Problem 1. Curves in $\mathbb{R}^3$

(a) Give an example of a curve with constant curvature that is not a circle.

(b) Prove that if a curve in $\mathbb{R}^3$ lies on a sphere and has constant curvature, then it is part of a circle.

Problem 2. Integrate. Verify your answers on a computer.

(a) Compute
\[ \int_{\alpha} \left( y^2 + 3ze^{3xz} \right) dx + (2xy)dy + (3xe^{3xz})dz \]
where $\alpha : [0, 2\pi] \to \mathbb{R}^3$ is the helix given by $\alpha(t) = (\cos(t), \sin(t), t)$.

(b) Compute
\[ \int_{\alpha} (3y + 3x)dx + (2y - x)dy + z^2dz \]
where $\alpha : [0, 2\pi] \to \mathbb{R}^3$ is the circle given by $\alpha(t) = (\cos(t), \sin(t), 3)$.

(c) Compute
\[ \int_{\alpha} 3x \ dy \wedge dz - 2y \ dx \wedge dz \]
where $r : \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] \times [0, 2\pi] \to \mathbb{R}^3$ is the sphere given by
\[ r(u, v) = (3\cos(u)\cos(v), 3\cos(u)\sin(v), 3\sin(u)). \]

Problem 3. Let $M_{22}$ be the space of $2 \times 2$ real matrices. By identifying $M_{22} \simeq \mathbb{R}^4$, we can discuss functions to and from $M_{22}$ as being continuous, differentiable, etc...

(a) Consider the function $F : M_{22} \to \mathbb{R}$ defined by $F(A) = \det(A)$. Show that $F$ has one critical point and investigate its nature (find the eigenvalues and and eigenvectors of the Hessian and determine whether the critical point is an extremum).

(b) Given a matrix $A$, one may ask whether $A$ has a square root. That is, whether there exists a matrix $B$ with $A = B^2$. Consider the following variation. Given a matrix $A$, does there exist a matrix $B$ with $A = B^2 + B$. Your problem: use the Inverse Function Theorem to prove that there exists a neighborhood of the $2 \times 2$ zero matrix so that for every $A \in U$, there exists a matrix $B$ with $A = B^2 + B$. 

Problem 4. The second order Taylor approximation:

(a) Let $U$ be an open subset of $\mathbb{R}^n$ and suppose that $f : U \to \mathbb{R}$ is smooth function. Fix a point $p \in U$. Prove that for any $x \in U$, there exists a number $t \in [0, 1]$ so that

$$f(x) = f(p) + D_p(x - p) + \frac{1}{2}(x - p)^T(H_c)(x - p)$$

where $D_p$ is the derivative of $f$ at $p$, $c$ is the point $c = tp + (1 - t)x$ on the segment between $p$ and $x$, and $H_c$ is the Hessian of $f$ at $c$.

(b) Use the fact that $f(x) \approx f(p) + D_p(x - p) + \frac{1}{2}(x - p)^T(H_p)(x - p)$ to approximate $1.05^{2.02}$.

Problem 5. Consider the surface $S \subset \mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : 2x^2 + 2y^2 + z^2 - 8xz + z + 8 = 0\}.$$

For most points on this surface, $S$ is locally the graph of a function. Find all the critical points of the function $z$ defined implicitly by $S$ and classify them as either a local max, a local min, or neither. Here’s a sketch of the surface $S$. 
This exam is due in class on Tuesday, April 14.

Neatness counts! Make sure your answers are clearly and carefully written.

Document any resources you use.

Success!