Problem 1. Use a Monte Carlo method to approximate

$$\int_R xe^{\cos(xyz)}$$

where the region R is defined by

$$xyz \leq 1, \quad 0 \leq x \leq 5, \quad 0 \leq y \leq 5, \quad 0 \leq z \leq 5$$

(a) Generate n points in the cube $0 \leq x, y, z \leq 5$ and form a sum, weighted by $xe^{\cos(xyz)}$ for each point $(x, y, z) \in R$. Divide by n and multiply by the volume of the cube. Generate estimates using $n = 10,000$ points and $n = 1,000,000$ points. Determine the standard deviation in each case.

(b) How many points are needed to be 95% certain that your answer is accurate to .05?

Problem 2. *Example 3 from section 12.7 in Apostol*. Let $S = r(T)$ be a simple parametric surface defined by a smooth function r defined on a region T in the u-v plane and let F be a smooth vector field defined and bounded on S. At each point of S let n denote the unit normal

$$n = \frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\|}$$

(a) Prove that if the vector field $F = U\mathbf{i} + V\mathbf{j} + W\mathbf{k}$, then

$$\iint_{r(T)} F \cdot n \, dS = \int_{r(T)} \omega$$

where on the left hand side $\iint_{r(T)} F \cdot n \; dS$ is Apostol's surface integral (Section 12.7) and on the right had side ω is the two-form $U dy \wedge dz - V dx \wedge dz + W dx \wedge dy$ and $\int_{r(T)} \omega$ is the ordinary integral $\int_T r^*(\omega)$ defined in the u-v plane.

(b) Apostol defines the divergence of a vector field $F = U\mathbf{i} + V\mathbf{j} + W\mathbf{k}$ to be the function

$$\text{Div}(F) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$

Use the equation in part (a) and the differential form version of Stoke's theorem to prove (easily!) the *Divergence Theorem* which states that

$$\iint_{r(T)} F \cdot n \; dS = \iiint_V \text{Div}(F) \, dx \, dy \, dz$$

where V is the the solid whose boundary is the surface S.

Problem 2. Continued.

(c) The Wikipedia article on *Divergence* begins:

> *In vector calculus, divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.*

If we consider a vector field F as defining the flux density flow of a fluid (think of velocity vectors of moving air particles), then the total mass of fluid flowing out of a sphere S in a unit time is given by

$$\int_S F \cdot ndS$$

where n is the outward pointing unit vector perpendicular to the sphere S and dividing by the volume of the sphere $|V|$ defines the mass flowing out per unit volume per unit time. The divergence according to the Wikipedia definition of a vector field F at a point p is this infinitesimal change in volume:

$$\mathrm{Div}(F)(p) = \lim_{t \to 0} \frac{1}{|V(t)|} \int_{S(t)} F \cdot ndS$$

where $V(t)$ is a solid sphere of radius $t > 0$ centered at p, $S(t)$ is the boundary of the sphere. Prove it.

Problem 3. Let

$$\omega = \left(xz - \frac{1}{3}\right) dy \wedge dz + \left(yz - 2ze^{x^2+y^2+z^2}\right) dx \wedge dz + \left(\frac{1}{3} - 2ye^{x^2+y^2+z^2}\right) dx \wedge dy$$

and H be the northern hemisphere of the unit sphere in $\mathbb{R}^3$. Compute $\displaystyle\int_H \omega$ four ways:

(a) Directly from the definition.

(b) Use Stokes theorem.

(c) Identify the two-form ω with a vector field F as in the previous problem and use Apostol's formula for $\int_H F \cdot ndS$.

(d) Identify the two-form ω with a vector field F as in the previous problem and use Apostol's formula for $\int_D F \cdot ndS$ where D is the unit disc in the xy-plane. Note the formula for the normal n is quite simple.

EXAM

Final Exam

Math 208

May 20, 2015

- This exam is due at 8:00 am on Thursday, May 21.
- Neatness counts! Make sure your answers are clearly and carefully written.
- Document any resources you use.

Success!