Problem 1. Curves in $\mathbb{R}^3$

(a) Give an example of a curve with constant curvature that is not a circle.

(b) Prove that if a curve in $\mathbb{R}^3$ lies on a sphere and has constant curvature, then it is part of a circle.

Problem 2. Integrate. Verify your answers on a computer.

(a) Compute

$$\int_\alpha \left(y^2 + 3ze^{3xz}\right) dx + (2xy)dy + \left(3xe^{3xz}\right) dz$$

where $\alpha : [0, 2\pi] \to \mathbb{R}^3$ is the helix given by $\alpha(t) = (\cos(t), \sin(t), t)$.

(b) Compute

$$\int_\alpha (3y + 3x)dx + (2y - x)dy + z^2 dz$$

where $\alpha : [0, 2\pi] \to \mathbb{R}^3$ is the circle given by $\alpha(t) = (\cos(t), \sin(t), 3)$.

(c) Compute

$$\int_r 3x\, dy \wedge dz - 2y\, dx \wedge dz$$

where $r : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi] \to \mathbb{R}^3$ is the sphere given by

$$r(u, v) = (3\cos(u)\cos(v), 3\cos(u)\sin(v), 3\sin(u)).$$

Problem 3. Let M_{22} be the space of 2×2 real matrices. By identifying $M_{22} \simeq \mathbb{R}^4$, we can discuss functions to and from M_{22} as being continuous, differentiable, etc...

(a) Consider the function $F : M_{22} \to \mathbb{R}$ defined by $F(A) = \det(A)$. Show that F has one critical point and investigate its nature (find the eigenvalues and and eigenvectors of the Hessian and determine whether the critical point is an extremum).

(b) Given a matrix A, one may ask whether A has a square root. That is, whether there exists a matrix B with $A = B^2$. Consider the following variation. Given a matrix A, does there exist a matrix B with $A = B^2 + B$. Your problem: use the *Inverse Function Theorem* to prove that there exists a neighborhood of the 2×2 zero matrix so that for every $A \in \mathcal{U}$, there exists a matrix B with $A = B^2 + B$.

Problem 4. The second order Taylor approximation:

(a) Let U be an open subset of $\mathbb{R}^n$ and suppose that $f : U \to \mathbb{R}$ is smooth function. Fix a point $p \in U$. Prove that for any $x \in U$, there exists a number $t \in [0, 1]$ so that

$$f(x) = f(p) + D_p(x-p) + \frac{1}{2}(x-p)^T(H_c)(x-p)$$

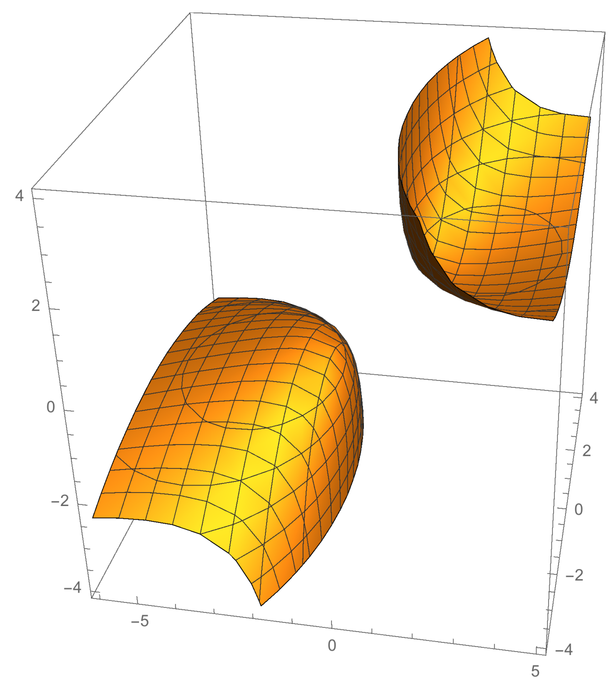
where D_p is the derivative of f at p, c is the point $c = tp + (1-t)x$ on the segment between p and x, and H_c is the Hessian of f at c.

(b) Use the fact that $f(x) \approx f(p) + D_p(x-p) + \frac{1}{2}(x-p)^T(H_p)(x-p)$ to approximate $1.05^{2.02}$.

Problem 5. Consider the surface $S \subset \mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : 2x^2 + 2y^2 + z^2 - 8xz + z + 8 = 0\}.$$

For most points on this surface, S is locally the graph of a function. Find all the critical points of the function z defined implicitly by S and classify them as either a local max, a local min, or neither. Here's a sketch of the surface S.



EXAM

Midterm

Math 208

April 12, 2015

- This exam is due in class on Tuesday, April 14.
- Neatness counts! Make sure your answers are clearly and carefully written.
- Document any resources you use.

Success!