
EXAM

Midterm

Math 157

November 14, 2013

ANSWERS

Problem 1. Define. **[2 points each]**

(a) Let $s : [a, b] \to \mathbb{R}$ be a function. What does it mean to say that s is a *step function*?

Answer:

A function $s : [a, b] \to \mathbb{R}$ is a step function if and only if there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$ for which s is constant on the open subintervals (x_{k-1}, x_k).

(b) Let $s : [a, b] \to \mathbb{R}$ be a step function. Define $\displaystyle\int_a^b s$.

Answer:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ for which s is constant on the open subintervals (x_{k-1}, x_k). Let s_k be the constant value of s on (x_{k-1}, x_k). Then

$$\int_a^b s := \sum_{k=1}^{n} s_k(x_k - x_{k-1}).$$

(c) Let $f : [a, b] \to \mathbb{R}$ be any bounded function. Define the statement f *is integrable*, and the expression $\displaystyle\int_a^b f$.

Answer:

If there exists one and only one number I satisfying

$$\int_a^b s < I < \int_a^b t$$

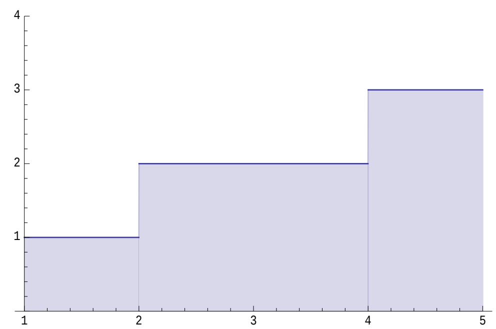
for all step functions s and t with $s < f < t$, then we say f is integrable and we denote the number I by $\displaystyle\int_a^b f$.

Problem 2. **[2 points each]** Compute. Do three out of four, or do all four for a bonus.

(a) $\displaystyle\int_1^5 \left[\frac{1}{2}x+1\right] dx$

Answer:

The function $s(x) = \left[\frac{1}{2}x+1\right]$ is a step function taking the value 1 on $(1,2)$, the value 2 on $(2,4)$ and the value 3 on $(4,5)$.



So,

$$\int_1^5 \left[\frac{1}{2}x+1\right] dx = 1(1)+2(2)+3(1) = 8.$$

(b) $\displaystyle\int_0^5 f$ where $f(x) = \begin{cases} 1 & \text{if } 3 \le |x^2-4| \le 5 \\ 0 & \text{otherwise.} \end{cases}$.

Answer:

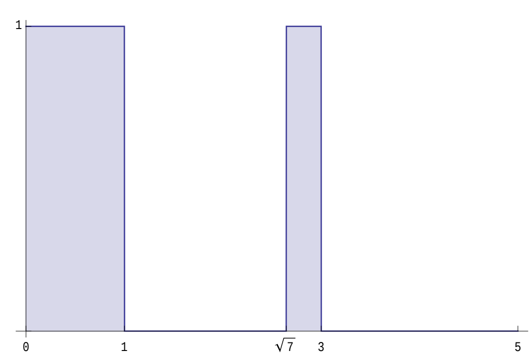
Note that

$$|x^2-4| \le 5 \Leftrightarrow -3 \le x \le 3.$$

and

$$3 \le |x^2-4| \Leftrightarrow -1 \le x \le 1 \text{ or } x \le -\sqrt{7} \text{ or } x \ge \sqrt{7}.$$

Therefore, $3 \le |x^2-4| \le 5 \Leftrightarrow -3 \le x \le -\sqrt{7}$ or $-1 \le x \le 1$ or $\sqrt{7} \le x \le 3$.



So,

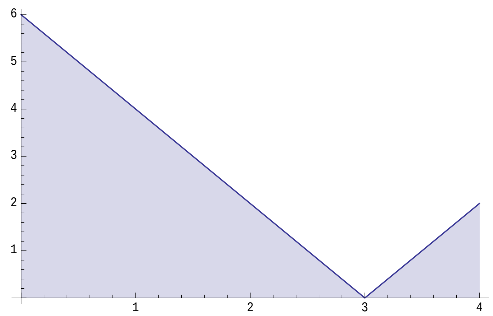
$$\int_0^5 f = 1(1-0)+1(3-\sqrt{7}) = 4-\sqrt{7}.$$

Problem 2. Continued.

(c) $\int_0^4 |2x-6|dx$

Answer:

Since the function $f : [0,4] \to \mathbb{R}$ given by $f(x) = |2x-6|$ is piecewise monotonic, it is integrable and since it's nonegative, the integral is the area of the ordinate set of f. Here's a picture of the ordinate set, which is the union of two triangles of area $\frac{1}{2}(6)(3) = 9$ and $\frac{1}{2}(2)(1) = 1$.



So,

$$\int_0^4 |2x-6|dx = 10.$$

(d) $\int_1^3 (3x^2-5x)dx$ We compute:

$$\begin{aligned}\int_1^3 (3x^2-5x)dx &= \int_0^3 (3x^2-5x)dx - \int_0^1 (3x^2-5x)dx \\ &= 3\int_0^3 x^2dx - 5\int_0^3 xdx - 3\int_0^1 x^2dx + 5\int_0^1 xdx \\ &= 3\frac{3^3}{3} - 5\frac{3^2}{2} - 3\frac{1^3}{3} + 5\frac{1^2}{2} \\ &= 6.\end{aligned}$$

Problem 3. True or false : **[Right answer $+1$, wrong answer -1, no answer $+\frac{1}{2}$]**

(a) For all functions $f, g, h : \mathbb{R} \to \mathbb{R}$, $f \circ (g + h) = f \circ g + f \circ h$.

Answer:

False. For example, if $f(x) = 1$, $g(x) = 2$, $h(x) = 2$, then $f \circ (g + h)(x) = 1$ and $f \circ g + f \circ h = 2$.

(b) For every function $f : [a, b] \to \mathbb{R}$, there exist step functions $s, t : [a, b] \to \mathbb{R}$ with $s \leq f \leq t$.

Answer:

False. The function $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 5$ is unbounded and there are no step functions s with $s \leq f$ and no step function t with $f \leq t$.

(c) If f is integrable and even, then $\displaystyle\int_{-b}^{b} f(x)dx = 2\int_{0}^{b} f(x)dx$.

Answer:

True.

(d) If $s : [a, b] \to \mathbb{R}$ is a step function, then $s([a, b])$ is a finite set.

Answer:

True. By definition, there's a partition $\{x_0, \ldots, x_n\}$ of $[a, b]$ for which s is constant on the n open subintervals (x_{k-1}, x_k). Accounting for the possibly distinct values of s at the $n + 1$ endpoints shows that s takes on at most $2n + 1$ values.

(e) If $s : [a, b] \to \mathbb{R}$ is a step function, then for any $y \in \mathbb{R}$, the set $s^{-1}(\{y\})$ is either empty or an interval.

Answer:

False. For example, if $s : [0, 3] \to \mathbb{R}$ is given by $s(x) = 1$ if $0 \leq x < 1$, $x(x) = 2$ if $1 \leq x \leq 2$ and $s(x) = 1$ if $2 < x \leq 3$. Then we have $s^{-1}(\{1\}) = [0, 1) \cup (2, 3]$ which is not an interval.

Problem 3. Continued.

(f) Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the canonical projections. Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be the ordinate set of a nonegative function f. Then

$$\pi_1(A) = \text{the domain of } f \quad \text{and} \quad \pi_2(A) = \text{the range of } f.$$

Answer:

False. Say $f : [0,1] \to \mathbb{R}$ is given by $f(x) = 2$. Then the ordinate set A of f is the rectangle $A = \{(x,y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2\}$. We see $\pi_1(A) = [0,1] =$ the domain of f but $\pi_2(A) = [0,2] \neq$ the range of $f = \{2\}$.

(g) For every $\epsilon > 0$, there exists a step function $s : [0,1] \to \mathbb{R}$ with

$$s(x) \leq x^2 \text{ for all } x \in [0,1] \text{ and } \int_0^1 s > \frac{1}{3} - \epsilon.$$

Answer:

True. We know that the function given by $f(x) = x^2$ is integrable on $[0,1]$ and $\int_0^1 x^2 = \frac{1}{3}$. So, the lower integral $\underline{I}(f) = \frac{1}{3}$. Since $\frac{1}{3} - \epsilon$ is less than $\frac{1}{3}$, which is the least upper bound of

$$S = \left\{ \int_0^1 s : \ s \text{ is a step function with } s \leq f \right\}$$

there must be an element of S greater than $\frac{1}{3} - \epsilon$. That is, a step function s with $s \leq f$ and $\int_0^1 s > \frac{1}{3} - \epsilon$.

(h) If $f, g : [a,b] \to \mathbb{R}$ are integrable, then so is fg and $\displaystyle\int_a^b fg = \left(\int_a^b f\right)\left(\int_a^b g\right)$

Answer:

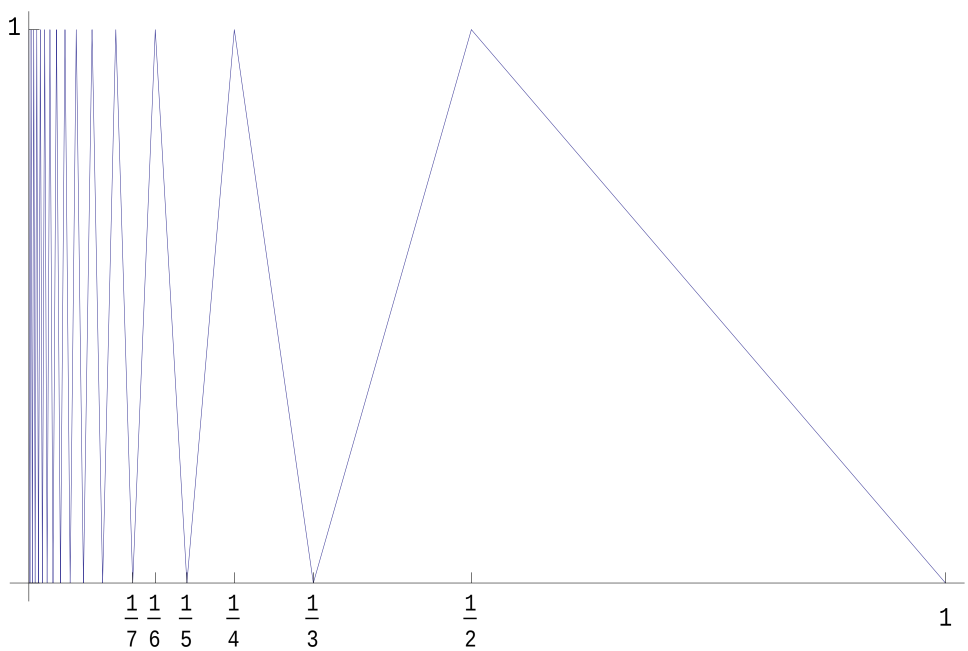
False. If $f = g =$ the constant function 1 and $[a,b] = [0,3]$, then

$$\int_a^b fg = \int_0^3 1 = 3 \text{ and } \left(\int_a^b f\right)\left(\int_a^b g\right) = \left(\int_0^3 1\right)\left(\int_0^3 1\right) = (3)(3) = 9.$$

Problem 4. Consider the function $f : [0,1] \to \mathbb{R}$ whose graph consists of straight lines connecting

$$(1,0) \text{ and } \left(\frac{1}{2},1\right)$$
$$\left(\frac{1}{2},1\right) \text{ and } \left(\frac{1}{3},0\right)$$
$$\left(\frac{1}{3},0\right) \text{ and } \left(\frac{1}{4},1\right)$$
$$\left(\frac{1}{4},1\right) \text{ and } \left(\frac{1}{5},0\right)$$
$$\vdots$$

The value of f at zero is immaterial, say $f(0) = 0$ if you like. One could give a formula for f on each interval $\left(\frac{1}{n}, \frac{1}{n+1}\right)$ but a picture is worth a thousand words:



(a) **[2 points]** Is f is piecewise monotonic on $[0,1]$?

Answer:

No.

(b) **[2 points]** Is f integrable on $[0,1]$?

Answer:

Yes.

Problem 4. Continued.

(c) **[2 bonus points]** Determine $\underline{I}(f)$ and $\overline{I}(f)$.

Answer:

f is integrable, so $\underline{I}(f) = \overline{I}(f) = \int_0^1 f = \frac{1}{2}$.

To see that $\int_0^1 f = \frac{1}{2}$, first note that the ordinate set of f on any interval of the form $\left[\frac{1}{n}, 1\right]$ consists of the union of a finite number of triangles, hence $\int_{\frac{1}{n}}^1 f$ is the sum of the areas of these finitely many rectangles. Adding them up (from right to left) we have

$$\begin{aligned}\int_{\frac{1}{n}}^1 f &= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{n}\right).\end{aligned}$$

Intuitively, as n gets larger and larger, these ordinates sets exhaust more and more of the ordinate set of f and we guess that $\int_0^1 f = \frac{1}{2}$.

To make this idea rigorous, we use step functions. Since $\frac{1}{2}\left(1 - \frac{1}{n}\right) - \frac{1}{n} < \int_{\frac{1}{n}}^1 f$, we can find a step function s with $s < f$ and $\int_{\frac{1}{n}}^1 s \geq \frac{1}{2}\left(1 - \frac{1}{n}\right) - \frac{1}{n}$. By extending s to take the value 0 on $\left[0, \frac{1}{n}\right)$, we've found a step function $\tilde{s}$ less than f with

$$\int_0^1 \tilde{s} \geq \frac{1}{2}\left(1 - \frac{1}{n}\right) - \frac{1}{n} = \frac{1}{2} - \frac{3}{2n}.$$

Similarly, we can find a step function t with $t > f$ on $\left[\frac{1}{n}, 1\right]$ with $\int_{\frac{1}{n}} t < \frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{n}$. By extending t to take the value 1 on $\left[0, \frac{1}{n}\right)$, we've found a step function $\tilde{t}$, greater than f with

$$\int_0^1 \tilde{t} \leq \frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{n} + \frac{1}{n} = \frac{1}{2} + \frac{3}{2n}.$$

Since $\frac{1}{2}$ is the only number that lies between $\frac{1}{2} - \frac{3}{2n}$ and $\frac{1}{2} + \frac{3}{2n}$ for every n, we conclude that f is integrable on $[0, 1]$ and $\int_0^1 f = \frac{1}{2}$.

Problem 5. Short answer. **[2 points each]** Do three out of four. Or do all four for a bonus.

(a) Find a rational number and an irrational number in the interval $\left(0, \frac{1}{2}\right)$.

Answer:

$\frac{1}{4}$ is a rational number, and $\frac{\sqrt{3}}{4}$ is an irrational number, in $\left(0, \frac{1}{2}\right)$. The number $\frac{\sqrt{3}}{4}$ is irrational since $\sqrt{3}$ is irrational and $\frac{1}{4}$ is rational and the product of an irrational number and a nonzero rational number is irrational. The number $\frac{\sqrt{3}}{4}$ lies in $\left(0, \frac{1}{2}\right)$ since

$$0 < \sqrt{3} < \sqrt{4} \Rightarrow 0 < \frac{\sqrt{3}}{4} \leq \frac{\sqrt{4}}{2} = \frac{1}{2}.$$

(b) If possible, find a left inverse and a right inverse of the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 2$. If either is not possible, be sure to say so.

Answer:

Since f is injective, it has a left inverse. Since f is not surjective (the number 1 is not in the image of f) it does not have a right inverse.

The function $g : \mathbb{N} \to \mathbb{N}$ given by

$$g(n) = \begin{cases} n - 2 & \text{if } n \geq 3 \\ 8 & \text{if } n = 1, 2 \end{cases}$$

is a left inverse of f.

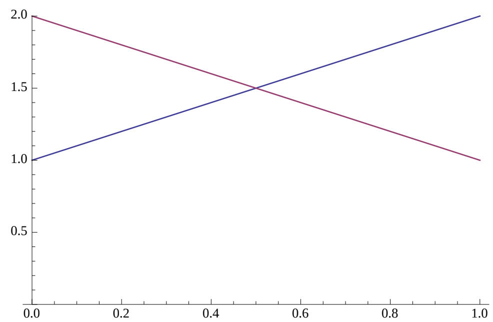
Problem 5. Continued.

(c) Compute $\underline{I}(f)$ and $\overline{I}(f)$ for $f : [0,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1+x & \text{if } x \in \mathbb{Q}, \\ 2-x & \text{if } x \notin \mathbb{Q}. \end{cases}$

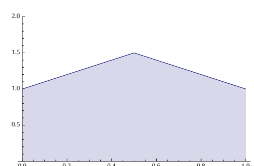
Answer:

$$\underline{I}(f) = \frac{5}{4} \text{ and } \overline{I}(f) = \frac{7}{4}.$$

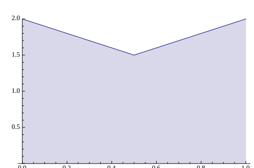
To see this, consider the lines $\{(x,y) : y = 1+x\}$ and $\{(x,y) : y = 2-x\}$.



Any step function less than f has an ordinate set contained in the region pictured below, and whose area is less than, but can be as close as desired, to the area of this region. Therefore $\underline{I}(f)$ is the area of this region, which is $\frac{5}{4}$.

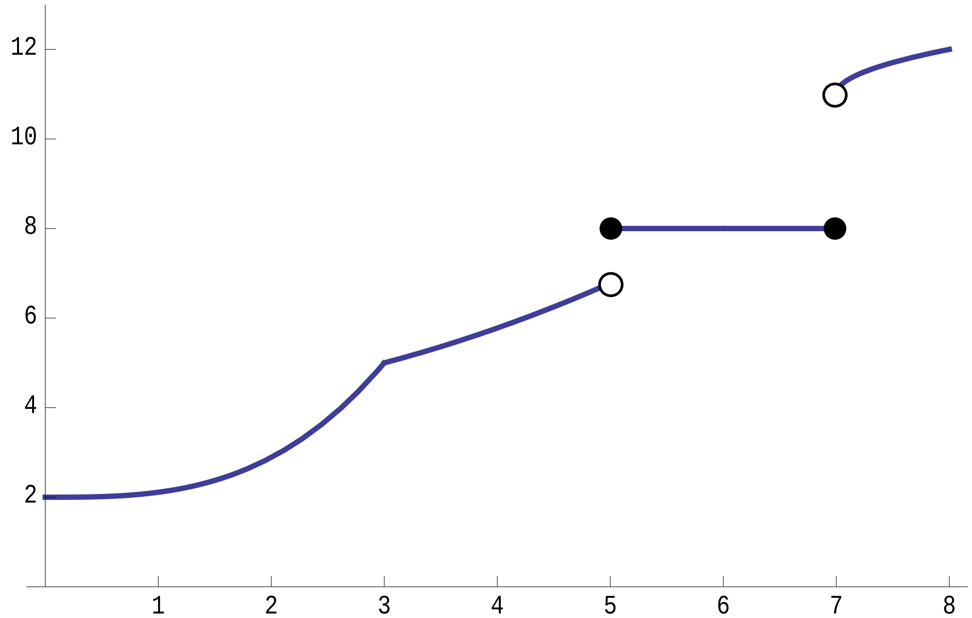


Any step function greater than f has an ordinate set containing the region pictured below and whose area is greater than, but can be as close as desired, to the area of this region. Therefore $\overline{I}(f)$ is the area of this region, which is $\frac{7}{4}$.



Problem 5. Continued.

(d) Pictured below is the graph of an increasing function g.



Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0, 8]$ into n subintervals of equal length and let

$$A = \sum_{k=1}^{n} g(x_{k-1})(x_k - x_{k-1}) \text{ and } B = \sum_{k=1}^{n} g(x_k)(x_k - x_{k-1}).$$

Note that

$$A \leq \int_0^8 g \leq B.$$

How large must n be in order for $B - A \leq \dfrac{1}{10}$.

Answer:

The difference

$$B - A = (g(x_n) - g(x_0))\left(\frac{8}{n}\right) = (12 - 2)\left(\frac{8}{n}\right) = \frac{80}{n}.$$

Setting

$$\frac{80}{n} \leq \frac{1}{10} \Rightarrow n \geq 800.$$