Curves

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Let I be an open interval in $\mathbb{R}$ and let $\beta : I \to \mathbb{R}^3$ be a smooth unit speed curve. By *smooth* I mean that the component functions β_i, $i = 1, 2, 3$, are infinitely differentiable. By *unit speed* I mean that $\|\beta'(s)\| = 1$ for all $s \in I$.

Definition 1. We call β' the tangent vector of β and denote it by T. The fuction $\kappa : I \to \mathbb{R}$ defined by $\kappa(s) = \|T'(s)\| = \|\beta''(s)\|$ is called the *curvature* of β.

The curvature κ measures how β differs from a straight line.

Theorem 1. *β is a part of straight line iff $\kappa = 0$.*

Proof. If $\kappa = 0$ then $\beta_i''(s) = 0$ for each $s \in I$ and $i = 1, 2, 3$. Therefore, $\beta_i(s) = a_i + b_i s$ for some fixed $a_i, b_i \in \mathbb{R}$. Therefore, $\beta = a + bs$ where $a = (a_1, a_1, a_3)$ and $b = (b_1, b_2, b_3)$ and we see that β is part of the straight line through the points a and b in $\mathbb{R}^3$. $\square$

Curves with $\kappa = 0$ are perfectly straight. If β is not part of a straight line, then we may go on to define further invariants, such as the normal to the curve. From now on, assume that $\kappa > 0$.

Definition 2. We call $\frac{1}{\kappa}T'$ the normal vector of β and denote it by N. We call $T \times N$ the binormal vector of β and denote it by B.

For each $s \in I$, the vectors $T(s), N(s)$, and $B(s)$ define an orthonormal basis of $\mathbb{R}^3$. The tangent vector T is assumed to be normal—that's the meaning of β being unit speed. Then N is defined to be the normalized derivative of T. The normal N and tangent T are orthogonal since $\langle T, T\rangle = 1 \Rightarrow \langle T', T\rangle = 0 \Rightarrow \langle N, T\rangle = 0$. Because B is the cross-product of T and N, B is orthogonal to both T and N and since both T and N have unit length, so does B. Picture the vectors $T(s), N(s)$, and $B(s)$ as defining an orthonormal basis of $\mathbb{R}^3$ at the point $\beta(s)$ and so as the parameter s varies, T, N, and B defines a moving frame along the curve β.

It turns out that the derivatives of T, N, and B satisfy special conditions:

Theorem 2. *There is a unique function $\tau : I \to \mathbb{R}$, called the* torsion *of β, so that*

$$\begin{aligned} T' &= \quad\quad\quad \kappa N \\ N' &= -\kappa T \quad\quad\quad\quad +\tau B \\ B' &= \quad\quad\quad -\tau N \end{aligned}$$

Sketch of proof. The way to get information is to differentiate the orthogonality equations and the normality equations. First show that B' is a mutliple of N. This defines τ ... ☐

The torsion τ measures how β twists in space, how near to planar β is.

Theorem 3. *β is a plane curve line iff $\tau = 0$.*

Proof. First observe that a plane is determined by a point $p \in \mathbb{R}^3$ (which is in the plane) and a unit vector v (which is orthogonal to the plane) via

$$q \text{ lies in the plane} \quad \Longleftrightarrow \quad \langle p - q, v\rangle = 0.$$

Now, suppose that β is a plane curve. Then there exist fixed $p, v \in \mathbb{R}^3$ with $\|v\| = 1$ so that $\langle \beta(s) - p, v\rangle = 0$ for all s. Differentiating a couple of times gives $\langle \beta'(s), v\rangle = 0$ and $\langle \beta''(s), v\rangle = 0$. Therefore, v is orthogonal to $T(s)$ and $N(s)$ for all s. This implies that $v = B(s)$ or $v = -B(s)$, either way we find that B is constant. Therefore, $B'(s) = 0 \Rightarrow \tau(s) = 0$.

Conversely, suppose that $\tau = 0$. Then $B'(s) = 0$ and B is constant. Now, fix one point $\beta(a)$ on the curve β and look at the plane that contains $\beta(a)$ and is orthogonal to B. To show that $\beta(s)$ is in the plane for all s, consider $\langle \beta(s) - \beta(a), B\rangle$. Differentiating with respect to s Give $\langle \beta'(s), B\rangle = \langle T, B\rangle = 0$. Therefore, the scalar function $\langle \beta(s) - \beta(a), B\rangle$ is constant.. Since this constant is zero when $s = a$, we must have $\langle \beta(s) - \beta(a), B\rangle = 0$ for all s, as needed. ☐

Theorem 4. *Suppose $\tau = 0$. Then β is part of a circle iff κ is constant.*

Proof. Exercise. ☐

Theorem 5. *Suppose that* $\beta_1, \beta_2 : I \to \mathbb{R}^3$ *are two smooth, unit speed curves for which* $\kappa_1, \kappa_2 > 0$*. Then, there exists a unique orthogonal transformation* $O : \mathbb{R}^3 \to \mathbb{R}^3$ *and a point* $p \in \mathbb{R}^3$ *such that* $\beta_1 = O \circ \beta_2 + p$ *iff* $\kappa_1 = \kappa_2$ *and* $\tau_1 = \pm\tau_2$*.*

Lemma 1. *Suppose that* $\beta_1, \beta_2 : I \to \mathbb{R}^3$ *are two smooth, unit speed curves that have the same curvature and the same torsion everywhere. If their Frenet frames agree at one point, then they are identical everywhere.*

Proof. To show that $T_1 = T_2$, $N_1 = N_2$, and $B_1 = B_2$, we differentiate $\langle T_1, T_2\rangle + \langle N_1, N_2\rangle + \langle B_1, B_2\rangle$ to get

$$\begin{aligned}
&\langle T_1', T_2\rangle + \langle T_1, T_2'\rangle + \langle N_1', N_2\rangle + \langle N_1, N_2'\rangle + \langle B_1', B_2\rangle + \langle B_1, B_2'\rangle \\
=&\kappa\langle N_1, T_2\rangle + \kappa\langle T_1, N_2\rangle - \kappa\langle T_1, N_2\rangle + \tau\langle B_1, N_2\rangle \\
&\qquad - \kappa\langle N_1, T_2\rangle + \tau\langle N_1, B_2\rangle - \tau\langle N_1, B_2\rangle - \tau\langle B_1, N_2\rangle \\
=&0
\end{aligned}$$

This proves that the scalar function $\langle T_1, T_2\rangle + \langle N_1, N_2\rangle + \langle B_1, B_2\rangle$ is constant. If $T_1(a) = T_2(a)$, $N_1(a) = N_2(a)$, and $B_1(a) = B_2(a)$ for some point $a \in I$, then $\langle T_1(a), T_2(a)\rangle + \langle N_1(a), N_2(a)\rangle + \langle B_1(a), B_2(a)\rangle = 3$ and we must have

$$\langle T_1(s), T_2(s)\rangle + \langle N_1(s), N_2(s)\rangle + \langle B_1(s), B_2(s)\rangle = 3 \text{ for all } s \in I.$$

The Schwarz inequality finishes the proof since, for example, $\langle N_1, N_2\rangle \leq \|N_1\|\|N_2\| = 1$ with equality iff $N_1 = N_2$. □

Proof of Theorem 5. First, some preliminaries. For any function $f : \mathbb{R} \to \mathbb{R}^3$, and any linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$, $(S \circ f)' = S{\circ}f'$. By definition, if S is orthogonal, $\langle S(v), S(w)\rangle = \langle v, w\rangle$. Note, in addition, that if S is orthogonal, then $\|S(v)\| = \|v\|$ and $S(v \times w) = \pm\,(S(v) \times S(w))$.

Suppose that $\beta_1 = O \circ \beta_2 + p$ for an orthogonal linear transformation O and a fixed p. Differentiating $\beta_1 = O \circ \beta_2 + p$ once gives

$$T_1 = O(T_2). \tag{1}$$

Differentiating again gives

$$\kappa_1 N_1 = \kappa_2 O(N_2). \tag{2}$$

Because O is orthogonal, $\|O(N_2)\| = \|N_2|$, implying that $\kappa_1 = \kappa_2$. Since $\kappa_1, \kappa_2 \neq 0$, Equation (2) says

$$N_1 = O(N_2). \tag{3}$$

From this it follows that $T_1 \times N_1 = O(T_2) \times O(N_2) = \pm O(T_2 \times N_2)$. Therefore

$$B_1 = \pm O(B_2). \tag{4}$$

Differentiating Equation 4 gives that $\tau_1 = \pm\tau_2$.

Conversely, suppose $\kappa_1 = \kappa_2$ and $\tau_1 = \tau_2$. (The case that $\tau_1 = -\tau_2$ is basically the same.) Let us call these common values κ and τ. In order to find right orthogonal tranformation and the translation that sends β_2 onto β_1, we pick one point $a \in I$ and use the two Frenet frames at a to define an orthogonal transformation, and translate so the images line up. The idea is that we define the map correctly at one point and because the curvature and torsion are the same, the lemma gaurantess that the map is correct at every point. The details follow.

Fix a number $a \in I$. The linear tranformation $O : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $O(T_2) = T_1$, $O(N_2) = N_1$, and $O(B_2) = B_1$ is orthogonal since both $\{T_2(a), N_2(a), B_2(a)\}$ and $\{T_1(a), N_1(a), B_1(a)\}$ are orthonormal bases of $\mathbb{R}^3$. Let $p = \beta_1(a) - O(\beta_2(a))$. We claim that

$$\beta_1 = O \circ \beta_2 + p. \tag{5}$$

From the first part of this proof, the curvature of $O \circ \beta_2 + p$ is κ, and the torsion is either plus or minus τ (and we can assume it's plus, for otherwise we can just redefine O so that $O(B_2) = -B_1$). By construction, $O(\beta_2(a)) + p = \beta_1(a)$ and the Frenet frames of the curve $O \circ \beta_2 + p$ and β_1 are the same at $a \in I$. Therefore, by the lemma, the Frenet frames of the $O \circ \beta_2 + p$ and β_1 agree everywhere. In particular, the velocity vectors are the same for all $s \in I$, which by Problem 7 proves our claim. $\square$

Problem 1. Prove Theorem 4 and complete the proof of Theorem 2.

Problem 2. Define $\beta : (-1, 1) \to \mathbb{R}^3$ by

$$\beta(s) = \left(\frac{(1+s)^{\frac{3}{2}}}{3}, \frac{(1-s)^{\frac{3}{2}}}{3}, \frac{s}{\sqrt{2}} \right).$$

Compute the Frenet frame. As a check, I get $\tau(0) = -\frac{1}{2\sqrt{2}}$ and $\tau\left(\frac{1}{2}\right) = -\frac{1}{\sqrt{6}}$.

Problem 3. Let $\beta : I \to \mathbb{R}^3$ is a smooth unit speed curve and suppose that $\beta(s) + N(s) = p$ for a fixed $p \in \mathbb{R}^3$. Prove that β is part of a circle.

Problem 4. Let $\beta : \mathbb{R} \to \mathbb{R}^2$ be a smooth unit speed curve satisfying $\beta(s) = \beta(s+1)$ for all $s \in \mathbb{R}$. Prove that there is a point on the image of β that is furthest, say distance d, from the origin. Prove that the curvature at this point is at least $\frac{1}{d}$.

Problem 5. Consider the unit sphere $S^2 = \{p \in \mathbb{R}^3 \text{ such that } \|p\| = r\}$. Prove that if $\beta : I \to S$ is a smooth unit speed curve, then $\kappa \geq \frac{1}{r}$.

Problem 6. The curve α defined by

$$\alpha(t) = \left(t + \sqrt{3}\sin(t), 2\cos(t), \sqrt{3}t - \sin(t)\right)$$

is a helix. Prove it. In fact, be explicit: find a unit speed curve β of the form

$$\beta(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right),$$

an orthogonal operator O, and a point p so that $O \circ \alpha + p = \beta$.

Problem 7. Prove that if $\alpha'(s) = \beta'(s)$ for all $s \in I$ and for one particular $a \in I$, $\alpha(a) = \beta(a)$, then $\alpha(s) = \beta(s)$ for all s.

References

[1] Manfredo P. Do Carmo. *Differential Geometry of Curves and Surfaces.* Pearson, 1976.

[2] Barrett O'Neil. *Elementary Differential Geometry.* Academic Press, 1966.