EXAM

Midterm

Math 208

April 15, 2015

ANSWERS

Problem 1. Curves in $\mathbb{R}^3$

(a) Give an example of a curve with constant curvature that is not a circle.

Answer:

The helix $\alpha : [0, 2\pi] \to \mathbb{R}^3$ defined by $\alpha(t) = \frac{1}{\sqrt{2}}(\cos(t), \sin(t), t)$ is a unit speed curve. A quick computation shows that the curvature $\kappa(t) = \frac{1}{\sqrt{2}}$, which is constant.

(b) Prove that if a curve in $\mathbb{R}^3$ lies on a sphere and has constant curvature, then it is part of a circle.

Answer:

Let $\alpha : I \to \mathbb{R}^3$ be a curve that lies on a sphere or radius r and has constant curvature. Without loss of generality, assume that α has unit speed. The condition that α lies on the sphere means that $\alpha(t) \cdot \alpha(t) = r^2 \Rightarrow \alpha'(t) \cdot \alpha(t) = 0$. We conclude that

$$T \cdot \alpha = 0. \tag{1}$$

We differentiate again to get $T' \cdot \alpha + T \cdot \alpha' = \kappa N \cdot \alpha + 1 = 0$. We conclude that

$$-1 = \kappa N \cdot \alpha. \tag{2}$$

Differentiating again, assuming the curvature κ is constant, gives

$$0 = \kappa N' \cdot \alpha + \kappa N \cdot T = \kappa(-\kappa T + \tau B) \cdot \alpha + 0 = 0.$$

We already know $T \cdot \alpha = 0$, so we conclude that

$$\alpha \cdot B = 0. \tag{3}$$

Since T, N, B form an orthonormal basis for $T_{\alpha(t)}\mathbb{R}^3$ for each t, equations (1), (2), (3) imply that

$$\alpha = -\frac{1}{\kappa}N. \tag{4}$$

Since $N' = -\kappa T + \tau B$ from the Frenet formulas, and equation (4) implies that $N' = -\kappa T$, we conclude that $\tau = 0$. This says that α is a plane curve. Therefore, α lies on the intersection of a plane and a sphere, which means it lies on a circle.

Problem 2. Integrate. Verify your answers on a computer.

(a) Compute

$$\int_\alpha \left(y^2 + 3ze^{3xz}\right) dx + (2xy)dy + \left(3xe^{3xz}\right) dz$$

where $\alpha : [0, 2\pi] \to \mathbb{R}^3$ is the helix given by $\alpha(t) = (\cos(t), \sin(t), t)$.

Answer:

Note that $\left(y^2 + 3ze^{3xz}\right) dx+(2xy)dy+\left(3xe^{3xz}\right) dz = df$ where $f(x, y, z) = xy^2+e^{3xz}$. Thus, by Stokes theorem the answer is

$$\int_\alpha \left(y^2 + 3ze^{3xz}\right) dx + (2xy)dy + \left(3xe^{3xz}\right) dz = \int_\alpha df$$

$$= \int_{\partial\alpha} f = f(\alpha(2\pi)) - f(\alpha(0)) = e^{6\pi} - 1.$$

(b) Compute

$$\int_\alpha (3y + 3x)dx + (2y - x)dy + z^2 dz$$

where $\alpha : [0, 2\pi] \to \mathbb{R}^3$ is the circle given by $\alpha(t) = (\cos(t), \sin(t), 3)$.

Answer:

Notice that $d\left((3y + 3x)dx + (2y - x)dy + z^2dz\right) = 3dy\wedge dx - 1dx\wedge dy = -4dx\wedge dy$ and that α parametrizes the boundary of D, a disc of radius 1. Therefore by Stokes theorem:

$$\int_r \left((3y + 3x)dx + (2y - x)dy + z^2dz\right) = \int_C -4dx \wedge dy = -4\pi.$$

(c) Compute

$$\int_r 3x\, dy \wedge dz - 2y\, dx \wedge dz$$

where $r : [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3$ is the sphere given by

$$r(u, v) = (3\cos(v)\cos(u), 3\cos(v)\sin(u), 3\sin(v)).$$

Answer:

Note that $d\left(3x\, dy \wedge dz - 2y\, dx \wedge dz\right) = 5dx \wedge dy \wedge dz$ and r parametrizes the boundary of the solid sphere V of radius 3 centered at the origin. So, by Stokes theorem

$$\int_r 3x\, dy \wedge dz - 2y\, dx \wedge dz = \int_V 5dx \wedge dx \wedge dy \wedge dz =$$

$$5\int_V dx \wedge dx \wedge dy \wedge dz = 5\left(\frac{4}{3}\pi 3^3\right) = 180\pi.$$

Problem 3. Let M_{22} be the space of 2×2 real matrices. By identifying $M_{22} \simeq \mathbb{R}^4$, we can discuss functions to and from M_{22} as being continuous, differentiable, etc...

(a) Consider the function $F : M_{22} \to \mathbb{R}$ defined by $F(A) = \det(A)$. Show that F has one critical point and investigate its nature (find the eigenvalues and and eigenvectors of the Hessian and determine whether the critical point is an extremum).

Answer:

By identifying $M_{22} \simeq \mathbb{R}^4$ via

$$\begin{bmatrix} x & y \\ z & u \end{bmatrix} \leftrightsquigarrow (x, y, z, u),$$

we have $F(x, y, z, u) = xu - yz$. We compute the derivative and Hessian

$$D = [u, -z, -y, x] \text{ and } H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We see there is one critical point of F at $(0, 0, 0, 0)$ and a quick computation gives the characteristic polynomial

$$\begin{aligned} \chi_H(\lambda) &= \det(H - \lambda I) \\ &= \det \begin{bmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & -1 & 0 \\ 0 & -1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{bmatrix} \\ &= \lambda^2(\lambda^2 - 1) - 1(\lambda^2 - 1) \\ &= (\lambda - 1)^2(\lambda + 1)^2 \end{aligned}$$

yields the eigenvalues $\lambda = -1$ and $\lambda = 1$. Since one is positive and one is negative, the critical point is a saddle point, neither a max nor a min. The eigenvectors are:

$$\{(-1, 0, 0, 1), (0, 1, 1, 0), (0, -1, 1, 0), (1, 0, 0, 1)\}.$$

The first two correspond to $\lambda = -1$ and the second two to $\lambda = 1$. So, the function curves

Problem 3.

(b) Given a matrix A, one may ask whether A has a square root. That is, whether there exists a matrix B with $A = B^2$. Consider the following variation. Given a matrix A, does there exist a matrix B with $A = B^2 + B$. Your problem: use the *Inverse Function Theorem* to prove that there exists a neighborhood of the 2×2 zero matrix so that for every $A \in \mathcal{U}$, there exists a matrix B with $A = B^2 + B$.

Answer:

Let $F : M_{22} \to M_{22}$ be defined by $F(B) = B^2 + B$. we express $F : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$F(x, y, z, u) = (x + x^2 + yz, y + uy + xy, z + uz + xz, u + u^2 + yz).$$

We compute the derivative of F and get

$$D = \begin{bmatrix} 1+2x & z & y & 0 \\ y & 1+u+x & 0 & y \\ z & 0 & 1+u+x & z \\ 0 & z & y & 1+2u \end{bmatrix}.$$

Notice that at the two by two zero matrix $(x, y, z, u) = (0, 0, 0, 0)$, we have

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is invertible. Therefore, the inverse function theorem says there exists a neighborhood $\mathcal{U}$ of $(0, 0, 0, 0)$ (in the domain) and a neighborhood $\mathcal{V}$ of $F(0, 0, 0, 0) = (0, 0, 0, 0)$ (in the range) so that $F : \mathcal{U} \to \mathcal{V}$ is bijective and the inverse, call it $G : \mathcal{V} \to \mathcal{U}$ is differentiable, with derivative $D_{F(A)}G = (D_A F)^{-1}$. Therefor, for each $A \in \mathcal{V}$, there exists a $B = G(A) \in \mathcal{U}$ so that $F(G(B)) = A$. That is, $A = B^2 + B$.

Problem 4. The second order Taylor approximation:

(a) Let U be an open subset of $\mathbb{R}^n$ and suppose that $f : U \to \mathbb{R}$ is smooth function. Fix a point $p \in U$. Prove that for any $x \in U$, there exists a number $t \in [0, 1]$ so that

$$f(x) = f(p) + D_p(x - p) + \frac{1}{2}(x - p)^t(H_c)(x - p)$$

where D_p is the derivative of f at p, c is the point $c = tp + (1 - t)x$ on the segment between p and x, and H_c is the Hessian of f at c.

Answer:

Let x be given. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(s) = f(sp + (1 - s)x)$. Taylor's theorem for one variable functions says that there is a point $t \in (0, s)$ so that

$$g(s) = g(0) + g'(0)s + \frac{1}{2}g''(t).$$

To finish the problem, express the right hand side in terms of f. We'll have to use the chain rule to differentiate g, so look at g as the composition:

$$\mathbb{R} \xrightarrow{\quad \phi \quad} U \xrightarrow{\qquad f \qquad} \mathbb{R}$$

$$s \longmapsto sx + (1 - s)p \longmapsto f(sx + (1 - s)p)$$

So, $g'(s) = Df_{sx+(1-s)p}(x - p) \Rightarrow g'(0) = D_p(x - p)$ and $g''(s) = (x - p)^T Hf_{sx+(1-s)p}(x - p)$, which proves the result.

(b) Use the fact that $f(x) \approx f(p) + D_p(x - p) + \frac{1}{2}(x - p)^t(H_p)(x - p)$ to approximate $1.05^{2.02}$.

Answer:

Let $f(x, y) = x^y$. We have

$$f(1.05, 2.02) \approx f(1, 2) + D_{(1,2)} \begin{bmatrix} .05 \\ .02 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} .05 & .02 \end{bmatrix} H_{(1,2)} \begin{bmatrix} .05 \\ .02 \end{bmatrix}.$$

We compute

$$D_{(x,y)} = \begin{bmatrix} yx^{y-1} & \log(x)x^y \end{bmatrix} \Rightarrow D_{(1,2)} = \begin{bmatrix} 2 & 0 \end{bmatrix}.$$

and

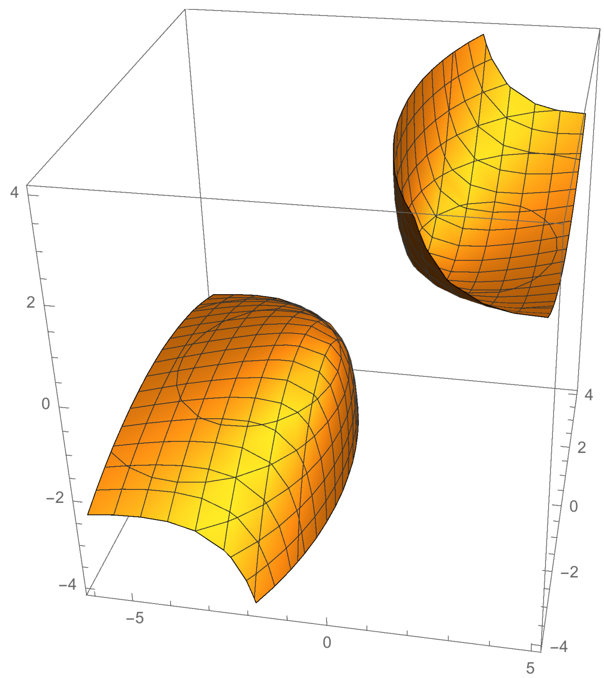
$$H_{(x,y)} = \begin{bmatrix} y(y-1)x^{y-2} & x^{y-1} + \log(x)yx^{y-1} \\ x^{y-1} + \log(x)yx^{y-1} & (\log(x))^2 x^y \end{bmatrix} \Rightarrow H_{(1,2)} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

This gives gives $1.05^{2.02} \approx 1^2 + 0.1 + \frac{1}{2}(0.007) = 1.1035$

Problem 5. Consider the surface $S \subset \mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : 2x^2 + 2y^2 + z^2 - 8xz + z + 8 = 0\}.$$

For most points on this surface, S is locally the graph of a function. Find all the critical points of the function z defined implicitly by S and classify them as either a local max, a local min, or neither. Here's a sketch of the surface S.



Problem 5.

Answer:
Let $F(x,y,z) = 2x^2 + 2y^2 + z^2 - 8xz + z + 8$. Then, we assume $F(x,y,z) = 0$ defines z implicitly as a function of x and y. Then $0 = F(x,y,z)$ implies

$$0 = \frac{\partial F}{\partial x} = 4x + 2z\frac{\partial z}{\partial x} - 8z - 8x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \text{ and } 0 = \frac{\partial F}{\partial y} = 4y + 2z\frac{\partial z}{\partial y} - 8x\frac{\partial z}{\partial y} + \frac{\partial z}{\partial y}.$$

So, for the function z defined implicitly, we have the derivative

$$Dz = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-4x+8z}{2z-8x+1} & \frac{-4y}{2z-8x+1} \end{bmatrix}. \tag{5}$$

Note that we have critical points where

$$Dz = 0 \Rightarrow -4x + 8z = 0 \text{ and } -4y = 0.$$

That is, we have critical points at the points on S where $(2z, 0, z)$. To find these points, we solve

$$F(2z,0,z) = 0 \Rightarrow 8z^2 + z^2 - 16z^2 + z + 8 = 0 \Rightarrow -7z^2 + z + 8 = 0 \Rightarrow z = -1, \frac{8}{7}.$$

Therefore, we have two critical points at

$$p = (-2, 0, -1) \text{ and } q = \left(\frac{16}{7}, 0, \frac{8}{7}\right).$$

To classify these points, we look at the second derivative of z by differentiating the expression for Dz in Equation (5):

$$\frac{\partial^2 z}{\partial x^2} = \frac{(2z-8x+1)\left(-4+8\frac{\partial z}{\partial x}\right) - (-4x+8z)\left(2\frac{\partial z}{\partial x}\right)}{(2z-8x+1)^2}$$
$$\frac{\partial^2 z}{\partial y \partial x}\frac{(2z-8x+1)\left(8\frac{\partial z}{\partial y}\right) - (-4x+8z)\left(2\frac{\partial z}{\partial y}\right)}{(2z-8x+1)^2}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(2z-8x+1)(0) - (-4y)\left(2\frac{\partial z}{\partial y}\right)}{(2z-8x+1)^2}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{(2z-8x+1)(-4) - (-4y)\left(2\frac{\partial z}{\partial y}\right)}{(2z-8x+1)^2}$$

which, at the critical points of z simplifies since $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$, giving

$$Hz = \frac{1}{(2z-8x+1)}\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}.$$

Finally, we have

$$H_p = \begin{bmatrix} -\frac{4}{15} & 0 \\ 0 & -\frac{4}{15} \end{bmatrix} \text{ and } H_q = \begin{bmatrix} \frac{4}{15} & 0 \\ 0 & \frac{4}{15} \end{bmatrix}.$$

The conclusion is that $p = (-2, 0, -1)$ is a local maximum and $q = \left(\frac{16}{7}, 0, \frac{8}{7}\right)$ is a local minimum.