EXAM

Exam 2

Math 157

Thursday, October 17, 2013

ANSWERS

Problem 0. **Bonus**. "lapidate" means kill by throwing stones at.

Problem 1. Let $x \geq -1$. Prove that for all $n \in \mathbb{N}$ we have $(1+x)^n \geq 1 + xn$.

Answer:
We use induction. When $n = 1$, the statement $(1+x)^n \geq 1 + xn$ is the statement

$$(1+x)^1 \geq 1 + x,$$

which is true since both sides are equal.
Now, assume the statement is true for some natural number k. That is, assume

$$(1+x)^k \geq 1 + xk.$$

Consider $(1+x)^{k+1}$:

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) && \text{because } (1+x)^k \geq 1 + xk \text{ and } (1+x) \geq 0 \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x && \text{because } kx^2 \geq 0.
\end{aligned}$$

This proves that if the statement is true for k, then the statement is true for $k+1$, completing our proof by induction.

Problem 2. Let X and Y be sets and $f : X \to Y$ be a function. Consider the following argument:

> *Suppose that* $f : X \to Y$ *is not injective. Then there exist elements* $x, z \in X$ *with* $f(x) = f(z)$ *and* $x \neq z$*. Let* $A = \{x\}$*. Note that* $z \in X \setminus A$ *so*
>
> $$f(z) \in f(X \setminus A).$$
>
> *Since* $x \in A$*,* $f(x) \in f(A)$*, so*
>
> $$f(x) \notin Y \setminus f(A).$$
>
> *Since* $f(x) = f(z)$*, we've found an element in* $f(X \setminus A)$ *that is not in* $Y \setminus f(A)$*. Therefore,* $f(X \setminus A) \not\subseteq Y \setminus f(A)$*.*

Which of the following propositions does the argument above prove?

(a) If f is injective, then for all $A \subseteq X$ we have $f(X \setminus A) \subseteq Y \setminus f(A)$.

(b) If f is injective, then there exists a set $A \subseteq X$ with $f(X \setminus A) \subseteq Y \setminus f(A)$.

(c) If for all subsets $A \subseteq X$ we have $f(X \setminus A) \subseteq Y \setminus f(A)$, then f is injective.

(d) If f is injective, then for all $A \subseteq X$ we have $Y \setminus f(A) \subseteq f(X \setminus A)$.

(e) If for all sets $A \subseteq X$ we have $Y \setminus f(A) \subseteq f(X \setminus A)$, then f is injective.

(f) If f is not injective, then for all $A \subseteq X$ we have $f(X \setminus A) \not\subseteq Y \setminus f(A)$.

Answer:

(c). The argument proves that if f is not injective, there exists a set $A \subseteq X$ for which $f(X \setminus A) \not\subseteq Y \setminus f(A)$. This proves that if for all sets $A \subset X$ we have $f(X \setminus A) \not\subseteq Y \setminus f(A)$, then f is injective.

Problem 3. Give an example of a function $f : X \to Y$ and a set $A \subseteq X$ for which

$$f^{-1}(f(A)) \neq A.$$

Answer:
Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ and define

$$\begin{aligned} f : X &\to Y \\ 1 &\mapsto a \\ 2 &\mapsto b \\ 3 &\mapsto b \\ 4 &\mapsto c \end{aligned}$$

Let $A = \{1, 2\}$. Then

$$f^{-1}(f(A)) = f^{-1}(\{a, b\}) = \{1, 2, 3\} \neq A.$$

Problem 4. True or False. Right answer $+1$, wrong answer -1, no answer 0.

(a) The field axioms of $\mathbb{R}$ imply that $1 + 1 \neq 0$.

Answer:

False. To see that it's false, note that $\{0, 1\}$ with addition defined by

$$0 + 0 = 0, \qquad 0 + 1 = 1 + 0 = 1, \qquad 1 + 1 = 0$$

and multiplication defined by

$$0 \times 0 = 0, \qquad 0 \times 1 = 1 \times 0 = 0, \qquad 1 \times 1 = 1$$

satisfies all the field axioms that $\mathbb{R}$ satisfies and $1 + 1 = 0$.

(b) For all functions $f : X \to Y$ and for all sets $C \subseteq Y$, we have $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$.

Answer:

True. Here's a proof.

Let $f : X \to Y$ be a function and $C \subseteq Y$. To show that $f^{-1}(Y \setminus C) \subseteq X \setminus f^{-1}(C)$, let $x \in f^{-1}(Y \setminus C)$. This means that $f(x) \in Y \setminus C$. So, $f(x) \notin C$. Since $f(x) \notin C$, we know $x \notin f^{-1}(C)$, implying that $x \in X \setminus f^{-1}(C)$.

To show that $X \setminus f^{-1}(C) \subseteq f^{-1}(Y \setminus C)$, let $x \in X \setminus f^{-1}(C)$. So $x \notin f^{-1}(C)$. This implies that $f(x) \notin C$. Therefore $f(x) \in Y \setminus C$. This says that $x \in f^{-1}(Y \setminus C)$ as needed.

Problem 4. Continued.

(c) For all functions $f : X \to Y$ and for all sets $C \subseteq Y$, we have $f(f^{-1}(C)) = C$.

Answer:

False. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ and define

$$\begin{aligned} f : X &\to Y \\ 1 &\mapsto a \\ 2 &\mapsto b \\ 3 &\mapsto b \\ 4 &\mapsto c \end{aligned}$$

Let $C = \{b, d\}$. Then $f^{-1}(C) = \{2, 3\}$ and $f(f^{-1}(C)) = \{b\}$.

(d) For all injective functions $f : X \to Y$ and for all sets $A \subseteq X$ we have $f^{-1}(f(A)) = A$.

Answer:

True. Here's a proof.

First, we'll prove that for all functions $f : X \to Y$ and all subsets $A \subset X$, we have $A \subseteq f^{-1}(f(A))$. So, suppose $f : X \to Y$ is a function and $A \subseteq X$. Let $x \in A$. Then $f(x) \in f(A)$. Since $f^{-1}(f(X))$ consists of all elements $x \in X$ with $f(x) \in f(A)$, we have $x \in f^{-1}(f(A))$.

Now we will prove that if f is injective, we have $f^{-1}(f(A)) = A$. So, let $x \in f^{-1}(f(A))$. This means that $f(x) \in f(A)$. This means that there exists a $z \in A$ with $f(z) = f(x)$. Since f is injective, $x = z$, and we find $x \in A$.