Problem 1. Define. **[2 points each]**

(a) Let $s : [a, b] \to \mathbb{R}$ be a function. What does it mean to say that s is a *step function*?

(b) Let $s : [a, b] \to \mathbb{R}$ be a step function. Define $\int_a^b s$.

(c) Let $f : [a, b] \to \mathbb{R}$ be any bounded function. Define the statement f *is integrable*, and the expression $\int_a^b f$.

Problem 2. [2 points each] Compute. Do three out of four, or do all four for a bonus.

(a) $\displaystyle\int_1^5 \left[\frac{1}{2}x + 1\right] dx$

(b) $\displaystyle\int_0^5 f$ where $f(x) = \begin{cases} 1 & \text{if } 3 \leq |x^2 - 4| \leq 5 \\ 0 & \text{otherwise.} \end{cases}$.

Problem 2. Continued.

(c) $\int_0^4 |2x - 6| dx$

(d) $\int_1^3 (3x^2 - 5x) dx$

Problem 3. True or false : **[Right answer $+1$, wrong answer -1, no answer $+\frac{1}{2}$]**

(a) For all functions $f, g, h : \mathbb{R} \to \mathbb{R}$, $f \circ (g + h) = f \circ g + f \circ h$.

(b) For every function $f : [a, b] \to \mathbb{R}$, there exist step functions $s, t : [a, b] \to \mathbb{R}$ with $s \le f \le t$.

(c) If f is integrable and even, then $\int_{-b}^{b} f(x)dx = 2\int_{0}^{b} f(x)dx$.

(d) If $s : [a, b] \to \mathbb{R}$ is a step function, then $s([a, b])$ is a finite set.

(e) If $s : [a, b] \to \mathbb{R}$ is a step function, then for any $y \in \mathbb{R}$, the set $s^{-1}(\{y\})$ is either empty or an interval.

(f) Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the canonical projections. Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be the ordinate set of a nonegative function f. Then

$$\pi_1(A) = \text{the domain of } f \quad \text{and} \quad \pi_2(A) = \text{the range of } f.$$

(g) For every $\epsilon > 0$, there exists a step function $s : [0, 1] \to \mathbb{R}$ with

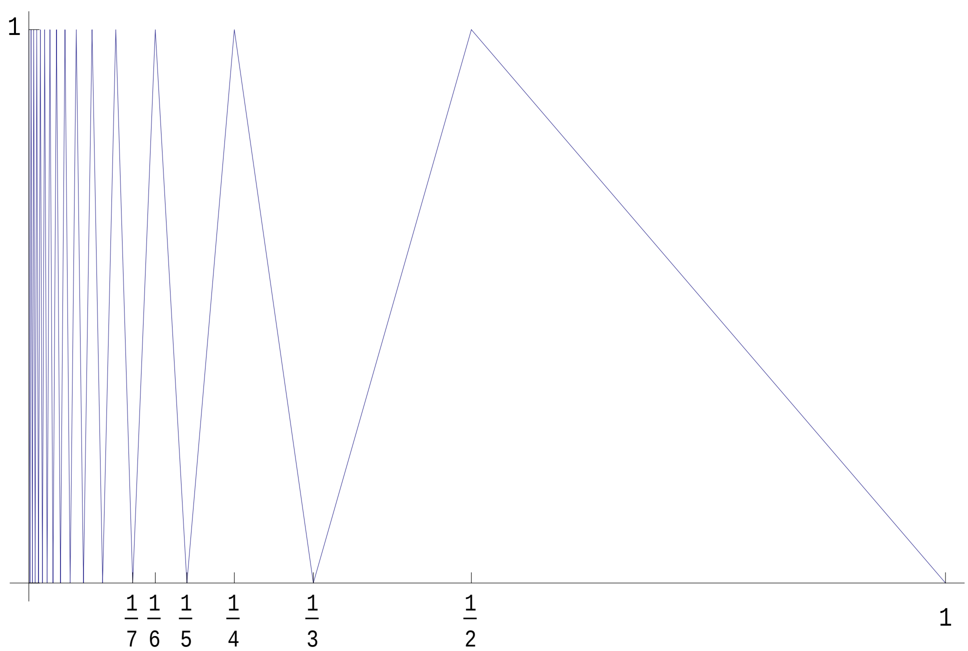
$$s(x) \le x^2 \text{ for all } x \in [0, 1] \text{ and } \int_0^1 s > \frac{1}{3} - \epsilon.$$

(h) If $f, g : [a, b] \to \mathbb{R}$ are integrable, then so is fg and $\int_a^b fg = \left(\int_a^b f\right)\left(\int_a^b g\right)$

Problem 4. Consider the function $f : [0,1] \to \mathbb{R}$ whose graph consists of straight lines connecting

$$
\begin{gathered}
(1,0) \text{ and } \left(\frac{1}{2},1\right) \\
\left(\frac{1}{2},1\right) \text{ and } \left(\frac{1}{3},0\right) \\
\left(\frac{1}{3},0\right) \text{ and } \left(\frac{1}{4},1\right) \\
\left(\frac{1}{4},1\right) \text{ and } \left(\frac{1}{5},0\right) \\
\vdots
\end{gathered}
$$

The value of f at zero is immaterial, say $f(0) = 0$ if you like. One could give a formula for f on each interval $\left(\frac{1}{n}, \frac{1}{n+1}\right)$ but a picture is worth a thousand words:



(a) **[2 points]** Is f is piecewise monotonic on $[0,1]$?

(b) **[2 points]** Is f integrable on $[0,1]$?

(c) **[2 bonus points]** Determine $\underline{I}(f)$ and $\overline{I}(f)$.

Problem 5. Short answer. **[2 points each]** Do three out of four. Or do all four for a bonus.

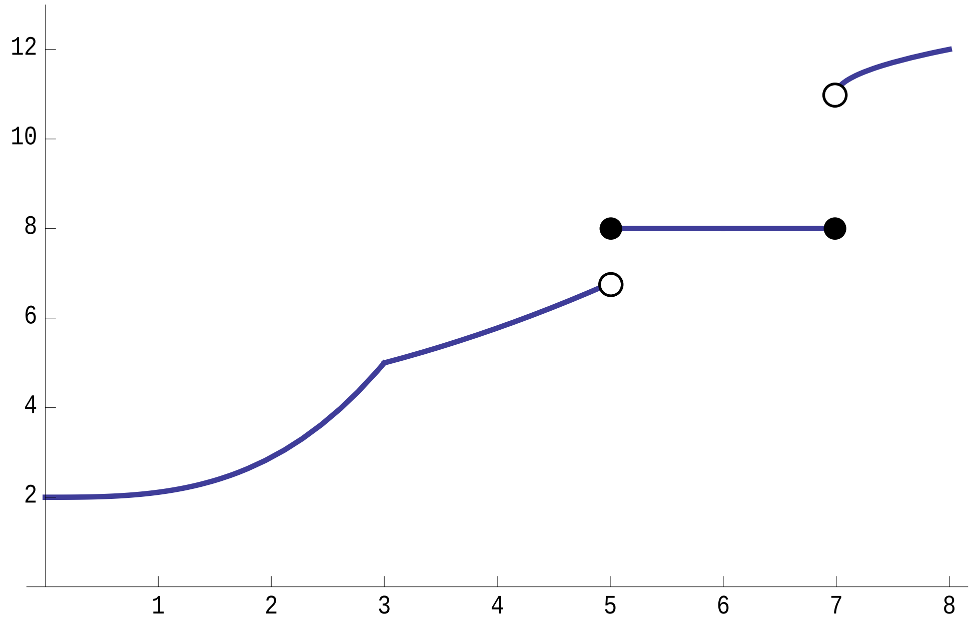
(a) Find a rational number and an irrational number in the interval $\left(0, \frac{1}{2}\right)$.

(b) If possible, find a left inverse and a right inverse of the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 2$. If either is not possible, be sure to say so.

(c) Compute $\underline{I}(f)$ and $\overline{I}(f)$ for $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1 + x & \text{if } x \in \mathbb{Q}, \\ 2 - x & \text{if } x \notin \mathbb{Q}. \end{cases}$

Problem 5. Continued.

(d) Pictured below is the graph of an increasing function g.



Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0, 8]$ into n subintervals of equal length and let

$$A = \sum_{k=1}^{n} g(x_{k-1})(x_k - x_{k-1}) \text{ and } B = \sum_{k=1}^{n} g(x_k)(x_k - x_{k-1}).$$

Note that

$$A \leq \int_0^8 g \leq B.$$

How large must n be in order for $B - A \leq \frac{1}{10}$.

EXAM

Midterm

Math 157

November 14, 2013

- Make sure your solutions are clearly and carefully written. Proofread.
- Show your work, but not your scratchwork. Neatness counts.
- There are 30 points and 6 bonus points (a total of 36 possible).

Success!