EXAM

Final Exam

Math 157

Tuesday, December 17, 2013

ANSWERS

Definitions and theorems [2 points each]

Problem 1. Let f be a bounded function defined on $[a, b]$. Define the statement "f is integrable" and the number $\int_a^b f$.

Answer:
If there exists one and only one number I satisfying

$$\int_a^b s \leq I \leq \int_a^b t$$

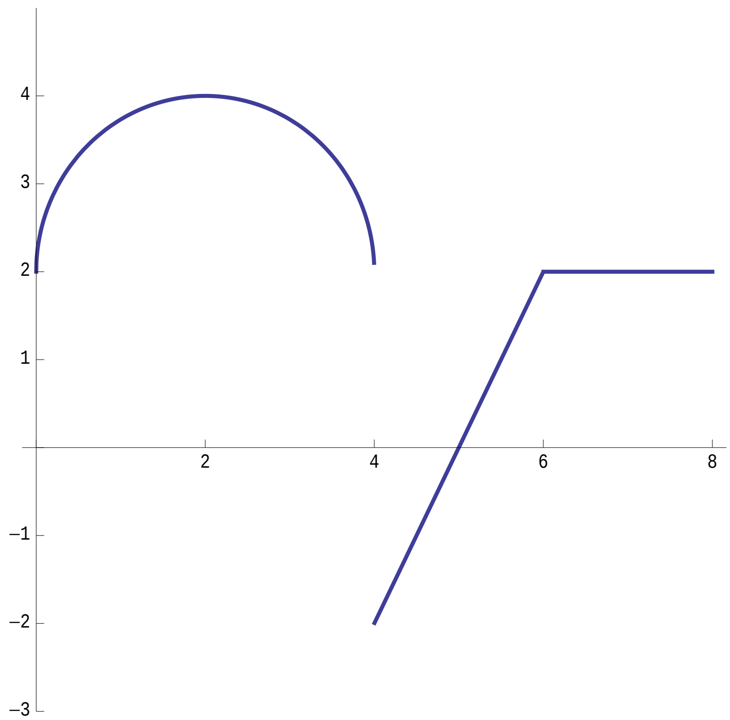
for all step functions s and t with $s \leq f \leq t$, then we say f is integrable and we let $\int_a^b f = I$.

Problem 2. Let f be a function defined on an open neighborhood of c. Define the statement "f is continuous at c."

Answer:
f is continuous at c if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ so that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

Multiple Choice [1 point each]

Problem 3. Below the graph of a function f is sketched



$$\int_2^8 f(t)dt =$$

(a) $\pi + 6$

(b) $\pi + 8$

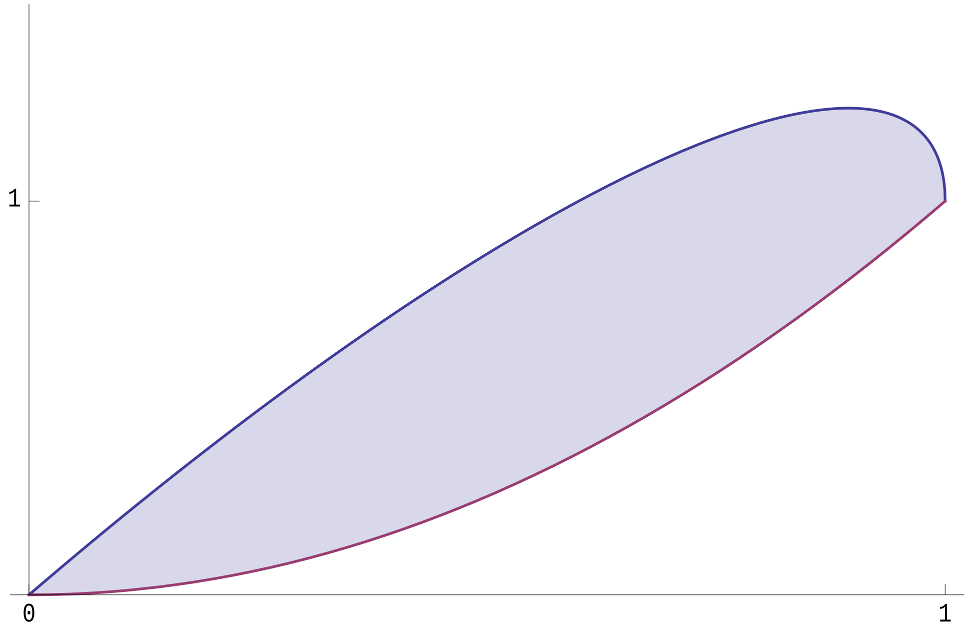
(c) $\pi + 10$

(d) $\pi + 12$

(e) $\pi + 14$

Answer:
(b). $\int_2^8 f(t)dt$ is the area of the region above the x-axis minus the area below the x-axis. The region above consists of two rectangles each of area 4, a quarter circle of radius 2 and a triangle of area 1. The region below consists of a triangle of area 1.

Problem 4. Consider the region sketched below.



The curve on top is defined by $y = 2x + \sqrt{1-x^2} - 1$ and the curve on bottom is $y = x^2$. The area of this region is

(a) $\frac{\pi}{4} - \frac{2}{3}$

(b) $\frac{\pi}{4} - \frac{1}{2}$

(c) $\frac{\pi}{4} - \frac{1}{3}$

(d) $\frac{\pi}{4} + \frac{1}{4}$

(e) $\frac{\pi}{4} + \frac{1}{2}$

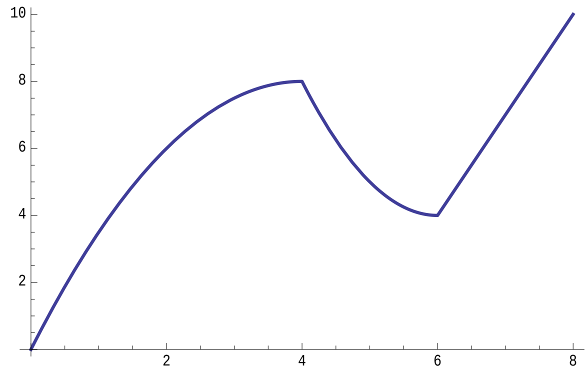
Answer:
(c). The area of the region is given by

$$\int_0^1 (2x + \sqrt{1-x^2} - 1) - x^2.$$

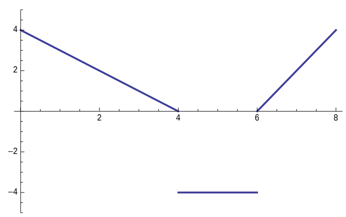
We compute

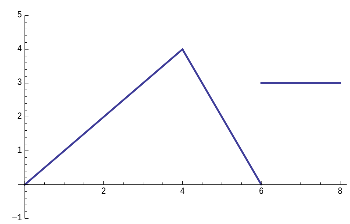
$$\begin{aligned}\int_0^1 (2x + \sqrt{1-x^2} - 1) - x^2 &= \int_0^1 2x + \int_0^1 \sqrt{1-x^2} - \int_0^1 1 - \int_0^1 x^2 \\ &= 1 + \frac{\pi}{4} - 1 - \frac{1}{3}\end{aligned}$$

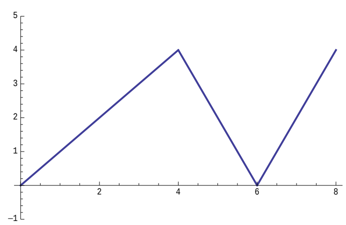
Problem 5. Here's the graph of $A(x) = \int_0^x f(t)dt$:

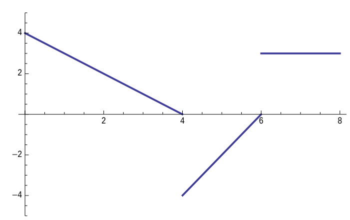


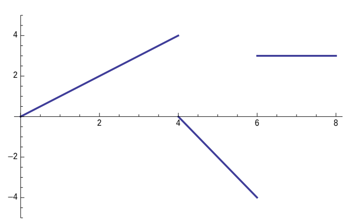
Which is the graph of f?

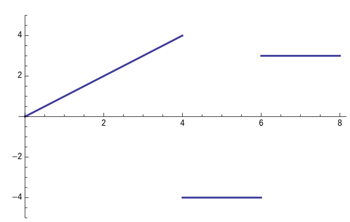
(a)


(d)


(b)


(e)


(c)


(f)


Answer:
(c). The following determines the answer conclusively. A is concave down and increasing on $(0, 4)$ so f is decreasing and positive on $(0, 4)$. A is concave up and decreasing on $(4, 6)$ so f is increasing and negative on $(4, 6)$.

Problem 6. On the interval $[0, 4\pi^2]$ the function defined by $A(x) = \int_0^x \sin(\sqrt{t})dt$ has a global maximum at

(a) $\frac{1}{4}\pi^2$

(b) π^2

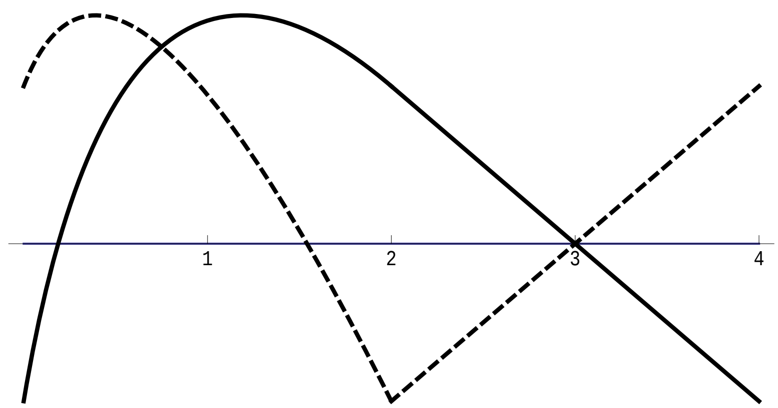
(c) $4\pi^2$

(d) $\frac{25}{4}\pi^2$

(e) $\sqrt{2\pi}$

Answer:
(b). Since $\sin(\sqrt{t}) > 0$ on $(0, \pi^2)$, A increases on $(0, \pi^2)$. Since $\sin(\sqrt{t}) > 0$ on $(\pi^2, 2\pi^2)$, A decreases on $(\pi^2, 2\pi^2)$. Therefore, the maximum is attained at π^2. For similar reasons, one checks that A has a global minimum at $4\pi^2$ on the interval $[0, 9\pi^2]$.

Problem 7. The graphs of two functions are sketched below.



The graph of f is solid and the graph of g is dashed. $\lim_{x\to 3} \dfrac{f(x)}{g(x)} =$

(a) 0

(b) 1

(c) -1

(d) 2

(e) does not exist

Answer:
(c). There's a neighborhood of 3 on which $g(x) = -f(x)$, so $\frac{f(x)}{g(x)} = -1$ for all $x \neq 3$ in this neighborhood. Therefore $\lim_{x\to 3} \dfrac{f(x)}{g(x)} = \lim_{x\to 3} -1 = -1.$

Problem 8. Which statement is false?

(a) There exists a continuous injection $f : (0, 1) \to [0, 1]$.

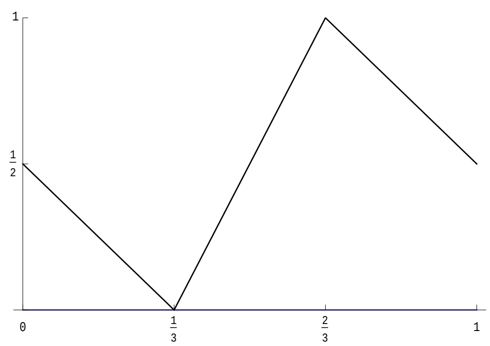
(b) There exists a continuous surjection $f : (0, 1) \to [0, 1]$.

(c) There exists a continuous surjection $f : [0, 1] \to (0, 1)$.

(d) There exists a continuous injection $f : [0, 1] \to (0, 1)$.

Answer:
(c). To see that there exist no continuous surjection $f : [0, 1] \to (0, 1)$, recall that a continuous function on a closed bounded interval must have a global maximum and a global minimum, but a surjection $[0, 1] \to (0, 1)$ has neither a maximum nor a minimum.

It's a good exercise to sketch the graph of functions that show the other statements are possible, but some are easy to find formulas for. For example, $f(x) = x$ defines a continuous injection $(0, 1) \to [0, 1]$. And $f(x) = \frac{1}{2}x + \frac{1}{4}$ defines a continuous injection $[0, 1] \to (0, 1)$. Finally, here's a picture of the graph of a continuous surjection $(0, 1) \to [0, 1]$:



Problem 9. Which statements about a function $f : X \to Y$ could be false?

(a) If f is injective, then for all sets $A, B \subseteq X$ we have $f(A \cap B) \subseteq f(A) \cap f(B)$.

(b) If f is injective, then for all sets $A, B \subseteq X$ we have $f(A) \cap f(B) \subseteq f(A \cap B)$.

(c) If $f(A \cap B) = f(A) \cap f(B)$ for all sets $A, B \subseteq X$ then f is injective.

(d) If f is not injective then there exist sets $A, B \subseteq X$ with $f(A \cap B) \neq f(A) \cap f(B)$

(e) If f is surjective then for all sets A we have $f(X \setminus A) \subseteq Y \setminus f(A)$.

Answer:
(e). For example, let $f : \{1, 2, 3\} \to \{a, b\}$ be given by $f(1) = a$, $f(2) = a$, and $f(3) = b$. Let $A = \{1\}$. Then $f(X \setminus A) = \{a, b\}$ is not a subset of $Y \setminus f(A) = \{3\}$.

All the statements were proved true in class.

Problem 10. Let

$$g(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ -x & \text{if } x \text{ is rational.} \end{cases}$$

Which of the following statements is false?

(a) g is continuous at 0

(b) $\lim_{x \to 0} g(x) = 0$

(c) g is invertible

(d) g is bounded on $[-1, 1]$

(e) g is piecewise monotonic

Answer:
(e). The function g is not monotonic on any interval. Here's a direct proof: Let I be any interval and choose two rational numbers $p, q \in I$ with $p < q$. We have $g(p) < g(q)$ so g is not decreasing on I. Choose two irrational numbers $a, b \in I$ with $a < b$. We have $g(a) > g(b)$ so g is not increasing on I.

It's worthwhile mentioning reasons the others are true. g is continuous at 0 since $g(0) = 0$ and Since $-x \leq g(x) \leq x$ for all x, the squeezing principle says $\lim_{x \to 0} g(x) = 0$. Since in addition, $g(0) = 0$, g is continuous at 0. It's not hard to see that g is bijective so g is invertible—in fact, $g(g(x)) = x$ for all $x \in \mathbb{R}$ so g is its own inverse. Also, $-1 \leq g(x) \leq 1$ for all $x \in [-1, 1]$ so g is bounded on $[-1, 1]$.

Problem 11. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x \text{ is rational and } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Which of the following statements is false?

(a) $f(0) = 1$

(b) $\lim_{x \to p} f(x) = 0$ for every number p

(c) f is continuous at every irrational number

(d) f is invertible

(e) f is integrable on $[0, 1]$

Answer:
(d). Since f is not one-to-one (note $g(\sqrt{2}) = g(\sqrt{3}) = 0$) it is not invertible.

We proved in class that $\lim_{x \to p} f(x) = 0$ for every number p so (b) is true. Since $f(x) = 0$ for every irrational number x, f is continuous at every irrational number so (c) is true. We proved $\lim_{x \to p} f(x) = 0$ by observing that for every $\epsilon > 0$, the value $f(p) > \epsilon$ for only finitely many $p \in [0, 1]$. That observation also proves that f is integrable and $\int_0^1 f = 0$ since it's possible to engineer a step function s with $s < f$ and $\int_0^1 s < \epsilon$. So (e) is true.
Oh yeah, (a) is true since the number 0 is rational and $0 = \frac{0}{1}$, so $f(0) = 1$.

Problem 12. Suppose that $f : [0,1] \to [0,1]$ is a continuous function satisfying $f(0) = \frac{1}{2}$ and $f(1) = \frac{1}{2}$. Which of the following statements *must be* false?

(a) $0 < \int_0^1 f < 1$

(b) f is invertible

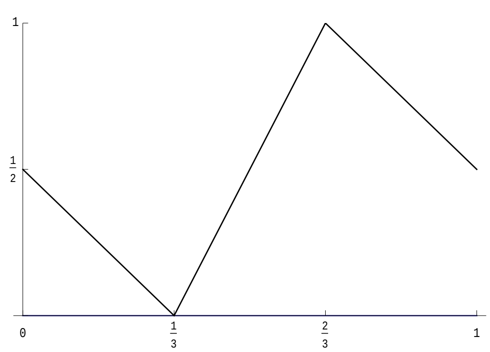
(c) there is a number $c \in [0,1]$ with $f(c) = 0$

(d) f is bounded

(e) there is a number c with $f(c) = c$

Answer:
(b). Since $f(0) = f(1)$, f cannot be one-to-one and therefore is not invertible.

It's worthwhile checking that the other choices are wrong. First, (d) is always true since the range of $f \subseteq [0,1]$ implies f is bounded. If f is continuous, f is integrable. Since $0 \leq f(x) \leq 1$, we have $0 \leq \int_0^1 f \leq 1$. But the only way for $\int_0^1 f = 0$ is iif $f(x) = 0$ for all x and the only way for $\int_0^1 f = 0$ is if $f(x) = 1$ for all x. So, (a) is always true. Also (e) is always true. You can see this graphically by observing that any curve connecting $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ must cross the line $y = x$. Or, apply the intermediate value theorem to $h(x) = f(x) - x$ which is a continuous function for which $h(0)$ and $h(1)$ have opposite signs.
(c) might be false, but (c) might be true, as this picture shows:



Problem 13. Suppose that $f : [0,1] \to [0,1]$ is a continuous function satisfying $f(0) = \frac{1}{2}$ and $f(1) = \frac{1}{2}$. Which of the following statements *might be* false?

(a) $0 < \int_0^1 f < 1$

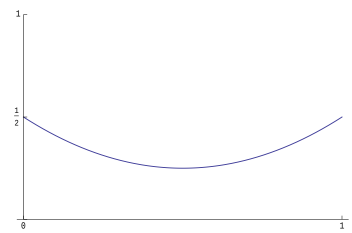
(b) f is invertible

(c) there is a number $c \in [0,1]$ with $f(c) = 0$

(d) f is bounded

(e) there is a number c with $f(c) = c$

Answer:
(c) might be false. For example, the function $f(x) = \frac{1}{4} + \left(x - \frac{1}{2}\right)^2$ has no zero in $[0,1]$:



See the answer to the problem above to see why the other choices are wrong.

Matching comutations [1 point each]

14. $\lim_{h\to 0} \frac{1}{h}\left(\cos\left(\frac{\pi}{6}+h\right)-\frac{\sqrt{3}}{2}\right)$

Answer:

We have

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}\left(\cos\left(\frac{\pi}{6}+h\right)-\frac{\sqrt{3}}{2}\right) &= \lim_{h\to 0} \frac{1}{h}\left(\cos\left(\frac{\pi}{6}\right)\cos(h)-\sin\left(\frac{\pi}{6}\right)\sin(h)-\frac{\sqrt{3}}{2}\right)\\ &= \lim_{h\to 0} \frac{\sqrt{3}}{2}\frac{\cos(h)-1}{h}-\frac{1}{2}\frac{\sin(h)}{h}\\ &= \frac{\sqrt{3}}{2}(0)-\frac{1}{2}(1)\\ &= -\frac{1}{2}\end{aligned}$$

15. $\int_0^\pi \left|\cos(t)+\frac{1}{2}\right| dt$

Answer:

Since $\cos(t) \geq -\frac{1}{2}$ for $0 \leq t \leq \frac{2\pi}{3}$ and $\cos(t) \leq -\frac{1}{2}$ for $\frac{2\pi}{3} \leq t \leq \pi$, we have

$$\begin{aligned}\int_0^\pi \left|\cos(t)+\frac{1}{2}\right| dt &= \int_0^{\frac{2\pi}{3}} \cos(t)+\frac{1}{2}dt - \int_{\frac{2\pi}{3}}^{\pi} -\cos(t)-\frac{1}{2}dt\\ &= \sqrt{3}+\frac{\pi}{6}\end{aligned}$$

16. $\int_0^6 [\sqrt{x}]dx$

Answer:

$f(x) = [\sqrt{x}]$ is a step function with value 0 on $[0, 1)$, value 1 on $[1, 4)$ and value 2 on $[4, 9)$. So,

$$\int_0^6 [\sqrt{x}]dx = 0(1) + 1(3) + 2(2) = 7.$$

17. $\sin\left(\frac{\pi}{12}\right)$

Answer:

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

18. $\int_0^{\pi^2} \sqrt{x}dx$

Answer:

Since $\int_a^b \sqrt{x}dx = \frac{2}{3}b^{\frac{3}{2}} - \frac{2}{3}a^{\frac{3}{2}}$, we have

$$\int_0^{\pi^2} \sqrt{x}dx = \frac{2}{3}(\pi^2)^{\frac{3}{2}} = \frac{2\pi^3}{3}.$$

Bonus [1 point]

Let $f(x) = x^3$. Give a rigorous, epsilon-delta proof that the function f is continuous at 1.

Answer:

Let $\epsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\epsilon}{9}\right\}$ and suppose $|x-1| < \delta$. Then we have

$$|x-1| < 1 \Rightarrow 0 < x < 2.$$

We also have $|x-1| < \frac{\epsilon}{9}$ so

$$\begin{aligned}
|x^3 - 1| &= |x-1||x^2 - 2x + 1| \\
&\leq |x-1|\left(|x^2| + |2x| + 1\right) \\
&\leq \frac{\epsilon}{9}(4+4+1) \\
&= \epsilon
\end{aligned}$$

as needed.