Problem 1. Let $\{e_1, \ldots, e_n\}$ be a basis for a vector space V and let $x \in V$. Prove that if

$$x = \alpha_1 e_1 + \cdots \alpha_n e_n \text{ and } x = \beta_1 e_1 + \cdots + \beta_n e_n$$

then $\alpha_i = \beta_i$ for $i = 1, \ldots, n$.

Problem 2. Determine which of the following are vector spaces

In Exercises 1 through 28, determine whether each of the given sets is a real linear space, if addition and multiplication by real scalars are defined in the usual way. For those that are not, tell which axioms fail to hold. The functions in Exercises 1 through 17 are real-valued. In Exercises 3, 4, and 5, each function has domain containing 0 and 1. In Exercises 7 through 12, each domain contains all real numbers.

1. All rational functions.
2. All rational functions f/g, with the degree off $\leq$ the degree ofg (including $f = 0$).
3. All f with $f(0) = f(1)$.
4. All f with $2f(0) = f(1)$.
5. All f with $f(1) = 1 + f(O)$.
6. All step functions defined on [0, 1].
7. All f with $f(x) \to 0$ as $x \to +\infty$.
8. All even functions.
9. All odd functions.
10. All bounded functions.
11. All increasing functions.
12. All functions with period 2π.
13. All f integrable on [0, 1] with $\int_0^1 f(x)\,dx = 0$.
14. All f integrable on [0, 1] with $\int_0^1 f(x)\,dx \geq 0$.
15. All f satisfying $f(x) = f(1 - x)$ for all x.
16. All Taylor polynomials of degree $\leq n$ for a fixed n (including the zero polynomial).
17. All solutions of a linear second-order homogeneous differential equation' y" + $P(x)y'$ + $Q(x)y$ = 0, where P and Q are given functions, continuous everywhere.
18. All bounded real sequences.
19. All convergent real sequences.
20. All convergent real series.
21. All absolutely convergent real series.
22. All vectors (x, y, z) in V_3 with z = 0.
23. All vectors (x, y, z) in V_3 with x = 0 or y = 0.
24. All vectors (x, y, z) in V_3 with y = 5x.
25. All vectors (x, y, z) in V_3 with 3x + $4y$ = 1, z = 0.
26. All vectors (x, y, z) in V_3 which are scalar multiples of (1, 2, 3).
27. All vectors (x, y, z) in V_3 whose components satisfy a system of three linear equations of the form :

$$a_{11}x + a_{12}y + a_{13}z = 0, \qquad a_{21}x + a_{22}y + a_{23}z = 0, \qquad a_{31}x + a_{32}y + a_{33}z = 0.$$