Vector Calculus and Differential Forms

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1 Vector fields

Definition 1. Let U be an open subest of $\mathbb{R}^n$. A *tangent vector* in U is a pair $(p, v) \in U \times \mathbb{R}^n$. We think of (p, v) as consisting of a vector $v \in \mathbb{R}^n$ lying at the point $p \in U$. Often, we denote a tangent vector by v_p instead of (p, v). For $p \in U$, the set of all tangent vectors at p is denoted by $T_p(U)$. The set of all tangent vectors in U is denoted $T(U)$

For a fixed point $p \in U$, the set $T_p(U)$ is a vector space with addition and scalar multiplication defined by $v_p + w_p = (v + w)_p$ and $\alpha v_p = (\alpha v)_p$. Note that as a vector space, $T_p(U)$ is isomorphic to $\mathbb{R}^n$.

Definition 2. A vector field on U is a function $X : U \to T(\mathbb{R}^n)$ satisfying $X(p) \in T_p(U)$.

Remark 1. Notice that any function $f : U \to \mathbb{R}^n$ defines a vector field X by the rule $X(p) = f(p)_p$. Denote the set of vector fields on U by $\mathfrak{Vect}(U)$.

Note that $\mathfrak{Vect}(U)$ is a vector space with addition and scalar multiplication defined pointwise (which makes sense since $T_p(U)$ is a vector space):

$$(X + Y)(p) := X(p) + Y(p) \text{ and } (\alpha X)(p) = \alpha(X(p)).$$

Definition 3. Denote the set of functions on the set U by $\mathfrak{Fun}(U) = \{f : U \to \mathbb{R}\}$. Let $C^1(U)$ be the subset of $\mathfrak{Fun}(U)$ consisting of functions with continuous derivatives and let $C^\infty(U)$ be the subset of $\mathfrak{Fun}(U)$ consisting of smooth functions, i.e., infinitely differentiable functions.

In a sense, $\mathfrak{Vect}(U)$ is like a vector space over $\mathfrak{Fun}(U)$ in that there is a map

$$\mathfrak{Fun}(U) \times \mathfrak{Vect}(U) \to \mathfrak{Vect}(U)$$

defined by $(f \times X)(p) = f(p)X(p)$ that works like scalar multiplication. Because in this case the "scalars" are functions instead of real numbers, the correct terminology is that $\mathfrak{Vect}(U)$ is a *module* over $\mathfrak{Fun}(U)$—the expression "vector space" only applies if the scalars have multiplicative inverses, which functions do not always have. Still, as a module, $\mathfrak{Vect}(U)$ has a nice $\mathfrak{Fun}(U)$ basis.

Definition 4. Let $U_i \in \mathfrak{Vect}(U)$ be defined by

$$\begin{aligned} U_1(p) &:= (1, 0, \dots, 0)_p \\ U_2(p) &:= (0, 1, \dots, 0)_p \\ &\vdots \\ U_n(p) &:= (0, \dots, 0, 1)_p. \end{aligned}$$

Then, for any vector field $X \in \mathfrak{Vect}(U)$, there exist unique functions $f_1, \dots, f_n \in \mathfrak{Fun}(\mathbb{R}^n)$ so that

$$X = f_1 U_1 + f_2 U_2 + \cdots + f_n U_n.$$

Now, we defined the directional derivative of a function f at a point p in the direction v. It is more efficient to define the derivative a function f with respect to a tangent vector v_p.

Definition 5. Let $v_p \in T_p(U)$ and $f \in C^1(U) \subset \mathfrak{Fun}(U)$. We define $v_p[f]$ to be the directional derivative of f at the point p in the direction v, provided it exists.

This definition gives a way for a vector field to act on a function.

Definition 6. Let $X \in \mathfrak{Vect}(U)$ and $f \in C^1(U)$. We define $X(f) \in \mathfrak{Fun}(U)$ by $X(f)(p) = X(p)[f]$.

So, there is a map

$$\begin{aligned} \mathfrak{Vect}(U) \times C^1(U) &\to \mathfrak{Fun}(U) \\ X, f &\mapsto X(f) \end{aligned}$$

as defined above.

Problem 1. Let $X, Y \in \mathfrak{Vect}(U)$, $f, g, h \in C^1(U)$, $\alpha, \beta \in \mathbb{R}$. Prove that

(a) $(fX + gY)[h] = fX[h] + gY[h]$

(b) $X[\alpha f + \beta g] = \alpha X[f] + \beta X[g]$

(c) $X[fg] = X[f] \cdot g + fX[g]$

(d) $X = \sum_i X[x_i]U_i$.

One can (and maybe should!) think of vector fields as functions $C^1(U) \to \mathfrak{Fun}(U)$ satisfying properties (a), (b), and (c) above.

Note that $U_i[f] = \frac{\partial f}{\partial x_i}$. For this reason, the vector field U_i is sometimes denoted by "$\frac{\partial}{\partial x_i}$."

Definition 7. Suppose $F : U \to \mathbb{R}^k$ is differentiable. We define

$$\begin{aligned} F_* : TU &\to T\mathbb{R}^k \\ v_p &\mapsto F_*(v_p) \end{aligned}$$

by $F_*(v_p) := (D_p F(v))_{F(p)}$.

Then, we have nice theorems such as

Inverse Function Theorem. Let $F : U \to \mathbb{R}^n$ and suppose that $F_{*p} : T_p U \to T_{F(p)}\mathbb{R}^n$ is an isomorphism at some point p. Then there exists a neighborhood $\mathcal{U}$ containing p and a neighborhood $\mathcal{V}$ containing $F(p)$, such that $F : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism (i.e., a differentiable bijection with a differentiable inverse).

and nice problems such as

Problem 2. Show that F_* "preserves velocity vectors." That is, let $U \subset \mathbb{R}^n$ and let $I \subset \mathbb{R}$ be open sets. Suppose $\alpha : I \to U$ is a curve in $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^k$. Let $\beta = F \circ \alpha : \mathbb{R} \to \mathbb{R}^k$ be the image of α under F. Prove that $F_*(\alpha') = \beta'$. Here, we interpret α' as a tangent vector $\alpha'(t)_{\alpha(t)}$.

2 One-forms

For any real vector space V, the vector space $V^* = \hom(V, \mathbb{R})$ is called *the dual space* of V. The dual space consists of all linear functions from V to the ground field $\mathbb{R}$.

Definition 8. A *cotangent vector* in U is a pair $(p, v^*) \in U \times (\mathbb{R}^n)^*$. We think of (v^*, p) as consisting of a linear functional $v^* : \mathbb{R}^n \to \mathbb{R}$ at the point p. Often, we denote a cotangent vector by v_p^* instead of (p, v^*). For $p \in U$, the set of all cotangent vectors at p is denoted by $T_p^*(U)$. The set of all cotangent vectors in U is denoted $T^*(U)$

For a fixed point p, $T_p^*(U)$ is a vector space with addition and scalar multiplication defined pointwise. The vector space $T_p^*(U)$ is naturally isomorphic to $(T_p(U))^*$.

Definition 9. A 1-form on U is a function $\phi : U \to T^*(U)$ satisfying $\phi(p) \in T_p^*(U)$ for all $p \in U$.

Just as with vector fields, the space of 1-forms defines a module over $\mathfrak{Fun}(U)$: for a function $f \in \mathfrak{Fun}(U)$ and a 1-form ϕ, we define a 1-form $f\phi$ by $f\phi(p) = f(p)\phi(p)$.

One may think of a 1-form as a function $\phi : T(U) \to \mathbb{R}$ satisfying $\phi(\alpha v_p + \beta w_p) = \alpha\phi(v_p) + \beta\phi(w_p)$ for all $\alpha, \beta \in \mathbb{R}$, $v_p, w_p \in T_p(U)$. Here, for simplicity, we've written $\phi(v_p)$ instead of $\phi(p)(v_p)$. In this sense, one forms are dual to vector fields. We have a map

$$\begin{aligned} \{1 \text{ forms } \} \times \mathfrak{Vect} &\to \mathfrak{Fun} \\ (\phi, X) &\mapsto \phi(X) \end{aligned}$$

where $\phi(X)(p) := \phi_p(X(p))$.

Definition 10. Given a function $f \in C^1(U)$, we define a 1-form df by $df(v_p) = v_p[f]$ for all $v_p \in T(U)$.

Just as with vector fields, at each point $p \in \mathbb{R}^n$, the cotangent space $T_p^*(U)$ is naturally a vector space and is naturally isomorphic to $(T_p(U))^*$. So, you can add one forms and get one forms and multiply functions by one forms to get one forms. This makes the space of 1-forms into a module over $\mathfrak{Fun}(U)$, which means

$$f(g\phi) = (fg)\phi \text{ and } (f+g)\phi = f\phi + g\phi \text{ and } f(\phi + \psi) = f\phi + f\psi$$

for functions f, g and 1-forms ϕ, ψ.

For the functions $x_1, \ldots, x_n$, the one forms $dx_1, \ldots, dx_n$ are dual to the vector fields $U_1, \ldots, U_n$. As the next problem shows, $\{dx_1, \ldots, dx_n\}$ define a $\mathfrak{Fun}(\mathbb{R}^n)$ basis for the space of 1-forms.

Problem 3. Show that for any 1-form ϕ, there exist unique functions $f_1, \ldots, f_n$ with $\phi = f_1 dx_1 + \cdots f_n dx_n$.

Problem 4. Prove that if $f \in \mathfrak{Fun}(U)$ is differentiable, then $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

Problem 5. Let $f, g \in \mathfrak{Fun}(U)$, let $h : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that

(a) $d(f + g) = df + dg$

(b) $d(fg) = (df)(g) + f(dg)$

(c) $d(h \circ f) = (h' \circ f)df$

Definition 11. Let $F : U \to \mathbb{R}^k$ be differentiable. We define a map, called the pullback,

$$F^* : \{1 - \text{forms on } \mathbb{R}^k\} \to \{1 - \text{forms on } U\}$$
$$\phi \mapsto F^*(\phi)$$

where $F^*(\phi)(v_p) = \phi_{F(p)}(F_*(v_p))$.

Problem 6. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x, y, z) = (2x^2y^2, xye^{4z})$. Let $\phi = x^3 y dx + \sin(xy)dy$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2 + 2xy$. Compute $F^*(df)$ and $F^*(\phi)$.

Problem 7. Prove that

(a) $F^*(\phi + \psi) = F^*(\phi) + F^*(\psi)$

(b) $F^*(df) = d(f(F))$

Now we can define the integral of a one form over a curve.

Definition 12. Let $\alpha : [a, b] \to U \subset \mathbb{R}^n$ be a curve and ϕ be a one form in U. We define

$$\int_\alpha \phi = \int \alpha^*(\phi).$$

This definition summarizes, quite neatly, the concept of line integral that appeared in the textbook (and supplies meaning for the mysterious "dt" that appears throughout calculus). To explain the connection between the integral of a one-form (Definition 12) and the definition of a line integral (defined in Apostol, Vol II, on page 324), one should first understand how the pullback works in coordinates. Suppose $\alpha : [a,b] \to \mathbb{R}^n$ is defined by $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t)$ and $\phi = f_1 dx_1 + \cdots + f_n dx_n$. Then the pullback $\alpha^*(\phi)$ is a one form on $\mathbb{R}$. Therefore, it must have the form $g(t)dt$ for some function $g(t)$. To determine $g(t)$, apply $\alpha^*(\phi)$ to the vector field $\frac{\partial}{\partial t}$:

$$\alpha^*(\phi)\left(\frac{\partial}{\partial t}\right) = \phi\left(\alpha_*\left(\frac{\partial}{\partial t}\right)\right) = \phi(\alpha'(t)).$$

Therefore, in coordinates, we have

$$\alpha^*(\phi) = (f_1(\alpha_1(t), \ldots, \alpha_n(t))\alpha_1'(t) + \cdots + f_n(\alpha_1(t), \ldots, \alpha_n(t))\alpha_n'(t))\, dt. \quad (1)$$

So, by definition of $\int_\alpha \phi$, we have

$$\int_\alpha \phi = \int_{[a,b]} \alpha^*(\phi) = \int_{[a,b]} f(\alpha(t)) \cdot \alpha'(t) dt.$$

Which is precisely the definition of the line integral $\int_\alpha f$ where f is the vector field defined by the function $(f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$.

Example 1. Let's do an example in $\mathbb{R}^2$ (well, in $\mathbb{R}^2 \setminus \{0,0\}$). Let $\alpha : [0, 2\pi] \to \mathbb{R}^2$ be the unit circle $\alpha(t) = (\cos(t), \sin(t))$. Consider the 1-form

$$\phi = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Then $\alpha^*(\phi)$ will be a 1-form in $\mathbb{R}$ which by Equation (1) is given by

$$\alpha^*(\phi) = -\frac{\sin(t)}{\cos^2(t)+\sin^2(t)}(-\sin(t)dt) + \frac{\cos(t)}{\cos^2(t)+\sin^2(t)}(\cos(t)dt) = dt.$$

Thus,

$$\int_\alpha \phi = \int_{[0,2\pi]} \alpha^*(\phi) = \int_0^{2\pi} dt = 2\pi.$$

In particular, we have the following beautiful theorem.

Theorem 1. *If $\phi = df$ for some function f, then*

$$\int_\alpha \phi = f(q) - f(p)$$

where $p = \alpha(a)$ and $q = \alpha(b)$.

Proof.

$$\int_\alpha \phi = \int_\alpha df = \int_a^b \alpha^*(df) = \int_a^b d(f\circ\alpha)$$
$$= \int_a^b \frac{\partial(f\circ\alpha)}{\partial t}dt = f(\alpha(b)) - f(\alpha(a)) = f(q) - f(p).$$

□

Corollary 1. *If $\phi = df$ and α is a closed curve, then $\int_\alpha \phi = 0$.*

3 The exterior algebra

Before defining a k-form, I will review a little bit about linear algebra. Let V be a finite dimensional vector space. There exist larger algebraic structures in which V fits. Here we define ΛV, the free exterior algebra of V.

Without getting into too much detail, here is the universal mapping property defining ΛV.

Definition 13. The exterior algebra of V is defined to be the unital, associative algebra ΛV with an inclusion $i : V \to \Lambda V$ having the following universal property: for every algebra A and every linear map $\sigma : V \to A$ with $\sigma(v)\sigma(v) = 0$ for all $v \in V$, there exists a unique algebra homomorphism $\sigma' : \Lambda V \to A$ with $\sigma' \circ i = \sigma$.

In practical terms, the exterior algebra ΛV consists of elements that are linear combinations of "wedge" products of vectors in V, where the following rules hold: For all $u, v, w \in V$ and $\alpha \in k$, we have

- linearity over scalar multiplication: $\alpha v \wedge w = \alpha(v \wedge w) = v \wedge (\alpha w)$
- linearity over addition: $v \wedge (w + u) = v \wedge w + v \wedge u$.
- associativity: $(v \wedge w) \wedge u = v \wedge (w \wedge u)$
- skew-symmetry: $v \wedge v = 0$.

Problem 8. Show that $v \wedge v = 0$ for all $v \in V$ implies that

(a) $v_1 \wedge \cdots v_k = 0$ if $v_1, \ldots, v_k$ are linearly dependent.

(b) $v \wedge u = -u \wedge v$

Computations in the exterior algebra can be handled conveniently using a basis of V. First, for $1 \leq k \leq n$, we define the k-fold exterior product of V, which is a vector space $\Lambda^k V$, of dimension $\binom{n}{k}$. If V has basis $\{e_1, \ldots, e_n\}$, then $\Lambda^k V$ is a vector space with basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. It is sometimes helpful to use a mutli-index notation. If I is the multi-index $I = \{i_1 < i_2 < \cdots < i_k\}$, then we may abbreviate $e_{i_1} \wedge \cdots \wedge e_{i_k}$ by e_I. For example, $e_1 \wedge e_2 \wedge e_5$ might be abbreviated by $e_{1,2,5}$.

Example 2. Suppose V is a four dimensional vector space with basis $\{e_1, e_2, e_3, e_4\}$, then $\Lambda^k V$ is a vector space with bases:

$$\begin{aligned}
\Lambda^1 V &= \operatorname{span}\{e_1, e_2, e_3, e_4\} \\
\Lambda^2 V &= \operatorname{span}\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\} \\
\Lambda^3 V &= \operatorname{span}\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e^4, e_2 \wedge e_3 \wedge e^4\} \\
\Lambda^4 V &= \operatorname{span}\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}.
\end{aligned}$$

Now, the wedge product defines a bilinear map, determined on bases by,

$$\wedge : \Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V$$

where one uses the alternation rule $e_i \wedge e_j = -e_j \wedge e_i$ to reorder the result in terms of the basis of $\Lambda^{k+l} V$. All together we set the exterior algebra to be the sum of the k-fold exterior products

$$\Lambda V = k \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^n V$$

and the wedge product

$$\wedge : \Lambda V \times \Lambda V \to \Lambda V$$

so that ΛV becomes an associated algebra over k. It is the "most general" skew-symmetric algebra containing V (this is the meaning of the universal property defining it).

Example 3. Here's a sample computation:

$$\begin{aligned}
(3e_1 \wedge e_3 + 5e_1 \wedge e_2) \wedge (2e_1 \wedge e_2 - e_2 + 2e_3) &= -3e_1 \wedge e_3 \wedge e_2 + 10e_1 \wedge e_2 \wedge e_3 \\
&= 3e_1 \wedge e_2 \wedge e_3 + 10e_1 \wedge e_2 \wedge e_3 \\
&= 13e_1 \wedge e_2 \wedge e_3.
\end{aligned}$$

Problem 9. Prove that $\wedge$ is associative and graded commutative. That is, prove that

(a) $(\phi \wedge \psi) \wedge \eta = \phi \wedge (\psi \wedge \eta)$ for all $\phi, \psi, \eta \in \Lambda V$

(b) $\phi \wedge \psi = (-1)^{kl} \psi \wedge \phi$ if $\phi \in \Lambda^k V$ and $\psi \in \Lambda^l V$.

3.1 Bilinear and multilinear functions

Let V be a vector space. A bilinear form on V is a function $f : V \times V \to k$ that is linear in each component. That is,

$$f(au + v, w) = af(u, w) + f(v, w) \text{ and } f(u, av + w) = af(u, v) + f(u, w).$$

A bilinear form is called symmetric if $f(u, v) = f(v, u)$ for all u, v and is called skew-symmetric if $f(v, v) = 0$ for all v (which implies that $f(u, v) = f(v, u)$). In general, a multilinear form is a function $f : V^{\times n} \to k$ that is linear in each component. A multilinear form is called symmetric if $f(v_1, \dots, v_n) = f(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ for every permutation $\sigma \in S_n$. A multilinear form is called alternating if $f(v_1, \dots, v_n) = 0$ if $v_i = v_j$ for any $i \neq j$ (which implies that $f(v_1, \dots, v_i, v_{i+1}, \dots v_n) = -f(v_1, \dots, v_{i+1}, v_i, \dots v_n)$.)

Now suppose that V is finite dimensional with basis $\{e_1, \dots, e_n\}$. There is a bijection between alternating multilinear forms on V and elements of ΛV^*. The correspondence

$$\{f : V^{\otimes n} \to V \text{ is an alternating } k\text{-linear form } \} \longleftrightarrow \Lambda^k V^*$$
$$f \leftrightarrow \omega$$

where $\omega = \frac{1}{k!} \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) e_{i_1} \wedge \cdots \wedge e_{i_k}$. So, for example if V is 4 dimensional and $f : V \times V \times V \to k$ is an alternating form defined by $f(e_1 e_2, e_3) = 10$, $f(e_1, e_3, e_4) = 0$, $f(e_1, e_2, e_4) = 0$, $f(e_2, e_3, e_4) = 0$, then we can associate f to the element $\frac{1}{3!}(10 e_1^* \wedge e_2^* \wedge e_3^* - 10 e_1^* \wedge e_3^* \wedge e_2^* + \cdots) = 10 e_1^* \wedge e_2^* \wedge e_3^*$. Likewise, the element

$$\phi = 7 e_1^* \wedge e_2^* \wedge e_4^* \in \Lambda^3 V^*$$

acts as a tri-linear form on V by

$$\omega(u, v, w) = 7u_1 v_2 w_4 - 7u_2 v_1 w_4 + 7u_2 v_4 w_1 - 7u_4 v_2 w_1 + 7u_4 v_1 w_2 - 7u_1 v_4 w_2.$$

Problem 10. (a) Let $B = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$. Define a bilinear form $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(v, w) = v^t B w$. Check that f is bilinear and alternating. Write down the associated element of $\Lambda^2((\mathbb{R}^2)^*)$ corresponding to f

(b) Let V be the vector space of real polynomials of degree 5 or smaller. Define $b : V \times V \to \mathbb{R}$ by $b(p, q) = p'(1) - q'(1)$. Check that b is bilinear and alternating. Choose a basis (wisely) for V and right down an element of $\Lambda^2 V^*$ corresponding to b.

4 Differential forms

For a fixed point p, the space $T_p^*(\mathbb{R}^n)$ is a vector space and we form the k-fold exterior power $\Lambda^k T_p^*(\mathbb{R}^n)$. We set $\Lambda^k T^*(\mathbb{R}^n) = \{(p, w) : p \in \mathbb{R}^n \text{ and } w \in \Lambda^k T_p^*(\mathbb{R}^n)\}$.

Definition 14. A k-form on $\mathbb{R}^n$ is a function $\phi : \mathbb{R}^n \to \Lambda^k T^*(\mathbb{R}^n)$ satisfying $\phi(p) \in \Lambda^k T_p^*(\mathbb{R}^n)$ for all $p \in \mathbb{R}^n$. Let us denote the set of k-forms by $\Omega^k(\mathbb{R}^n)$. A differential form is a sum of k-forms and we denote the set of differential forms by $\Omega^\bullet(\mathbb{R}^n)$.

Alternatively, one could define a k-form on $\mathbb{R}^n$ to be a function that assigns to each point $p \in \mathbb{R}^n$ a k-linear function function $\phi_p : (T_p\mathbb{R}^n)^{\times k} \to \mathbb{R}$. Or, taken over all points at once, a k-form defines a function

$$\phi : \mathfrak{Vect}(\mathbb{R}^n)^{\times k} \to \mathfrak{Fun}(\mathbb{R}^n)$$

by

$$\phi(X_1, \ldots, X_k)(p) = \phi_p(X_1(p), \ldots, X_k(p)).$$

Since ϕ_p is multilinear and alternating for each p, the k-form ϕ is multlinear over $\mathfrak{Fun}(\mathbb{R}^n)$. So, one has the alternative

Definition 15. A k-form ϕ is a function

$$\phi : \mathfrak{Vect}(\mathbb{R}^n)^{\times k} \to \mathfrak{Fun}(\mathbb{R}^n)$$

satisfying

- $\phi(X_1, \ldots, fX_i, \ldots, X_k) = f\phi(X_1, \ldots, X_i, \ldots, X_k)$ for all functions $f \in \mathfrak{Fun}(\mathbb{R}^n)$.

- $\phi(X_1, \ldots, X_i+Y_i, \ldots, X_k) = \phi(X_1, \ldots, Y_i, \ldots, X_k)+\phi(X_1, \ldots, Y_i, \ldots, X_k)$ for all vector fields $X_1, \ldots, X_k \in \mathfrak{Vect}(\mathbb{R}^n)$.

- $\phi(X_1, \ldots, X_k) = 0$ whenever $X_i = X_j$ for $i \neq j$.

Since at each point $\Lambda^k T_p^*(\mathbb{R}^n)$ is a vector space, $\Omega^k(\mathbb{R}^n)$ and $\Omega^\bullet(\mathbb{R}^n)$ become a module over $\mathfrak{Fun}(\mathbb{R}^n)$ (just as with vector fields and one-forms). In addition, $\Omega^\bullet(\mathbb{R}^n)$ becomes an associative algebra with the wedge product defined pointwise.

We already have a convenient $\mathfrak{Fun}(\mathbb{R}^n)$ basis for one forms. Every k-form can be expressed uniquely as a linear combination $\sum_I f_I dx_I$, where the sum is over all multi-indices $I = \{i_1 < i_2 < \cdots < i_k\}$. For example, every $\phi \in \Omega^2(\mathbb{R}^3)$ has a unique expression

$$\phi = f dx \wedge dy + g dx \wedge dz + h dy \wedge dz$$

for functions f, g, h. Sometimes it's handy, for example, to express the form $f dx \wedge dy$ as $-f dy \wedge dx$.

For any differentiable function f, we have defined a 1-form df. The function d extends to give a map

$$d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n).$$

The property determining the extension is

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^k \phi \wedge d\psi \text{ if } \psi \in \Omega^k V.$$

But, to be concrete, we make the following definition:

Definition 16. The exterior derivative is the linear map

$$d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}$$

defined by

$$d\left(\sum_I f_I dx_I\right) = \sum_I df_I \wedge dx_I = \sum_I \left(\sum_j \frac{\partial f_I}{\partial x_j} dx_j\right) \wedge dx_I.$$

Example 4. Let $\phi = e^{xy}dx + z^2x^2y^3dy \in \Omega^1(\mathbb{R}^n)$. Then

$$\begin{aligned} d\phi &= xe^{xy}dy \wedge dx + 2z^2xy^3dx \wedge dy + 2zx^2y^3dz \wedge dy \\ &= (-xe^{xy}dy + 2z^2xy^3)dx \wedge dy - 2zx^2y^3dy \wedge dz. \end{aligned}$$

Problem 11. Check, using the definition above, that $d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^k\phi \wedge d\psi$ if $\phi \in \Omega^kV$.

Problem 12. Check explicitly that $d(df) = 0$ for any function f, $d(d\phi) = 0$ for any one form ϕ, and $d(d\omega) = 0$ for any differential form ω.

By the problem above,

Theorem 2. *The exterior derivative d satisfies $d^2 = 0$.*

Definition 17. Differential forms in $\ker(d)$ are called closed, forms in $\text{im}(d)$ are called exact. In other words, we call ϕ closed iff $d\phi = 0$ and we call ϕ exact if there exists a differential form η with $d\eta = \phi$.

Since $d^2 = 0$, $\text{im}(d) \subset \ker(d)$. That is, if ϕ is exact, then ϕ is closed. This is the content of theorem 10.6 in Apostol Vol II. However, not all closed forms are exact. For example, the differential form

$$\phi = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is closed and not exact. (Why?) The question as to whether a closed form is exact depends importantly on the topology of the set on which ϕ is defined. Theorems 10.9 and 11.11 in Apostol Vol II begin to answer the question. The starting point is the following important theorem

The Poincare Lemma. If ϕ is a C^1 one form defined in an open rectange containing the origin, then ϕ is closed iff ϕ is exact.

Proof. We give the proof for a one form in $\mathbb{R}^2$. Let $\phi = gdx + hdy$ and suppose $d\phi = 0$. Since $d\phi = \frac{\partial g}{\partial y}dy \wedge dx + \frac{\partial h}{\partial x}dx \wedge dy$, $d\phi = 0$ means

$$\frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}.$$

We define a function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows. For $(x, y) \in \mathbb{R}^2$, define $\alpha : [0, 2] :\to \mathbb{R}^2$ by

$$\alpha(t) = \begin{cases} (xt, 0) & \text{for } 0 \le x \le 1, \\ (x, (t-1)y) & \text{for } 1 < t \le 2. \end{cases}$$

Then, set $f(x,y) = \int_\alpha \phi$. Now we compute $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ (in order to see that $df = \phi$). First, Use the definition of $\int_\alpha \phi$ and some one variable substitutions to write

$$f(x,y) = \int_0^1 g(tx,0)xdt + \int_1^2 h(x,(t-1)y)ydt = \int_0^x g(u,0)du + \int_0^y h(x,u)du.$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\int_0^x g(u,0)du + \frac{\partial}{\partial x}\int_0^y h(x,u)du \\
&= g(x,0) + \int_0^y \frac{\partial h(x,u)}{\partial x}du \\
&= g(x,0) + \int_0^y \frac{\partial g(x,u)}{\partial u}du \\
&= g(x,0) + g(x,y) - g(x,0) \\
&= g(x,y).
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\int_0^x g(u,0)du + \frac{\partial}{\partial y}\int_0^y h(x,u)du \\
&= h(x,y).
\end{aligned}$$

So, we have $df = g(x,y)dx + h(x,y)dy = \phi$, completing the proof. □

4.1 Surface integrals

Just as we define integrals of one forms over curves, we can define integrals of two forms over surfaces, integrals of three forms over 3-dimensional solids, etc...

First, we need to define the pullback of a j-form by a differentiable function. This definition naturally extends the definition for the pullback of a 1-form.

Definition 18. Let $F : \mathbb{R}^n \to \mathbb{R}^k$ be differentiable. We define a map, called the pullback,

$$\begin{aligned}
F^* : \Omega^j(\mathbb{R}^k) &\to \Omega^j(\mathbb{R}^n) \\
\phi &\mapsto F^*(\phi)
\end{aligned}$$

where $F^*(\phi)((v_1)_p, \ldots, (v_j)_p) = \phi_{F(p)}(F_*(v_1)_p, \ldots, F_*(v_j)_p)$.

Problem 13. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^k$ and ϕ, ψ be differential forms on $\mathbb{R}^k$.

(a) $F^*(\phi + \psi) = F^*(\phi) + F^*(\psi)$

(b) $F^*(\phi \wedge \psi) = F^*(\phi) \wedge F^*(\psi)$

(c) $F^*(d\phi) = d(F^*\phi)$.

Definition 19. Let $\beta : [a_1, b_1] \times \cdots \times [a_k, b_k] \to \mathbb{R}^n$ be differentiable and let η be an k-form. We define

$$\int_\beta \eta = \int_R \beta^*(\eta)$$

where R is the k-dimensional box $[a_1, b_1] \times \cdots \times [a_k, b_k] \subset \mathbb{R}^k$.

Let us analyze this definition for the integral of a two form over a surface in $\mathbb{R}^3$. Suppose $\beta : [a, b] \times [c, d] \to \mathbb{R}^3$ is defined by

$$\beta(s, t) = (f(s, t), g(s, t), h(s, t))$$

and ω is the two form defined by

$$\omega = F dx \wedge dy + G dx \wedge dz + H dy \wedge dz.$$

Then,

$$\begin{aligned} \int_\beta \omega = \int_R \beta^* \omega = \int_R F(x, y, z) \left(\frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt \right) \wedge \left(\frac{\partial g}{\partial s} ds + \frac{\partial g}{\partial t} dt \right) \\ + G(x, y, z) \left(\frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt \right) \wedge \left(\frac{\partial h}{\partial s} ds + \frac{\partial h}{\partial t} dt \right) \\ + H(x, y, z) \left(\frac{\partial g}{\partial s} ds + \frac{\partial g}{\partial t} dt \right) \wedge \left(\frac{\partial h}{\partial s} ds + \frac{\partial h}{\partial t} dt \right). \end{aligned} \tag{2}$$

Problem 14. Verify the formula in equation (2). *Hint:* You know that $\beta^*\omega$ must have the form $r(s, t) ds \wedge dt$ for some $r(s, t)$. To determine r, note that $r(s, t) ds \wedge dt \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) = r(s, t)$ So, look at $\beta^* \omega \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) = \omega \left(\beta_* \left(\frac{\partial}{\partial s} \right), \beta_* \left(\frac{\partial}{\partial t} \right) \right) =$ and keep going...

Problem 15. Consider the surface $\beta : [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ defined by

$$\beta(s, t) = (\cos(s) \cos(t), \sin(s) \cos(t), \sin(t)).$$

The image of β is the unit sphere in $\mathbb{R}^3$. Compute

$$\int_\beta xz(dy \wedge dz) + yz(dz \wedge dx) + x^2(dx \wedge dy).$$

(This is basically problem 7 on page 437 from the textbook).

And now we have the main theorem.

Stokes Theorem. If ϕ is a 1 form and $\beta : [a, b] \times [c, d] \to \mathbb{R}^n$ is a surface. Then

$$\int_\beta d\phi = \int_{\mathfrak{d}\,\beta} \phi.$$

Let me make a couple of remarks before the proof. In the case that $n = 2$, that is, β is a surface in $\mathbb{R}^2$, this theorem is sometimes called Green's theorem. That's theorem 11.10 in Apostol Vol II. For $n = 3$ (surface integrals in $\mathbb{R}^3$) it appears in Apostol as theorem 12.3. I didn't state the more general version that asserts that $\int_\beta d\phi = \int_{\partial\beta} \phi$ for any $k - 1$ form ϕ and any k-dimensional surface β. It's not hard to prove, it's just hard to define $\mathfrak{d}\,\beta$ with the correct signs. For a parametrized two dimensional surface:

$$\beta : [a, b] \times [c, d] \to \mathbb{R}^n$$

we define $\mathfrak{d}\,\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ where

$$\alpha_1 : [c, d] \to \mathbb{R}^n \text{ is defined by } \alpha_1(t) = \beta(b, t)$$
$$\alpha_2 : [a, b] \to \mathbb{R}^n \text{ is defined by } \alpha_2(s) = \beta(b + a - s), d)$$
$$\alpha_3 : [c, d] \to \mathbb{R}^n \text{ is defined by } \alpha_3(t) = \beta(a, d + c - t)$$
$$\alpha_4 : [a, b] \to \mathbb{R}^n \text{ is defined by } \alpha_4(s) = \beta(s, c)$$

Problem 16. Sketch a picture of β and $\mathfrak{d}\,\beta$

Now we prove Stokes theorem.

Proof.

$$\begin{aligned}\int_\beta d\phi &= \int_R \beta^*(d\phi) \\ &= \int_R d\beta^*(\phi) \\ &= \int_R \left(\frac{\partial \phi(\frac{\partial \beta}{\partial t})}{\partial s} - \frac{\partial \phi(\frac{\partial \beta}{\partial s})}{\partial t} \right) ds \wedge dt.\end{aligned}$$

Let's compute the first term:

$$\int_c^d \left(\int_a^b \frac{\partial \phi(\frac{\partial \beta}{\partial t})}{\partial s} ds \right) dt = \int_c^d \phi \left(\frac{\partial \beta(b,t)}{\partial t} \right) - \phi \left(\frac{\partial \beta(a,t)}{\partial t} \right) dt$$
$$= \int_{\alpha_1} \phi + \int_{\alpha_3} \phi.$$

Similarly, the second term gives

$$-\int_a^b \left(\int_c^d \frac{\partial \phi(\frac{\partial \beta}{\partial s})}{\partial t} dt \right) ds = \int_{\alpha_2} \phi + \int_{\alpha_4} \phi.$$

□

We have the expected corollary

Corollary 2. *If ω is exact then $\int_\beta \omega = 0$ for all closed surfaces β.*

And (although we don't have time to prove it this semester!) we also have the Poincare Lemma for n forms.

The Poincare Lemma. If ϕ is a C^1 form defined in an open, convex set contining the origin, the ϕ is closed iff ϕ is exact.

4.2 Connections with vector calculus

Finally, I'll supply a little dictionary between the language of differential forms and the language of vector calculus in $\mathbb{R}^3$.

First, there are two correspondences to be aware of. First, I'll state the correspondence here simply, without worrying about what kind of correspondence each is.

	vector field	differential form
correspondence A	$f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$	$fdx + gdy + hdz$
correspondence B	$f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$	$hdy \wedge dz - gdx \wedge dz + fdy \wedge dz$

In vector calculus one has the following three notions:

- **Gradient**. Given a function $f : \mathbb{R}^3 \to \mathbb{R}$, one has the gradient vector field

$$\nabla(f) = \frac{\partial f}{\partial x_i} U_i.$$

- **Curl**. Given a vector field $X = \sum_i f_i U_i$, one has a vector field called the curl of X:

$$\mathrm{curl}(X) = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) U_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) U_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) U_3.$$

- **Divergence**. Given a vector field $X = \sum_i f_i U_i$ one has a function called the divergence of X

$$\mathrm{Div}(X) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.$$

Problem 17. Express these three concepts in terms of the exterior derivatives and the two correspondences in the table.