Problem 1. In quantum mechanics one considers the three matrices $\sigma^x, \sigma^y, \text{and } \sigma^z$.

\[
\begin{align*}
\sigma^x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\sigma^y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
\sigma^z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{align*}
\]

In quantum computing theory, these matrices represent fundamental operations on a single quantum bit of quantum information. For example, $\sigma^x$ is the “Quantum NOT” operation.

(a) Compute the eight products: $\sigma^i \sigma^j$ for $i, j = x, y, z$.

(b) Find the eigenvalues of $\sigma^x, \sigma^y, \text{and } \sigma^z$.

Problem 2. Consider the inner product space $(V, (\cdot, \cdot))$ where $V$ is the space of all infinitely differentiable functions $f: \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = f(x + 1)$ for all $x \in \mathbb{R}$ and $(f, g) = \int_0^1 f(x)g(x)dx$. Let $\sigma: V \to V$ defined by $\sigma(f) = f'$.

(a) Prove that $\sigma$ is a skew-symmetric linear operator.

(b) Prove that $\sigma^2$ is a negative semi-definite symmetric operator.

Recall: A linear operator $\tau$ on an inner product space $V$ is negative semi-definite if and only if $(\tau(v), v) \leq 0$ for all $v \in V$.

The Matrix Exponential

Definition 1. For any $n \times n$ complex matrix $A$, define the matrix $e^A$ by

\[
e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{1}
\]

One can prove that the series defined above converges absolutely for all matrices $A$.

Problem 3. Compute $e^A$ and $e^B$ for

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3i & 0 & -2 + i \\ 3 - 2i & -2 & 1 + 2i \\ 0 & 0 & 6 \end{bmatrix}.
\]

Problem 4. Prove that if $A$ is similar to $B$ then $e^A$ is similar to $e^B$.

Problem 5. Prove that $e^\text{trace}(A) = \det(e^A)$. 