The problem from the previous homework

Problem 1. Compute e^B for

$$B = \begin{bmatrix} 3i & 0 & -2+i \\ 3-2i & -2 & 1+2i \\ 0 & 0 & 6 \end{bmatrix}.$$

For a little more practice (and it will be useful later!) compute e^A for

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Jordan canonical form

Problem 2. Find the characteristic polynomials, minimal polynomials, and Jordan canonical forms for the matrices below. First give the answer as efficiently as you can, then do all the details by finding a matrix P so that conjugating by P is the Jordan form.

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 3 & 1 & 1 \\ 2 & 2 & 2 & -2 \\ -1 & -1 & 0 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 0 & 1 & 3 \\ 0 & 2 & 0 & -1 \\ 2 & 2 & 2 & -2 \\ -1 & -1 & 0 & 3 \end{pmatrix}$$

Systems of differential equations

Theorem 1. *Let A be any $n \times n$ matrix of scalars. Then columns of e^{tA} form a basis for the solution space of the matrix differential equation $X'(t) = AX(t)$.*

Problem 3. Suppose that $A(t)$ and $B(t)$ are differentiable matrix valued functions of t. Prove the product rule:

$$\frac{d}{dt}\left(A(t)B(t)\right) = \frac{dA(t)}{dt}B(t) + A(t)\frac{dB(t)}{dt}.$$

Problem 4. Prove that for any A, $\frac{d}{dt}e^{tA} = Ae^{tA}$.

Problem 5. Prove Theorem 1. Use the result of problem 4 to prove that the columns of e^{tA} are solutions and use the product rule from problem 3 to prove uniqueness.

Problem 6. Solve $X'(t) = AX(t)$ for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

Problem 7. The second order (meaning two derivatives) differential equation $y''(t)+y(t) = 0$ is equivalent to the first order system of equations

$$Y'(t) = AY(t) \text{ for } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Solve this system of equations.

Problem 8. Here's an extra problem. Solve the two second order differential equations:

$$y_2'' = y_2 - 2y_1'$$
$$y_1'' = y_1 + 2y_2'$$

subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = 1.$$

By translating them into a single system of the form $Y'(t) = AY(t)$ for a 4×4 matrix A.