Correspondences obtained by using ordered bases

Using an ordered basis to represent a vector as columns of scalars

Using an ordered basis $B = \{b_1, \ldots, b_n\}$ for a vector space $V$, there is a bijective correspondence between the vectors in $V$ and vectors in $\mathbb{R}^n$. I will remind you of this correspondence and also define some convenient terminology.

Any vector $v \in V$ can be expressed uniquely as $v = \sum_{i=1}^{n} \alpha_i b_i$ where $\alpha_i \in \mathbb{R}$. Then $v$ corresponds to the column vector $\text{Rep}_B(v) \in \mathbb{R}^n$ defined by

$$\text{Rep}_B(p(x)) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

The correspondence

$$V \longleftrightarrow \mathbb{R}^n$$
$$v \longleftrightarrow \text{Rep}_B(v)$$

is one-to-one since every vector in the space can be expressed as a linear combination of the basis vectors uniquely and the correspondence is onto since every linear combination of the basis vectors is a vector in the space.

Note that the correspondence depends on the choice of ordered basis $B$, and that dependence is reflected in the notation $\text{Rep}_B(v)$.

**Problem 1.** Let $V$ be the space of polynomials of degree less than or equal to four. The set

$$B = \{1 + x, 1 - x, 1 - 2x^2, 1 + x - x^3, x^2 - x^4\}$$

is a basis for $V$. Use the ordered basis $B$ to obtain a correspondence between $V \leftrightarrow \mathbb{R}^5$.

(a) Give the polynomials in $V$ that correspond to the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

**Answer.**

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

means $1e_1 = 1 + x$. 

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
\[
\begin{pmatrix}
1 \\
0 \\
0 \\
-1 \\
0
\end{pmatrix}
\] means \(1e_1 - 1e_4 = 1 + x - (1 + x - x^3) = x^3\).

\[
\begin{pmatrix}
1 \\
0 \\
-2 \\
-1 \\
1
\end{pmatrix}
\] means \(1e_1 - 2e_3 - e_4 + e_5 = 1 + x - 2(1 - 2x^2) - (1 + x - x^3) + x^2 - x^4 = -2 + 5x^2 + x^3 - x^4\).

(b) Give the vectors in \(\mathbb{R}^5\) that correspond to the polynomials 
\(x^3, \ 3x^2, \ 2x^4 - 7, \text{ and } 4x^4 - 14\)

**Answer.** From above, we know

\[
x^3 \leftrightarrow \begin{pmatrix}
1 \\
0 \\
0 \\
-1 \\
0
\end{pmatrix}
\]

To find the vector that corresponds to \(3x^2\), write

\[
3x^2 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5
\]

\[
= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + (\alpha_1 - \alpha_2 + \alpha_4)x + (\alpha_5 - 2\alpha_3)x^2 - \alpha_4 x^3 - \alpha_5 x^4.
\]

Since \(3x^2\) has no \(x^3\) or \(x^4\) terms, \(\alpha_5 = \alpha_4 = 0\). So

\[
3x^2 = \alpha_1 + \alpha_2 + \alpha_3 + (\alpha_1 - \alpha_2)x - 2\alpha_3 x^2
\]

So, \(3 = -2\alpha_3 \Rightarrow \alpha_3 = -\frac{3}{2}\). Then, setting the constant and linear terms equal to zero yields

\[
\alpha_1 + \alpha_2 = -\frac{3}{2} = 0 \text{ and } \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \frac{3}{4} \text{ and } \alpha_2 = \frac{3}{4}.
\]

So,

\[
3x^2 \leftrightarrow \begin{pmatrix}
\frac{3}{4} \\
\frac{3}{4} \\
-\frac{3}{2} \\
0 \\
0
\end{pmatrix}
\]

To express \(2x^4 - 7 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5\), note that \(-2e_5 - e_3 = 2x^4 - 1\).

Also note that \((e_1 + e_2) = 2 \Rightarrow -3e_1 - 3e_2 = -6\). So, \(-2e_5 - e_3 - 3e_1 - 3e_2 = 2x^4 - 7\).
So,

$$2x^4 - 7 \leftrightarrow \begin{pmatrix} -3 \\ -3 \\ -1 \\ 0 \\ -2 \end{pmatrix}$$

Finally, since $4x^4 - 14 = 2(2x^4 - 7)$, we have

$$4x^4 - 14 \leftrightarrow \begin{pmatrix} -6 \\ -6 \\ -2 \\ 0 \\ -4 \end{pmatrix}$$

Using ordered bases to represent linear transformations as a matrix of scalars

Given an ordered basis $B_V = \{v_1, \ldots, v_k\}$ of a vector space $V$ and an ordered basis $B_W = \{w_1, \ldots, w_n\}$ of $W$, there's a bijective correspondence between linear transformations $T : V \rightarrow W$ and $n \times k$ matrices of scalars, defined as follows:

Given a linear transformation $T : V \rightarrow W$, define a matrix $\text{Rep}_{B_V, B_W}(T)$ by setting the $i$-th column to be $\text{Rep}_{B_W}(T(v_i))$.

To put it another way, for each basis vector $v_i \in B_V$, the vector $T(v_i)$ is an element of $W$ and so it can be expressed uniquely in terms of the basis $B_W$:

$$T(v_i) = \sum_{j=1}^{n} a_{ij} w_j.$$ 

The numbers $a_{ij}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$ define the $n \times k$ matrix

$$\text{Rep}_{B_V, B_W}(T) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix}.$$ 

Notice, the $i$-th column of the matrix $\text{Rep}_{B_V, B_W}(T)$ is the column of scalars

$$\begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} = \text{Rep}_{B_W}(T(v_i)).$$

**Problem 2.** As before, let $V$ be the space of polynomials of degree less than or equal to four and let

$$B = \{1 + x, 1 - x, 1 - 2x^2, 1 + x - x^3, x^2 - x^4\}$$

be a basis for $V$. The function $T : V \rightarrow V$ defined by $T(p) = p'(t) + p(0)$ defines a linear transformation.
(a) Using the correspondence $V \leftrightarrow \mathbb{R}^5$, express $T$ as a matrix $\text{Rep}_{B,B}(T)$.

**Answer.** Compute $T$ of each basis vector:

$$Te_1 = 2 = e_1 + e_2$$

$$Te_2 = 0$$

$$Te_3 = -4x + 1 = \frac{3}{2}e_1 - \frac{5}{2}e_2$$

$$Te_4 = 2 - 3x^2 = \frac{1}{4}e_1 + \frac{1}{4}e_2 + \frac{3}{2}e_3$$

$$Te_5 = 2x - 4x^3 = -3e_1 - e_2 + 4e_4$$

Thus,

$$\text{Rep}_{B,B}(T) = \begin{pmatrix}
1 & 0 & \frac{3}{2} & \frac{1}{4} & -3 \\
1 & 0 & -\frac{5}{2} & \frac{1}{4} & -1 \\
0 & 0 & 0 & \frac{3}{2} & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

(b) Compute the polynomial $T(x^3)$ and compute the following product of a matrix with a vector:

$$\text{Rep}_{B,B}(T) \begin{pmatrix}
1 \\
0 \\
0 \\
-1 \\
0
\end{pmatrix}$$

**Answer.** First, note that $T(x^3) = 3x^2$. Also that $\text{Rep}_B(x^3) = \begin{pmatrix}
1 \\
0 \\
-1 \\
0
\end{pmatrix}$. Computing the product

$$\text{Rep}_{B,B}(T) \begin{pmatrix}
1 \\
0 \\
-1 \\
0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \frac{3}{4} & \frac{1}{4} & -3 \\
1 & 0 & -\frac{5}{4} & \frac{1}{4} & -1 \\
0 & 0 & 0 & \frac{4}{2} & 0 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
-1 \\
0
\end{pmatrix} = \begin{pmatrix}
\frac{3}{4} \\
\frac{3}{4} \\
-3 \\
-3
\end{pmatrix}.$$
(c) Let $B' = \{1, t, t^2, t^3, t^4\}$ and compute the matrix $\text{Rep}_{B', B'}(T)$. Here, let $\{1, t, t^2, t^3, t^4\} = \{b_1, b_2, b_3, b_4, b_5\}$ and express

\[
Te_1 = 2 = 2b_1 \\
Te_2 = 0 \\
Te_3 = 1 - 4x = b_1 - 4b_2 \\
Te_4 = 2 - 3x^2 = 2b_1 - 3b_3 \\
Te_5 = 2x - 4x^3 = 2b_2 - 4b_4
\]

Thus,

\[
\text{Rep}_{B', B'}(T) = \begin{pmatrix}
2 & 0 & 1 & 2 & 0 \\
0 & 0 & -4 & 0 & 2 \\
0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

As a consistency check, we compute

\[
\text{Rep}_{B', B'}(T) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}
\]

which is the representative of $3x^2$ in the basis $B'$. Thus,

\[
\text{Rep}_{B', B'}(T) \text{Rep}_B(x^3) = \text{Rep}_{B'}(T(x^3))
\]

as expected.

(d) Find a basis $B''$ of $V$ so that the matrix $\text{Rep}_{B', B''}(T)$ is diagonal with either 1 or 0 on the diagonal.

**Answer.** Here’s one solution. Let $B'' = \{2, x^4, 1 - 4x, 2 - 3x^2, 4x^3\}$ then

\[
\text{Rep}_{B', B''}(T) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

**Composition of linear transformations**

**Problem 3.** This is a variation of problems 28 and 29 in section 2.8 of the Apostol’s book. Let $V$ be the space of all real polynomials. Define linear transformations by

\[
D : V \to V \\
S : V \to V \\
T : V \to V
\]

\[
p(t) \mapsto p'(t) \\
p(t) \mapsto tp(t) \\
p(t) \mapsto tp'(t)
\]
(a) Let \( p(t) = 2 + 3t - t^2 + 4t^3 \) Determine the image of \( p \) under
\[
D, S, T, DT, TD, DS, SD, ST, TS, DT - TD, T^2D^2 - D^2T^2.
\]

**Answer.** For \( SD \) and \( DS \), see the part (d) of this problem. For the others, here are some computations:
\[
Dp = 3 - 2t + 12t^2, \quad Sp = 2t + 3t^2 - t^3 + 4t^4, \quad Tp = 3t - 2t^2 + 12t^3.
\]
\[
DTp = D(3t-2t^2+12t^3) = 3-4t+36t^2, \quad TDP = T(3-2t+12t^2) = t(-2+24t) = -2t+24t^2
\]
\[
DSP = D(2t+3t^2-t^3+4t^4) = 2+6t-3t^2+16t^3, \quad SDP = S(3-2t+12t^2) = 3-2t^2+12t^3.
\]
et cetera. Note that \( DT - TD = (3 - 4t + 36t^2) - (-2t + 24t^2) = 3 - 2t + 12t^2 = Dp \).

For the last one:
\[
(T^2D^2 - D^2T^2)p = T^2D^2p - D^2T^2p
\]
\[
= T^2p'' - D^2T(tp')
\]
\[
= T(tp'''') - D^2(t(p' + tp''))
\]
\[
= T(tp'''') - D^2(tp'' + t^2p''')
\]
\[
= t(p''' + t^2p''') - D(p' + tp'' + 2tp'' + t^2p''')
\]
\[
= tp''' + t^3p''''' - (p'' + p'' + tp'' + 2tp''' + 2tp''' + 2tp'''' + 3tp''''')
\]
\[
= -4p'' - 4tp'''
\]

So
\[
(T^2D^2 - D^2T^2)(2 + 3t - t^2 + 4t^3) = -4(-2 + 24t) - 4t(24) = 8 - 192t.
\]

(b) Find the kernel of \( T, T - \text{Id} \), and \( DT - 2D \).

**Answer.** If \( p \in \ker(T) \) then \( tp' = 0 \Rightarrow p'' = 0 \Rightarrow p = c \) for some constant \( c \). Thus \( \ker(T) \) is the space of constant polynomials.

If \( p \in \ker(T - \text{Id}) \) then \( tp' - p = 0 \Rightarrow tp' = p \Rightarrow p(t) = ct \) for some constant \( t \). Thus, \( \ker(T - \text{Id}) \) is the one dimensional subspace of polynomials of the form \( ct \) for some constant \( c \); i.e., the span of \( \{t\} \).

A quick check reveals that \( (DT - 2D)(p) = tp'' - p' \), so to determine for which polynomials \( p \), we have \( (DT - 2D)(p) = 0 \), write \( p(t) = c_0 + c_1t + c_2t^2 + \cdots + c_nt^n \). Then
\[
\begin{align*}
tp'' - p'(t) &= 2c_2t + 6c_3t^2 + \cdots (n)(n - 1)c_nt^{n-1} - (c_1 + 2c_2t + 3c_3t^2 + \cdots + nc_nt^{n-1}) \\
&= -c_1 + 3c_2t^2 + \cdots + (n)(n - 2)c_nt^{n-1} \\
\end{align*}
\]

Setting \( tp'' - p'(t) = 0 \iff c_1 = c_2 = \cdots = c_n = 0 \). Thus, the kernel of \( DT - 2D \) consists of all polynomials of the form \( p(t) = c_0 + c_2t^2 \); i.e. the span of \( \{1, t^2\} \).

(c) Does \( D \) have a left inverse or a right inverse?
**Problem Set 1 - Answers**

**Answer.** Since $D$ has a nontrivial kernel, it does not have a left inverse. To see this, let $p$ be the constant polynomial $p(t) = 1$. For any linear map $F : V \to V$, we have

$$FD(p) = F(0) = 0 \neq p$$

and so $FD \neq \text{Id}$ for any linear $F : V \to V$.

The map $D$ does have a right inverse. Here’s one: Let $I : V \to V$ be defined by $I(p) = \int_0^t p(s)ds$. Then

$$DI(p(t)) = D\int_0^t p(s)ds = p(t).$$

The last equality follows from the fundamental theorem of calculus. So $DI = \text{Id}$ and we see that $I$ is a right inverse of $D$.

The fact that $D$ has a set-theoretic right inverse says that $D$ is onto; i.e. every polynomial is the derivative of another polynomial (certainly true — every polynomial has many antiderivatives). However, to say that $D$ has a right inverse that is a linear transformation says that an antiderivative for each polynomial can be chosen so the totality of choices defines a linear map (this is only true if we choose the “constant of integration” to be zero).

(d) Let $S : V \to V$ be defined by $p(t) \mapsto tp(t)$. Compute $DS - SD$ and, more generally, $DS^n - S^n D$.

**Answer.** $DS(p) = D(tp) = p + tp'$, $SD(p) = S(p') = tp'$, and $(DS - SD)(p) = p + tp' - tp' = p$. So $DS - SD = \text{Id}$.

More generally, $DS^n(p) = t^n p = nt^{n-1}p + t^n p'$, $S^n D(p) = S^n(p') = t^n p'$, and $(DS^n - S^n D)(p) = nt^{n-1}p$.

**Inner products and best approximation**

**Problem 4.** Let $V$ be the space of polynomials of degree less than or equal to three and let

$$\langle p, q \rangle = p(0)q(0) + p'(0)q'(0) + p(1)q(1) + p'(1)q'(1).$$

(a) Prove that $\langle p, q \rangle$ defines an inner product on $V$. In particular, prove that if $\langle p, p \rangle = 0$ then $p = 0$. **Hint:** first prove the lemma that if $p$ is a polynomial and $a$ is a number for which $p(a) = 0$ and $p'(a) = 0$, then there exists a polynomial $q$ so that $p(x) = (x - a)^2 q(x)$.

**Answer.** It’s straightforward to check that $(\cdot , \cdot)$ as defined is symmetric and bilinear, and it’s clear that as a sum of squares, $\langle p, p \rangle \geq 0$ for all $p \in V$. The one property that’s not straightforward to check is that $\langle p, p \rangle = 0$ only for the zero polynomial. To check this, we first prove the lemma in the hint.

**Lemma.** If $p$ is a polynomial and $a$ is a number for which $p(a) = 0$ and $p'(a) = 0$, then there exists a polynomial $q$ so that $p(t) = (t - a)^2 q(t)$.
Proof. First, recall that if \( p \) is any polynomial and \( a \) is a number for which \( p(a) = 0 \), then there exists a polynomial \( q \) so that that

\[
p(t) = (t - a)q(t).
\]

Now assume that for a particular polynomial \( p \) we have \( p(a) = 0 \) and \( p'(a) = 0 \). Since \( p(a) = 0 \) then there exists a polynomial \( q \) with

\[
p(t) = (t - a)q(t).
\]

Note that

\[
p(t) = (t - a)q'(t) + q(t).
\]

Since \( p'(a) = 0 \) we have

\[
0 = p'(a) = (a - a)q'(a) + q(a) = q(a).
\]

Since \( q \) is a polynomial for which \( q(a) = 0 \), there exists a polynomial \( r \) of degree so that

\[
q(t) = (t - a)r(t).
\]

Therefore,

\[
p(t) = (t - a)^2r(t).
\]

Now suppose that \( \langle p, p \rangle = 0 \). Then

\[
\frac{(2x^3 - 3x^2, 1)}{\sqrt{(2x^3 - 3x^2, 2x^3 - 3x^2)}\sqrt{(1, 1)}} = -\frac{1}{\sqrt{2}}.
\]

So the angle between these two polynomials is

\[
\arccos \left( -\frac{1}{\sqrt{2}} \right) = \frac{3\pi}{4}.
\]

(b) Compute the angle between the polynomials \( 2t^3 - 3t^2 \) and \( 1 \).

Answer. A quick computation gives

\[
\frac{(2x^3 - 3x^2, 1)}{\sqrt{(2x^3 - 3x^2, 2x^3 - 3x^2)}\sqrt{(1, 1)}} = -\frac{1}{\sqrt{2}}.
\]

So the angle between these two polynomials is

\[
\arccos \left( -\frac{1}{\sqrt{2}} \right) = \frac{3\pi}{4}.
\]

(c) Use Gram-Schmidt to replace the basis \( \{1, t, t^2, t^3\} \) by an orthonormal set.

Answer. There’s a lot of computation to do this, the result is

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{2t - 1}{\sqrt{10}} & \frac{(t - 1)t}{\sqrt{10}} & \frac{10t^3 - 15t^2 + t + 2}{\sqrt{10}}
\end{bmatrix}
\]

(d) The following polynomials form a very nice basis for \( V \).

\[
e_1 = t(t - 1)^2
\]

\[
e_2 = t^2(t - 1)
\]

\[
e_3 = (2t + 1)(t - 1)^2
\]

\[
e_4 = t^2(1 - 2(t - 1))
\]

Projecting an arbitrary differentiable function \( f \) onto \( V \) yields a linear combination of the basis vectors

\[
a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4.
\]

Describe (simply!) the coefficients \( a_i \) in terms of the function \( f \). The simplicity of this description explains why \( \{e_1, e_2, e_3, e_4\} \) is such a nice basis for \( V \).
**Answer.** In general, for any orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( V \), the projection of an any \( f \) onto \( V \) is obtained by

\[
\text{Proj}_V(f) = \sum_{i=1}^{n} \langle f, e_i \rangle e_i.
\]

Now, each term in the sum above \( \langle f, e_i \rangle \) is a sum of four products

\[e_i(0)f(0) + e_i'(0)f'(0) + e_i(1)f'(1) + e_i'(1)f''(1).\]

Now, here’s what’s so special about the given basis:

- \( e_1'(0) = 1 \) and \( e_1(0) = e_1'(1) = 0 \)
- \( e_2'(1) = 1 \) and \( e_2(0) = e_2'(0) = 0 \)
- \( e_3(0) = 1 \) and \( e_3(1) = e_3'(0) = e_3'(1) = 0 \)
- \( e_4(1) = 1 \) and \( e_4(0) = e_4'(0) = e_4'(1) = 0 \)

So, instead of a complicated express for \( \langle f, e_i \rangle \), we have simply

\[
\langle f, e_1 \rangle = f'(0), \quad \langle f, e_2 \rangle = f'(1), \quad \langle f, e_3 \rangle = f(0), \quad \langle f, e_4 \rangle = f(1).
\]

So,

\[
\text{Proj}_V(f) = f'(0)e_1 + f'(1)e_2 + f(0)e_3 + f(1)e_4.
\]

**Problem 5.** Each of the pictures below shows a polynomial approximation for \( \cos(2t) \). Five of them are degree four polynomials obtained by projection onto the subspace of quartic polynomials, using one of the following inner products.

\[
\langle f, g \rangle_1 = f(2)g(2) + f'(2)g'(2) + f''(2)g''(2) + f'''(2)g'''(2)
\]

\[
\langle f, g \rangle_2 = \int_0^4 f(t)g(t)dt
\]

\[
\langle f, g \rangle_3 = f(0)g(0) + f(1)g(1) + f(2)g(2) + f(3)g(3) + f(4)g(4)
\]

\[
\langle f, g \rangle_4 = f(0)g(0) + f(2)g(2) + f'(2)g'(2) + f''(2)g''(2) + f'''(2)g'''(2) + f(4)g(4)
\]

\[
\langle f, g \rangle_5 = f(0)g(0) + f'(0)g'(0) + f(2)g(2) + f(4)g(4) + f'(4)g'(4)
\]

The sixth approximation is obtained by “gluing together” two degree two polynomials defined on \([0, 2]\) and \([2, 4]\), each determined by projection using, respectively, the inner products

\[
\langle f, g \rangle_6 = f(0)g(0) + f(2)g(2) + f'(2)g'(2)
\]

\[
\langle f, g \rangle_7 = f(2)g(2) + f'(2)g'(2) + f(4)g(4)
\]
The answers are:

\[
\begin{array}{c|c|c|c}
\hline
5 & 1 & 2 \\
2 & 3 & 6/7 \\
\hline
\end{array}
\]