Problem 1. In quantum mechanics one considers the three matrices σ^x, σ^y, and σ^z.

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In quantum computing theory, these matrices represent fundamental operations on a single quantum bit of quantum information. For example, σ^x is the "Quantum NOT" operation.

(a) Compute the eight products: $\sigma^i \sigma^j$ for $i, j = x, y, z$.

(b) Find the eigenvalues of σ^x, σ^y, and σ^z.

Problem 2. Consider the inner product space $(V, \langle\, , \,\rangle)$ where V is the space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = f(x+1)$ for all $x \in \mathbb{R}$ and $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Let $\sigma : V \to V$ defined by $\sigma(f) = f'$.

(a) Prove that σ is a skew-symmetric linear operator.

(b) Prove that σ^2 is a negative semi-definite symmetric operator.

Recall: A linear operator τ on an inner product space V is negative semi-definite if and only if $\langle \tau(v), v \rangle \leq 0$ for all $v \in V$.

The Matrix Exponential

Definition 1. For any $n \times n$ complex matrix A, define the matrix e^A by

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{1}$$

One can prove that the series defined above converges absolutely for all matrices A.

Problem 3. Compute e^A and e^B for

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3i & 0 & -2+i \\ 3-2i & -2 & 1+2i \\ 0 & 0 & 6 \end{bmatrix}.$$

Problem 4. Prove that if A is similar to B then e^A is similar to e^B.

Problem 5. Prove that $e^{\text{trace}(A)} = \det(e^A)$.