**EXAM**

Exam 1

Math 157

Thursday, October 3, 2013

# ANSWERS

**Problem 1**. Let $S = \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$ and define two binary operations $+$ and $\times$ as follows:

| $+$ | $\heartsuit$ | $\diamondsuit$ | $\clubsuit$ | $\spadesuit$ |
|---|---|---|---|---|
| $\heartsuit$ | $\heartsuit$ | $\diamondsuit$ | $\clubsuit$ | $\spadesuit$ |
| $\diamondsuit$ | $\diamondsuit$ | $\heartsuit$ | $\spadesuit$ | $\clubsuit$ |
| $\clubsuit$ | $\clubsuit$ | $\spadesuit$ | $\heartsuit$ | $\diamondsuit$ |
| $\spadesuit$ | $\spadesuit$ | $\clubsuit$ | $\diamondsuit$ | $\heartsuit$ |

| $\times$ | $\heartsuit$ | $\diamondsuit$ | $\clubsuit$ | $\spadesuit$ |
|---|---|---|---|---|
| $\heartsuit$ | $\heartsuit$ | $\heartsuit$ | $\heartsuit$ | $\heartsuit$ |
| $\diamondsuit$ | $\heartsuit$ | $\diamondsuit$ | $\heartsuit$ | $\diamondsuit$ |
| $\clubsuit$ | $\heartsuit$ | $\heartsuit$ | $\clubsuit$ | $\clubsuit$ |
| $\spadesuit$ | $\heartsuit$ | $\diamondsuit$ | $\clubsuit$ | $\spadesuit$ |

(a) Which element of $S$ is an identity for the operation $+$?

***Answer***:

$\heartsuit$ is an identity for $+$. You can see this from the first row and column of the $+$ table.

(b) Solve the equation $\diamondsuit x + \clubsuit = \spadesuit$ for $x$.

***Answer***:

Since $\diamondsuit$ is the only element that when added to $\clubsuit$ results in $\spadesuit$, we must have $\diamondsuit x = \diamondsuit$. Looking at the multiplication table reveals that $\diamondsuit x = \diamondsuit$ if and only if $x = \diamondsuit$ or $x = \spadesuit$.

(c) Only one of the field axioms is not satisfied by $S$ with $+$ and $\times$. Which one?

***Answer***:

Note that $\spadesuit$ is an identity for $\times$. However, $\diamondsuit$ is a nonzero (i.e., not $\heartsuit$) element of $S$ that has no multiplicative inverse—there's no element of $S$ that when multiplied by $\diamondsuit$ yields $\spadesuit$.

*Remark:* Note that if $\diamondsuit$ had an inverse, call it $\diamondsuit^{-1}$, then the equation $\diamondsuit x + \clubsuit = \spadesuit$ would have a unique solution $x = \diamondsuit^{-1}(\spadesuit + -\clubsuit)$.

**Problem 2**. Let $A, B$ and $C$ be sets.

(a) One of the following conditions is sufficient for $(A \setminus B) \setminus C = A \setminus (B \setminus C)$. Which one?

- $A \subset (B \cup C)$
- $(B \cup C) \subset A$
- $A \cap B \cap C = \emptyset$
- $C \subset (B \setminus A)$
- $A \cap B = C \cap B$

(b) Prove that the condition you identified implies that $(A \setminus B) \setminus C = A \setminus (B \setminus C)$.

***Answer***:

The condition $C \subset (B \setminus A)$ is sufficient for $(A \setminus B) \setminus C = A \setminus (B \setminus C)$. That is, for all sets $A$, $B$, and $C$ with $C \subset (B \setminus A)$, we have $(A \setminus B) \setminus C = A \setminus (B \setminus C)$.

*Proof*. First we prove that for all sets $A, B, C$

$$(A \setminus B) \setminus C \subseteq A \setminus (B \setminus C).$$

Let $a \in (A \setminus B) \setminus C$. This means $a \in A \setminus B$ and $a \notin C$. The fact that $a \in A \setminus B$ implies that $a \in A$ and $a \notin B$. Since $a \notin B$, it follows that $a \notin B \setminus C$. Hence $a \in A \setminus (B \setminus C)$.

Now we prove that for all sets $A, B, C$ with $C \subset (B \setminus A)$,

$$A \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C.$$

So suppose $C \subset (B \setminus A)$ and let $a \in A \setminus (B \setminus C)$. So, $a \in A$ and $a \notin B \setminus C$. The fact that $a \notin B \setminus C$ means that

$$a \notin B \text{ or } a \in C.$$

But since $C \subset (B \setminus A)$, it's impossible for $a \in C$ since $a \in C$ implies $a \notin A$ and we know $a \in A$. Therefore, we have $a \in A$, $a \notin B$, and $a \notin C$. That is $a \in (A \setminus B) \setminus C$.

(c) Give an example to show that the identified condition is not *necessary* for $A \setminus B = A \setminus C$.

***Answer***:

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 5, 6, 7\}$, and $C = \{6, 7, 8, 9\}$. Note that $C$ is not a subset of $B \setminus A = \{5, 6, 7\}$. We do have

$$(A \setminus B) \setminus C = \{3, 4\} \setminus C = \{3, 4\} \text{ and } A \setminus (B \setminus C) = A \setminus \{1, 2, 5\} = \{3, 4\}.$$

**Problem 3**. True or False. Give brief, but conclusive evidence, to support your answer.

(a) For all sets $S$ and for all $A \subseteq S$ there exists a unique set $B \subseteq S$ with $A \cup B = S$.

***Answer***:

False. Let $S = \{1, 2, 3, 4\}$ and $A = \{1, 3\}$. Note that $B = \{2, 4\}$ and $B' = \{1, 2, 4\}$ are different sets with the property that $A \cup B = S$.

(b) For all sets $A \subseteq \mathbb{R}$, either $A$ or $\mathbb{R} \setminus A$ is bounded above.

***Answer***:

False. Let $A = \mathbb{N}$. Note $A$ is not bounded above, and $\mathbb{R} \setminus \mathbb{N}$ is not bounded above.

(c) For all $x, y \in \mathbb{R}$, if $x^2 < y^2$ then either $x < y$ or $-x < y$.

***Answer***:

False. Let $x = 3$ and $y = -4$. Then $x^2 = 9 < 16 = y^2$. However, neither $3 < -4$ nor $-3 < -4$ are true.

(d) For all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$ $\left( |2x - 6| < \frac{1}{n} \Rightarrow x = 3 \right)$.

***Answer***:

True. If $|2x - 6| < \frac{1}{n}$ for all $n \in \mathbb{N}$, then $2x - 6 = 0$, which implies that $x = 3$.

**Problem 4**.

(a) Use mathematical induction to prove that $\sum_{k=1}^{n} k^3 = \frac{(n)^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$.

***Answer***:

To prove that $\sum_{k=1}^{n} k^3 = \frac{(n)^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$, note that for $n = 1$, the statement is $1 = \frac{(1^2)(2^2)}{4}$, which is true.

Now suppose that $\sum_{k=1}^{m} k^3 = \frac{(m)^2(m+1)^2}{4}$ for some $m \in \mathbb{N}$ and consider $\sum_{k=1}^{m} k^3$

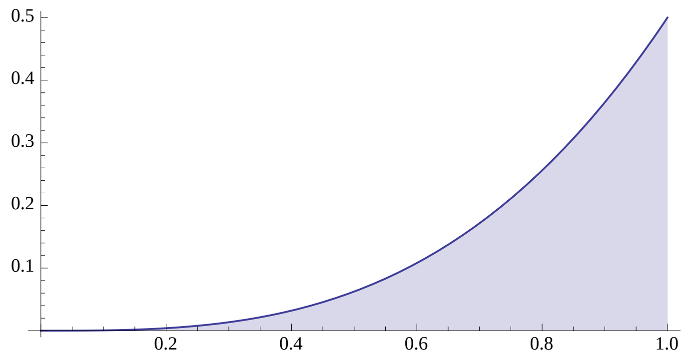
$$\begin{aligned}
\sum_{k=1}^{m+1} k^3 &= \left(\sum_{k=1}^{m} k^3\right) + (m+1)^3 \\
&= \frac{(m)^2(m+1)^2}{4} + (m+1)^3 \\
&= \frac{(m)^2(m+1)^2}{4} + \frac{4(m+1)^2(m+1)}{4} \\
&= \frac{(m+1)^2\left(m^2+4(m+1)\right)}{4} \\
&= \frac{(m+1)^2(m^2+4m+4)}{4} \\
&= \frac{(m+1)^2(m+2)^2}{4}
\end{aligned}$$

This proves that if the statement is true for $n = m$ then it's true for $n = m + 1$.

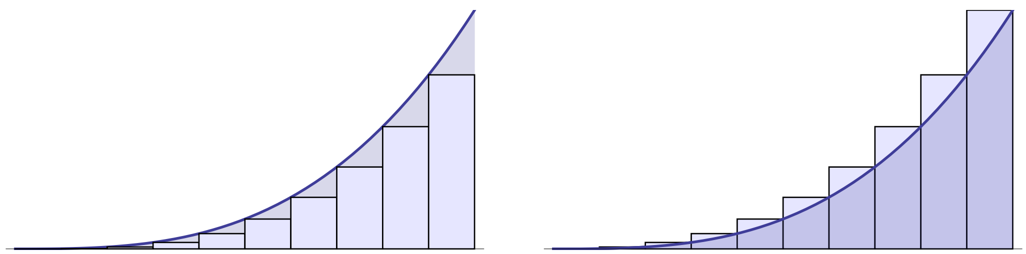
Since the statement is true for $n = 1$, and true for $m$ implies true for $m + 1$, the principle of mathematical induction, says it's true for all $n \in \mathbb{N}$.

**Problem 4**.

(b) Use this result to compute the area of the region pictured below (the vertical distance between the point $b$ units from 0 is $\frac{1}{2}b^3$).



***Answer***:
To compute the area of the region in question, we bound it by two rectangular regions obtained by inscribing $n$ rectangles of equal width and circumscribing $n$ rectangles of equal width, as pictured below:



Let $s_n$ be the area of the inscribed rectangular region, let $A$ be the area of the curved region, and $S_n$ be the area of the circumscribed rectangular region.

$$s_n < A < S_n.$$

We express the areas of the rectangular regions $s_n$ as a sum, the $i$-th terms of which is the area of an inscribed rectangle of width $\frac{1}{n}$ and height $\frac{1}{2}\left(\frac{i-1}{n}\right)^3$.

$$\begin{aligned} s_n &= \frac{1}{n}\left(\frac{1}{2}(0)^3\right) + \frac{1}{n}\left(\frac{1}{2}\left(\frac{1}{n}\right)^3\right) + \frac{1}{n}\left(\frac{1}{2}\left(\frac{2}{n}\right)^3\right) + \cdots + \frac{1}{n}\left(\frac{1}{2}\left(\frac{(n-1)}{n}\right)^3\right) \\ &= \frac{1}{2n^4}\left(1^3 + 2^3 + \cdots (n-1)^3\right) \end{aligned}$$

Similarly, we express the areas of the rectangular regions $S_n$ as a sum, the $i$-th terms of which is the the area of a circumscribed rectangle of width $\frac{1}{n}$ and height $\frac{1}{2}\left(\frac{i}{n}\right)^3$.

$$\begin{aligned} s_n &= \frac{1}{n}\left(\frac{1}{2}(1)^3\right) + \frac{1}{n}\left(\frac{1}{2}\left(\frac{2}{n}\right)^3\right) + \cdots + \frac{1}{n}\left(\frac{1}{2}\left(\frac{(n-1)}{n}\right)^3\right) + \frac{1}{n}\left(\frac{1}{2}\left(\frac{n}{n}\right)^3\right) \\ &= \frac{1}{2n^4}\left(1^3 + 2^3 + \cdots (n)^3\right) \end{aligned}$$

By the inequalities proved above, we have

$$s_n = \left(\frac{1}{2n^4}\right)\left(\frac{(n-1)^2(n)^2}{4}\right) = \frac{1}{8}\left(1 - \frac{1}{n}\right)^2 \text{ and } S_n = \left(\frac{1}{2n^4}\right)\left(\frac{(n)^2(n+1)^2}{4}\right) = \frac{1}{8}\left(1 + \frac{1}{n}\right)^2$$

So, for every $n \in \mathbb{N}$, we have

$$s_n < \frac{1}{8} < S_n \text{ and } s_n < A < S_n.$$

Since $S_n - s_n = \frac{1}{2n}$, we have $\left|A - \frac{1}{8}\right| \leq \frac{1}{2n}$ for every $n \in \mathbb{N}$. Thus, $\left|A - \frac{1}{8}\right| = 0$, and we conclude

$$A = \frac{1}{8}.$$