmatrix}
-1 & 8 & 2 \\
-1 & -7 & -1 \\
2 & 8 & -1
\end{pmatrix}\begin{pmatrix}
2 & -3 & -1 \\
-1 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
-3 & 1 & 0 \\
0 & -3 & 0 \\
0 & 0 & -3
\end{pmatrix}
\]

(c) Compute $e^A$ and $e^B$.
\[
e^B = \begin{pmatrix}
1 & 3 & 1 \\
1 & 4 & 1 \\
-2 & -6 & -1
\end{pmatrix}\begin{pmatrix}
e^2 & e^2 & \frac{e^2}{2} \\
e^2 & e^2 & \frac{e^2}{4} \\
0 & 0 & e^2
\end{pmatrix}\begin{pmatrix}
2 & -3 & -1 \\
-1 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
7e^2 & e^2 & \frac{7e^2}{2} \\
8e^2 & 2e^2 & \frac{9e^2}{2} \\
-12e^2 & -2e^2 & -6e^2
\end{pmatrix}
\]

Problem 11. Solve the two second order differential equations:
\[
y_2'' = y_2 - 2y_1' \\
y_1'' = y_1 + 2y_2'
\]
subject to the initial conditions
\[
y_1(0) = 0, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = 1.
\]
By translating them into a single system of the form $Y'(t) = AY(t)$ for a $4 \times 4$ matrix $A$. 

**Answer.** Here, we translate the system into the single, first order matrix equation

\[ Y' = AY, \quad Y(0) = B \]

where

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 2 \\
0 & 1 & -2 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
y_1 \\
y_2
\end{pmatrix}, \quad \text{and } Y = \begin{pmatrix}
y_1 \\
y_2 \\
y_1' \\
y_2'
\end{pmatrix}.
\]

We know the solution space for \( Y \) is the span of the columns of \( e^{tA} \) and the solution we are interested in is given by \( e^{tA}B \).

In order to find \( e^{tA} \), we study the matrix \( A \). We compute \( \chi_A(t) = t^4 + 2t^2 + 1 = (t^2 + 1)^2 \). Note that \( m_A(t) = \chi_A(t) \) does not have distinct factors, so \( A \) is not diagonalizable. However, we can change basis so as to put \( A \) in a form simple enough to compute the exponential. We have

\[
P^{-1}AP = A' \quad \text{where} \quad A' = \begin{pmatrix}
i & 1 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -i & 1 \\
0 & 0 & 0 & -i
\end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix}
-1 & -i & -1 & i \\
-i & 1 & i & 1 \\
-i & 0 & i & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}.
\]

Now, we compute \( e^{tA}B = e^{tPA'}P^{-1} = P e^{tA'}P^{-1}B \) and explicitly

\[
P e^{tA'}P^{-1}B = P \begin{pmatrix}
e^{ti} & t e^{it} & 0 & 0 \\
0 & e^{ti} & 0 & 0 \\
0 & 0 & e^{-ti} & t e^{-ti} \\
0 & 0 & 0 & e^{-ti}
\end{pmatrix} P^{-1} \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

\[
= \frac{1}{2} \begin{pmatrix}
(e^{it} + e^{-it}) - it(e^{it} - e^{-it}) & -i(e^{it} + e^{-it}) - t(e^{it} + e^{-it}) & t(e^{it} + e^{-it}) & -it(e^{it} - e^{-it}) \\
it(e^{it} - e^{-it}) + t(e^{it} + e^{-it}) & (e^{it} + e^{-it}) - it(e^{it} - e^{-it}) & (e^{it} + e^{-it}) & t(e^{it} - e^{-it}) \\
* & * & * & *
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

(We ignore the last two rows since they’re just the derivatives of the first two.)

Using the basic identities, \( \cos(t) = \frac{1}{2}(e^{it} + e^{-it}) \) and \( \sin(t) = \frac{1}{2i}(e^{it} - e^{-it}) \), we simplify:

\[
y_1(t) = t \sin(t) \\
y_2(t) = t \cos(t).
\]