Correspondences obtained by using ordered bases

Using an ordered basis to represent a vector as columns of scalars

Using an ordered basis $B = \{b_1, \ldots, b_n\}$ for a vector space V, there is a bijective correspondence between the vectors in V and vectors in $\mathbb{R}^n$. I will remind you of this correspondence and also define some convenient terminology.

Any vector $v \in V$ can be expressed uniquely as $v = \sum_{i=1}^n \alpha_i b_i$ where $\alpha_i \in \mathbb{R}$. Then v corresponds to the column vector $\mathrm{Rep}_B(v) \in \mathbb{R}^n$ defined by

$$\mathrm{Rep}_B(p(x)) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

The correspondence

$$\begin{aligned} V &\longleftrightarrow \mathbb{R}^n \\ v &\longleftrightarrow \mathrm{Rep}_B(v) \end{aligned}$$

is one-to-one since every vector in the space can be expressed as a linear combination of the basis vectors uniquely and the corrspondence is onto since every linear combination of the basis vectors is a vector in the space.

Note that the correspondence depends on the choice of ordered basis B, and that dependence is reflected in the notation $\mathrm{Rep}_B(v)$.

Problem 1. Let V be the space of polynomials of degree less than or equal to four. The set

$$B = \{1 + x, 1 - x, 1 - 2x^2, 1 + x - x^3, x^2 - x^4\}$$

is a basis for V. Use the ordered basis B to obtain a correspondence between $V \leftrightarrow \mathbb{R}^5$.

(a) Give the polynomials in V that correspond to the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

Answer.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ means } 1e_1 = 1 + x.$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \text{ means } 1e_1 - 1e_4 = 1 + x - (1 + x - x^3) = x^3.$$

$$\begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 1 \end{pmatrix} \text{ means } 1e_1 - 2e_3 - e_4 + e_5 = 1 + x - 2(1 - 2x^2) - (1 + x - x^3) + x^2 - x^4 = -2 + 5x^2 + x^3 - x^4.$$

(b) Give the vectors in $\mathbb{R}^5$ that correspond to the polynomials

$$x^3, \quad 3x^2, \quad 2x^4 - 7, \text{ and } 4x^4 - 14$$

Answer. From above, we know

$$x^3 \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

To find the vector that corresponds to $3x^2$, write

$$\begin{aligned} 3x^2 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 \\ &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + (\alpha 1 - \alpha 2 + \alpha_4)x + (\alpha_5 - 2\alpha_3)x^2 - \alpha_4 x^3 - \alpha_5 x^4. \end{aligned}$$

Since $3x^2$ has no x^3 or x^4 terms, $\alpha_5 = \alpha_4 = 0$. So

$$3x^2 = \alpha_1 + \alpha_2 + \alpha_3 + (\alpha 1 - \alpha 2)x - 2\alpha_3 x^2$$

So, $3 = -2\alpha_3 \Rightarrow \alpha_3 = -\frac{3}{2}$. Then, setting the constant and linear terms equal to zero yields

$$\alpha_1 + \alpha_2 = -\frac{3}{2} = 0 \text{ and } \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \frac{3}{4} \text{ and } \alpha_2 = \frac{3}{4}.$$

So,

$$3x^2 \leftrightarrow \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ -\frac{3}{2} \\ 0 \\ 0 \end{pmatrix}$$

To express $2x^4 - 7 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5$, note that $-2e_5 - e_3 = 2x^4 - 1$. Also note that $(e_1 + e_2) = 2 \Rightarrow -3e_1 - 3e_2) = -6$. So, $-2e_5 - e_3 - 3e_1 - 3e_2 = 2x^4 - 7$.

So,

$$2x^4 - 7 \leftrightarrow \begin{pmatrix} -3 \\ -3 \\ -1 \\ 0 \\ -2 \end{pmatrix}$$

Finally, since $4x^4 - 14 = 2(2x^4 - 7)$, we have

$$4x^4 - 14 \leftrightarrow \begin{pmatrix} -6 \\ -6 \\ -2 \\ 0 \\ -4 \end{pmatrix}$$

Using ordered bases to represent linear transformations as a matrix of scalars

Given an ordered basis $B_V = \{v_1, \dots, v_k\}$ of a vector space V and an ordered basis $B_W = \{w_1, \dots, w_n\}$ of W, there's a bijective correspondence between linear transformations $T : V \to W$ and $n \times k$ matrices of scalars, defined as follows:

Given a linear transformation $T : V \to W$, define a matrix $\mathrm{Rep}_{B_V, B_W}(T)$ by setting the i-th column to be $\mathrm{Rep}_{B_W}(T(v_i))$.

To put it another way, for each basis vector $v_i \in B_V$, the vector $T(v_i)$ is an element of W and so it can be expressed uniquely in terms of the basis B_W:

$$T(v_i) = \sum_{j=1}^{n} a_{ij} w_j.$$

The numbers a_{ij} for $i = 1, \dots, k$ and $j = 1, \dots, n$ define the $n \times k$ matrix

$$\mathrm{Rep}_{B_V, B_W}(T) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix}.$$

Notice, the i-th column of the matrix $\mathrm{Rep}_{B_V, B_W}(T)$ is the column of scalars

$$\begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} = \mathrm{Rep}_{B_W}(T(v_i)).$$

Problem 2. As before, let V be the space of polynomials of degree less than or equal to four and let

$$B = \{1 + x, 1 - x, 1 - 2x^2, 1 + x - x^3, x^2 - x^4\}$$

be a basis for V. The function $T : V \to V$ defined by $T(p) = p'(t) + p(0)$ defines a linear transformation.

(a) Using the correspondence $V \longleftrightarrow \mathbb{R}^5$, express T as a matrix $\mathrm{Rep}_{B,B}(T)$.

Answer. Compute T of each basis vector:

$$\begin{aligned}
Te_1 &= 2 = e_1 + e_2 \\
Te_2 &= 0 \\
Te_3 &= -4x + 1 = \frac{3}{2}e_1 - \frac{5}{2}e_2 \\
Te_4 &= 2 - 3x^2 = \frac{1}{4}e_1 + \frac{1}{4}e_2 + \frac{3}{2}e_3 \\
Te_5 &= 2x - 4x^3 = -3e_1 - e_2 + 4e_4
\end{aligned}$$

Thus,

$$\mathrm{Rep}_{B,B}(T) = \begin{pmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{4} & -3 \\ 1 & 0 & -\frac{5}{2} & \frac{1}{4} & -1 \\ 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Compute the polynomial $T(x^3)$ and compute the following product of a matrix with a vector:

$$\mathrm{Rep}_{B,B}(T) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

Answer. First, note that $T(x^3) = 3x^2$. Also that $\mathrm{Rep}_B(x^3) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$. Computing the product

$$\mathrm{Rep}_{B,B}(T) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{4} & -3 \\ 1 & 0 & -\frac{5}{2} & \frac{1}{4} & -1 \\ 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ -\frac{3}{2} \\ 0 \\ 0 \end{pmatrix}.$$

Note that the result $\begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ -\frac{3}{2} \\ 0 \\ 0 \end{pmatrix}$ is the vector that corresponds to $3x^2$. That is,

$$\mathrm{Rep}_{B,B}(T)\,\mathrm{Rep}_B(x^3) = \mathrm{Rep}_B(T(x^3)).$$

(c) Let $B' = \{1, t, t^2, t^3, t^4\}$ and compute the matrix $\operatorname{Rep}_{B,B'}(T)$. Here, let $\{1, t, t^2, t^3, t^4\} = \{b_1, b_2, b_3, b_4, b_5\}$ and express

$$\begin{aligned}
Te_1 &= 2 = 2b_1 \\
Te_2 &= 0 \\
Te_3 &= 1 - 4x = b_1 - 4b_2 \\
Te_4 &= 2 - 3x^2 = 2b_1 - 3b_3 \\
Te_5 &= 2x - 4x^3 = 2b_2 - 4b_4
\end{aligned}$$

$$\operatorname{Rep}_{B,B'}(T) = \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As a consistency check, we compute

$$\operatorname{Rep}_{B,B'}(T) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

which is the representatative of $3x^2$ in the basis B'. Thus,

$$\operatorname{Rep}_{B,B'}(T) \operatorname{Rep}_B(x^3) = \operatorname{Rep}_{B'}(T(x^3))$$

as expected.

(d) Find a basis B'' of V so that the matrix $\operatorname{Rep}_{B,B''}(T)$ is *diagonal* with either 1 or 0 on the diagonal.

Answer. Here's one solution. Let $B'' = \{2, x^4, 1 - 4x, 2 - 3x^2, 4x^3\}$ then

$$\operatorname{Rep}_{B,B''}(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Composition of linear transformations

Problem 3. This is a variation of problems 28 and 29 in section 2.8 of the Apostol's book. Let V be the space of all real polynomials. Define linear transformations by

$$\begin{aligned} D &: V \to V & S &: V \to V & T &: V \to V \\ p(t) &\mapsto p'(t) & p(t) &\mapsto tp(t) & p(t) &\mapsto tp'(t) \end{aligned}$$

(a) Let $p(t) = 2 + 3t - t^2 + 4t^3$ Determine the image of p under

$$D, S, T, DT, TD, DS, SD, ST, TS, DT - TD, T^2D^2 - D^2T^2.$$

Answer. For SD and DS, see the part (d) of this problem. For the others, here are some computations:

$$Dp = 3 - 2t + 12t^2, \quad Sp = 2t + 3t^2 - t^3 + 4t^4, \quad Tp = 3t - 2t^2 + 12t^3.$$

$$DTp = D(3t-2t^2+12t^3) = 3-4t+36t^2, \quad TDp = T(3-2t+12t^2) = t(-2+24t) = -2t+24t^2$$

$$DSp = D(2t+3t^2-t^3+4t^4) = 2+6t-3t^2+16t^3, \quad SDp = S(3-2t+12t^2) = 3t-2t^2+12t^3.$$

et cetera. Note that $DT - TD = (3 - 4t + 36t^2) - (-2t + 24t^2) = 3 - 2t + 12t^2 = Dp$. For the last one:

$$\begin{aligned}
(T^2D^2 - D^2T^2)p &= T^2D^2p - D^2T^2p \\
&= T^2p'' - D^2T(tp') \\
&= T(tp''') - D^2(t(p' + tp'')) \\
&= T(tp''') - D^2(tp' + t^2p'') \\
&= t(p''' + t^2p'''') - D(p' + tp'' + 2tp'' + t^2p''') \\
&= tp''' + t^3p'''' - (p'' + p'' + tp''' + 2p'' + 2tp''' + 2tp''' + t^3p'''') \\
&= -4p'' - 4tp'''
\end{aligned}$$

So

$$(T^2D^2 - D^2T^2)(2 + 3t - t^2 + 4t^3) = -4(-2 + 24t) - 4t(24) = 8 - 192t.$$

(b) Find the kernel of T, $T - \mathrm{Id}$, and $DT - 2D$.

Answer. If $p \in \ker(T)$ then $tp' = 0 \Rightarrow p' = 0 \Rightarrow p = c$ for some constant c. Thus $\ker(T)$ is the space of constant polynomials.

If $p \in \ker(T - \mathrm{Id})$ then $tp' - p = 0 \Rightarrow tp' = p \Rightarrow p(t) = ct$ for some constant t. Thus, $\ker(T - \mathrm{Id})$ is the one dimensional subspace of polynomials of the form ct for some constant c; i.e., the span of $\{t\}$.

A quick check reveals that $(DT - 2D)(p) = tp'' - p'$, so to determine for which polynomials p, we have $(DT - 2D)(p) = 0$, write $p(t) = c_0 + c_1t + c2_t^2 + \cdots + c_nt^n$. Then

$$\begin{aligned}
tp''(t) - p'(t) &= 2c_2t + 6c_3t^2 + \cdots (n)(n-1)c_nt^{n-1} - \left(c_1 + 2c_2t + 3c_3t^2 + \cdots + nc_nt^{n-1}\right) \\
&= -c_1 + 3c_2t^2 + \cdots + (n)(n-2)c_nt^{n-1}
\end{aligned}$$

Setting $tp''(t) - p'(t) = 0 \Leftrightarrow c_1 = c_3 = \cdots = c_n = 0$. Thus, the kernel of $DT - 2D$ consists of all polynomials of the form $p(t) = c_0 + c_2t^2$; i.e. the span of $\{1, t^2\}$.

(c) Does D have a left inverse or a right inverse?

Answer. Since D has a nontrivial kernel, it does not have a left inverse. To see this, let p be the constant polynomial $p(t) = 1$. For any linear map $F : V \to V$, we have

$$FD(p) = F(0) = 0 \neq p$$

and so $FD \neq \mathrm{Id}$ for any linear $F : V \to V$.

The map D does have a right inverse. Here's one: Let $I : V \to V$ be defined by $I(p) = \int_0^t p(s)ds$. Then

$$DI(p(t)) = D \int_0^t p(s)ds = p(t).$$

The last equality follows from the fundamental theorem of calculus. So $DI = \mathrm{Id}$ and we see that I is a right inverse of D.

The fact that D has a set-theoretic right inverse says that D is onto; i.e. every polynomial is the derivative of another polynomial (certainly true — every polynomial has many antiderivatives). However, to say that D has a right inverse that is a linear transformation says that an antiderivative for each polynomial can be chosen so the totality of choices defines a linear map (this is only true if we choose the "constant of integration" to be zero).

(d) Let $S : V \to V$ be defined by $p(t) \mapsto tp(t)$. Compute $DS - SD$ and, more generally, $DS^n - S^nD$.

Answer. $DS(p) = D(tp) = p + tp'$, $SD(p) = S(p') = tp'$, and $(DS - SD)(p) = p + tp' - tp' = p$. So $DS - SD = \mathrm{Id}$.

More generally, $DS^n(p) = t^np = nt^{n-1}p + t^np'$, $S^nD(p) = S^n(p') = t^np'$, and $(DS^n - S^nD)(p) = nt^{n-1}p$.

Inner products and best approximation

Problem 4. Let V be the space of polynomials of degree less than or equal to three and let

$$\langle p, q\rangle = p(0)q(0) + p'(0)q'(0) + p(1)q(1) + p'(1)q'(1).$$

(a) Prove that $\langle p, q\rangle$ defines an inner product on V. In particular, prove that if $\langle p, p\rangle = 0$ then $p = 0$. *Hint:* first prove the lemma that if p is a polynomial and a is a number for which $p(a) = 0$ and $p'(a) = 0$, then there exists a polynomial q so that $p(x) = (x - a)^2q(x)$.

Answer. It's straightforward to check that $\langle\,,\,\rangle$ as defined is symmetric and bilinear, and it's clear that as a sum of squares, $\langle p, p\rangle \geq 0$ for all $p \in V$. The one property that's not straightforward to check is that $\langle p, p\rangle = 0$ only for the zero polynomial. To check this, we first prove the lemma in the hint.

Lemma. If p is a polynomial and a is a number for which $p(a) = 0$ and $p'(a) = 0$, then there exists a polynomial q so that $p(t) = (t - a)^2q(t)$.

Proof. First, recall that if p is any polynomial and a is a number for which $p(a) = 0$, then there exists a polynomial q so that that

$$p(t) = (t - a)q(t).$$

Now assume that for a particular polynomial p we have $p(a) = 0$ and $p'(a) = 0$. Since $p(a) = 0$ then there exists a polynomial q with $p(t) = (t - a)q(t)$. Note that $p'(t) = (t-a)q'(t)+q(t)$. Since $p'(a) = 0$ we have $0 = p'(a) = (a-a)q'(a)+q(a) = q(a)$. Since q is a polynomial for which $q(a) = 0$, there exists a polynomial r of degree so that $q(t) = (t - a)r(t)$. Therefore,

$$p(t) = (t - a)^2 r(t).$$

□

Now suppose that $\langle p, p\rangle = 0$. Then $p(0)^2 + p'(0)^2 + p(1)^2 + p'(1)^2 = 0 \Rightarrow p(0) = p'(0) = p(1) = p'(1) = 0$. Since $p(0) = 0$ and $p'(0) = 0$, we know $p(t)$ is divisible by t^2. Since $p(1) = p'(1) = 0$ we know $p(t)$ is divisible by $(t - 1)^2$. The only polynomial of degree less than or equal to three that's divisible by both t^2 and $(t - 1)^2$ is zero.

(b) Compute the angle between the polynomials $2t^3 - 3t^2$ and 1.

Answer. A quick computation gives

$$\frac{\langle 2x^3 - 3x^2, 1\rangle}{\sqrt{\langle 2x^3 - 3x^2, 2x^3 - 3x^2\rangle}\sqrt{\langle 1, 1\rangle}} = -\frac{1}{\sqrt{2}}.$$

So the angle between these two polynomials is

$$\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}.$$

(c) Use Gram-Schmidt to replace the basis $\{1, t, t^2, t^3\}$ by an orthogonormal set.

Answer. There's a lot of computation to do this, the result is

$$\left\{\frac{1}{\sqrt{2}}, \frac{2t - 1}{\sqrt{10}}, \frac{(t - 1)t}{\sqrt{2}}, \frac{10t^3 - 15t^2 + t + 2}{\sqrt{10}}\right\}$$

(d) The following polynomials form a very nice basis for V.

$$\begin{aligned} e_1 &= t(t - 1)^2 \\ e_2 &= t^2(t - 1) \\ e_3 &= (2t + 1)(t - 1)^2 \\ e_4 &= t^2(1 - 2(t - 1)) \end{aligned}$$

Projecting an arbitrary differentiable function f onto V yields a linear combination of the basis vectors

$$a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4.$$

Describe (simply!) the coefficients a_i in terms of the function f. The simplicity of this description explains why $\{e_1, e_2, e_3, e_4\}$ is such a nice basis for V.

Answer. In general, for any orthonormal basis $\{e_1, \ldots, e_n\}$ of V, the projection of an any f onto V is obtained by

$$\text{Proj}_V(f) = \sum_{i=1}^{n} \langle f, e_i \rangle e_i.$$

Now, each term in the sum above $\langle f, e_i \rangle$ is a sum of four products

$$e_i(0)f(0) + e_i'(0)f'(0) + e_i(1)f'(1) + e_i'(1)f'(1).$$

Now, here's what's so special about the given basis:

$$\begin{aligned}
e_1'(0) = 1 &\text{ and } e_1(0) = e_1(1) = e_1'(1) = 0 \\
e_2'(1) = 1 &\text{ and } e_2(0) = e_2(1) = e_2'(0) = 0 \\
e_3(0) = 1 &\text{ and } e_3(1) = e_3'(0) = e_3'(1) = 0 \\
e_4(1) = 1 &\text{ and } e_4(0) = e_4'(0) = e_4'(1) = 0
\end{aligned}$$

So, instead of a complicated express for $\langle f, e_i \rangle$, we have simply

$$\langle f, e_1 \rangle = f'(0), \quad \langle f, e_2 \rangle = f'(1), \quad \langle f, e_3 \rangle = f(0), \quad \langle f, e_4 \rangle = f(1).$$

So,

$$\text{Proj}_V(f) = f'(0)e_1 + f'(1)e_2 + f(0)e_3 + f(1)e_4.$$

Problem 5. Each of the pictures below shows a polynomial approximation for $\cos(2t)$. Five of them are degree four polynomials obtained by projection onto the subspace of quartic polynomials, using one of the following inner products.

$$\begin{aligned}
\langle f, g \rangle_1 &= f(2)g(2) + f'(2)g'(2) + f''(2)g''(2) + f'''(2)g'''(2) \\
\langle f, g \rangle_2 &= \int_0^4 f(t)g(t)dt \\
\langle f, g \rangle_3 &= f(0)g(0) + f(1)g(1) + f(2)g(2) + f(3)g(3) + f(4)g(4) \\
\langle f, g \rangle_4 &= f(0)g(0) + f(2)g(2) + f'(2)g'(2) + f''(2)g''(2) + f(4)g(4) \\
\langle f, g \rangle_5 &= f(0)g(0) + f'(0)g'(0) + f(2)g(2) + f(4)g(4) + f'(4)g'(4)
\end{aligned}$$

The sixth approximation is obtained by "gluing together" two degree two polynomials defined on $[0, 2]$ and $[2, 4]$, each determined by projection using, respectively, the inner products

$$\begin{aligned}
\langle f, g \rangle_6 &= f(0)g(0) + f(2)g(2) + f'(2)g'(2) \\
\langle f, g \rangle_7 &= f(2)g(2) + f'(2)g'(2) + f(4)g(4)
\end{aligned}$$

Answer. The answers are

5	1
2	3
6/7	4