Correspondences obtained by using ordered bases

Using an ordered basis to represent a vector as columns of scalars

Using an ordered basis $B = \{b_1, \ldots, b_n\}$ for a vector space V, there is a bijective correspondence between the vectors in V and vectors in $\mathbb{R}^n$. I will remind you of this correspondence and also define some convenient terminology.

Any vector $v \in V$ can be expressed uniquely as $v = \sum_{i=1}^{n} \alpha_i b_i$ where $\alpha_i \in \mathbb{R}$. Then v corresponds to the column vector $\mathrm{Rep}_B(v) \in \mathbb{R}^n$ defined by

$$\mathrm{Rep}_B(p(x)) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

The correspondence

$$\begin{aligned} V &\longleftrightarrow \mathbb{R}^n \\ v &\longleftrightarrow \mathrm{Rep}_B(v) \end{aligned}$$

is one-to-one since every vector in the space can be expressed as a linear combination of the basis vectors uniquely and the corrspondence is onto since every linear combination of the basis vectors is a vector in the space.

Note that the correspondence depends on the choice of ordered basis B, and that dependence is reflected in the notation $\mathrm{Rep}_B(v)$.

Problem 1. Let V be the space of polynomials of degree less than or equal to four. The set

$$B = \{1+x, 1-x, 1-2x^2, 1+x-x^3, x^2-x^4\}$$

is a basis for V. Use the ordered basis B to obtain a correspondence between $V \leftrightarrow \mathbb{R}^5$.

(a) Give the polynomials in V that correspond to the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

(b) Give the vectors in $\mathbb{R}^5$ that correspond to the polynomials

$$x^3, \quad 3x^2, \quad 2x^4 - 7, \text{ and } 4x^4 - 14$$

Using ordered bases to represent linear transformations as a matrix of scalars

Given an ordered basis $B_V = \{v_1, \ldots, v_k\}$ of a vector space V and an ordered basis $B_W = \{w_1, \ldots, w_n\}$ of W, there's a bijective correspondence between linear transformations $T : V \to W$ and $n \times k$ matrices of scalars, defined as follows:

Given a linear transformation $T : V \to W$, define a matrix $\mathrm{Rep}_{B_V,B_W}(T)$ by setting the i-th column to be $\mathrm{Rep}_{B_W}(T(v_i))$.

To put it another way, for each basis vector $v_i \in B_V$, the vector $T(v_i)$ is an element of W and so it can be expressed uniquely in terms of the basis B_W:

$$T(v_i) = \sum_{j=1}^{n} a_{ij} w_j.$$

The numbers a_{ij} for $i = 1, \ldots, k$ and $j = 1, \ldots, n$ define the $n \times k$ matrix

$$\mathrm{Rep}_{B_V,B_W}(T) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix}.$$

Notice, the i-th column of the matrix $\mathrm{Rep}_{B_V,B_W}(T)$ is the column of scalars

$$\begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} = \mathrm{Rep}_{B_W}(T(v_i)).$$

Problem 2. As before, let V be the space of polynomials of degree less than or equal to four and let

$$B = \{1 + x, 1 - x, 1 - 2x^2, 1 + x - x^3, x^2 - x^4\}$$

be a basis for V. The function $T : V \to V$ defined by $T(p) = p'(t) + p(0)$ defines a linear transformation.

(a) Using the correspondence $V \longleftrightarrow \mathbb{R}^5$, express T as a matrix $\mathrm{Rep}_{B,B}(T)$.

(b) Compute the polynomial $T(x^3)$ and compute the following product of a matrix with a vector:

$$\mathrm{Rep}_{B,B}(T) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

(c) Let $B' = \{1, t, t^2, t^3, t^4\}$ and compute the matrix $\mathrm{Rep}_{B,B'}(T)$.

(d) Find a basis B'' of V so that the matrix $\mathrm{Rep}_{B,B''}(T)$ is *diagonal* with either 1 or 0 on the diagonal.

Composition of linear transformations

Problem 3. This is a variation of problems 28 and 29 in section 2.8 of the Apostol's book. Let V be the space of all real polynomials. Define linear transformations by

$$\begin{aligned} D : V &\to V & S : V &\to V & T : V &\to V \\ p(t) &\mapsto p'(t) & p(t) &\mapsto tp(t) & p(t) &\mapsto tp'(t) \end{aligned}$$

(a) Let $p(t) = 2 + 3t - t^2 + 4t^3$ Determine the image of p under

$$D, S, T, DT, TD, DS, SD, ST, TS, DT - TD, T^2D^2 - D^2T^2.$$

(b) Find the kernel of T, $T - \mathrm{Id}$, and $DT - 2D$.

(c) Does D have a left inverse or a right inverse?

(d) Let $S : V \to V$ be defined by $p(t) \mapsto tp(t)$. Compute $DS - SD$ and, more generally, $DS^n - S^nD$.

Inner products and best approximation

Problem 4. Let V be the space of polynomials of degree less than or equal to three and let

$$\langle p, q\rangle = p(0)q(0) + p'(0)q'(0) + p(1)q(1) + p'(1)q'(1).$$

(a) Prove that $\langle p, q\rangle$ defines an inner product on V. In particular, prove that if $\langle p, p\rangle = 0$ then $p = 0$. *Hint:* first prove the lemma that if p is a polynomial and a is a number for which $p(a) = 0$ and $p'(a) = 0$, then there exists a polynomial q so that $p(x) = (x - a)^2q(x)$.

(b) Compute the angle between the polynomials $2t^3 - 3t^2$ and 1.

(c) Use Gram-Schmidt to replace the basis $\{1, t, t^2, t^3\}$ by an orthogonormal set.

(d) The following polynomials form a very nice basis for V.

$$\begin{aligned} e_1 &= t(t-1)^2 \\ e_2 &= t^2(t-1) \\ e_3 &= (2t+1)(t-1)^2 \\ e_4 &= t^2(1 - 2(t-1)) \end{aligned}$$

Projecting an arbitrary differentiable function f onto V yields a linear combination of the basis vectors

$$a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4.$$

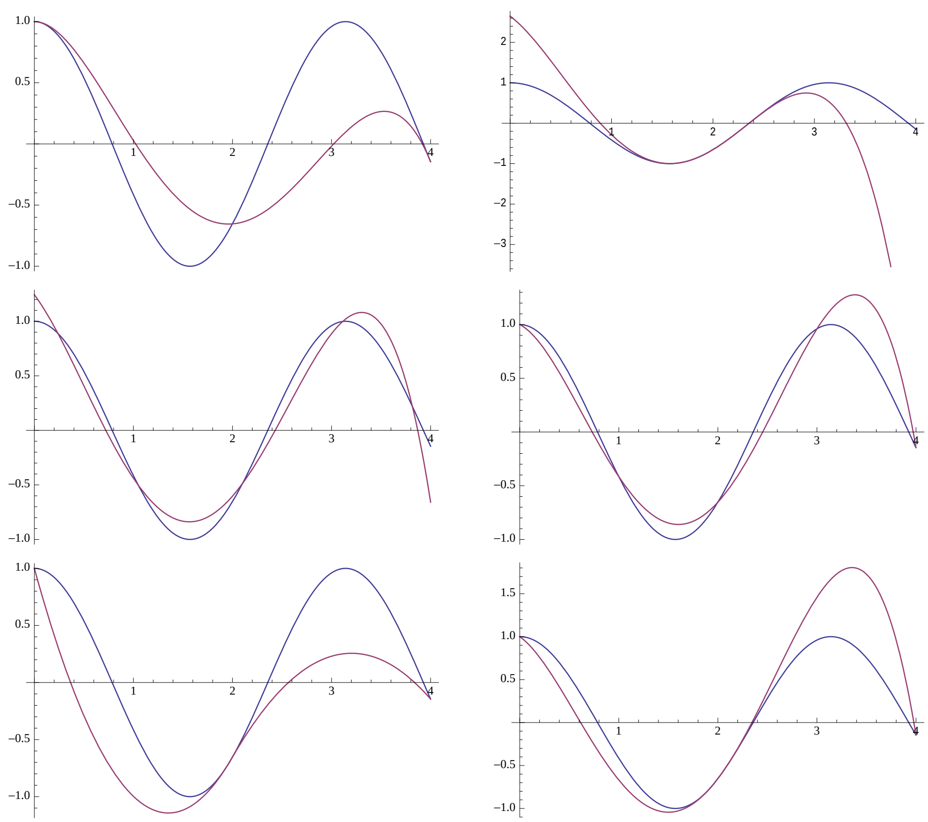
Describe (simply!) the coefficients a_i in terms of the function f. The simplicity of this description explains why $\{e_1, e_2, e_3, e_4\}$ is such a nice basis for V.

Problem 5. Each of the pictures below shows a polynomial approximation for $\cos(2t)$. Five of them are degree four polynomials obtained by projection onto the subspace of quartic polynomials, using one of the following inner products.

$$\begin{aligned}
\langle f, g\rangle_1 &= f(2)g(2) + f'(2)g'(2) + f''(2)g''(2) + f'''(2)g'''(2)\\
\langle f, g\rangle_2 &= \int_0^4 f(t)g(t)dt\\
\langle f, g\rangle_3 &= f(0)g(0) + f(1)g(1) + f(2)g(2) + f(3)g(3) + f(4)g(4)\\
\langle f, g\rangle_4 &= f(0)g(0) + f(2)g(2) + f'(2)g'(2) + f''(2)g''(2) + f(4)g(4)\\
\langle f, g\rangle_5 &= f(0)g(0) + f'(0)g'(0) + f(2)g(2) + f(4)g(4) + f'(4)g'(4)
\end{aligned}$$

The sixth approximation is obtained by "gluing together" two degree two polynomials defined on $[0, 2]$ and $[2, 4]$, each determined by projection using, respectively, the inner products

$$\begin{aligned}
\langle f, g\rangle_6 &= f(0)g(0) + f(2)g(2) + f'(2)g'(2)\\
\langle f, g\rangle_7 &= f(2)g(2) + f'(2)g'(2) + f(4)g(4)
\end{aligned}$$



Which pictures go with which approximations?