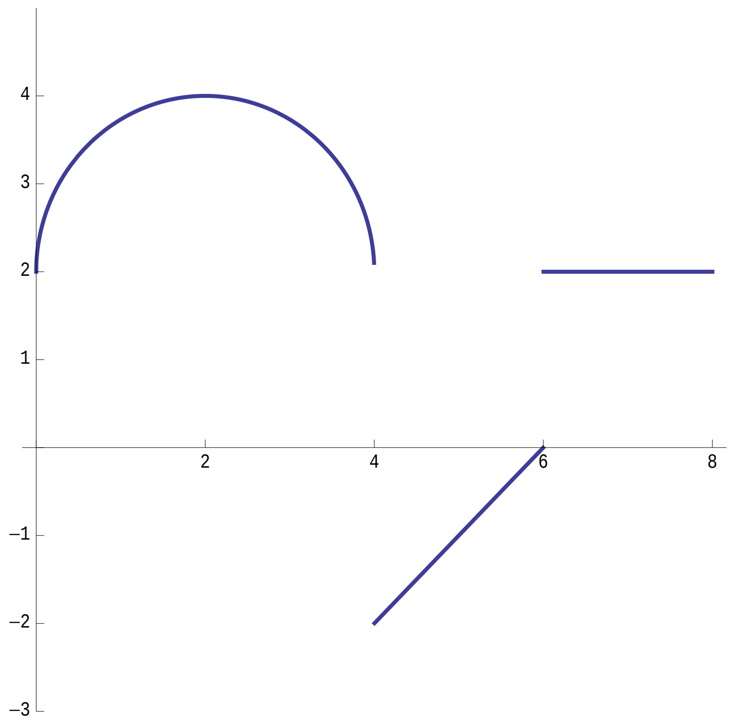
Definitions and theorems [2 points each]

Problem 1. Let f be a bounded function defined on $[a, b]$. Define the statement "f is integrable" and the number $\int_a^b f$.

Problem 2. Let f be a function defined on an open neighborhood of c. Define the statement "f is continuous at c."

Multiple Choice [1 point each]

Problem 3. Below the graph of a function f is sketched



$$\int_2^8 f(t)dt =$$

(a) $\pi + 6$

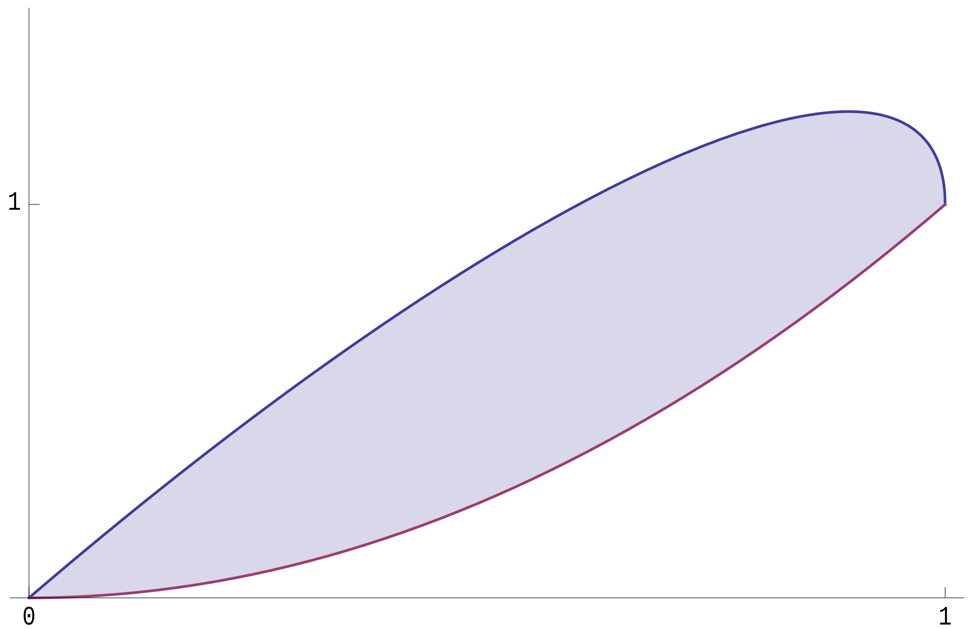
(b) $\pi + 8$

(c) $\pi + 10$

(d) $\pi + 12$

(e) $\pi + 14$

Problem 4. Consider the region sketched below.



The curve on top is defined by $y = 2x + \sqrt{1 - x^2} - 1$ and the curve on bottom is $y = x^2$. The area of this region is

(a) $\frac{\pi}{4} - \frac{1}{3}$

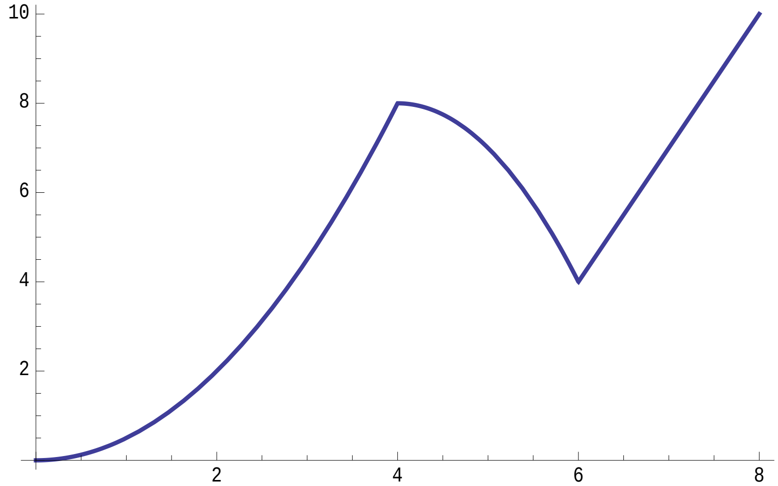
(b) $\frac{\pi}{4} + \frac{1}{4}$

(c) $\frac{\pi}{4} + \frac{1}{2}$

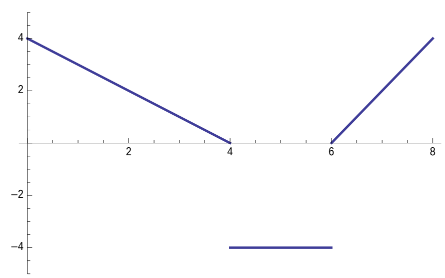
(d) $\frac{\pi}{4} - \frac{2}{3}$

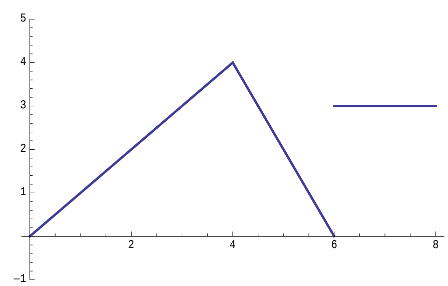
(e) $\frac{\pi}{4} - \frac{1}{2}$

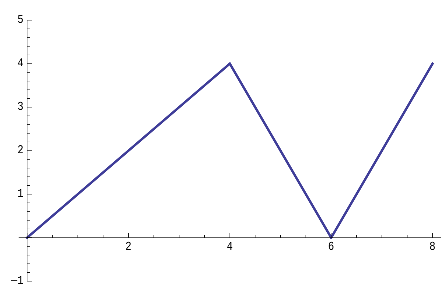
Problem 5. Here's the graph of $A(x) = \int_0^x f(t)dt$:

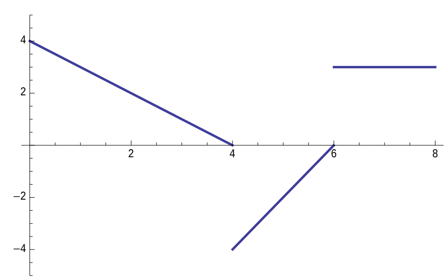


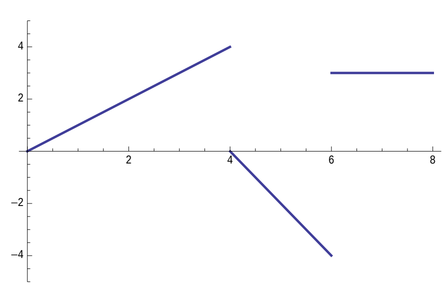
Which is the graph of f?

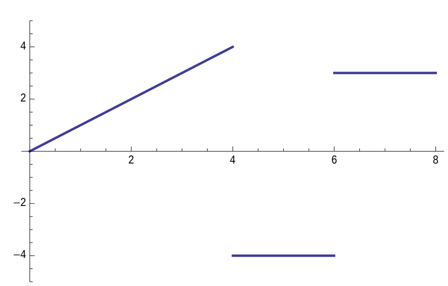
(a)


(d)


(b)


(e)


(c)


(f)


Problem 6. On the interval $[0, 4\pi^2]$ the function defined by $A(x) = \int_0^x \sin(\sqrt{t})dt$ has a global maximum at

(a) π^2

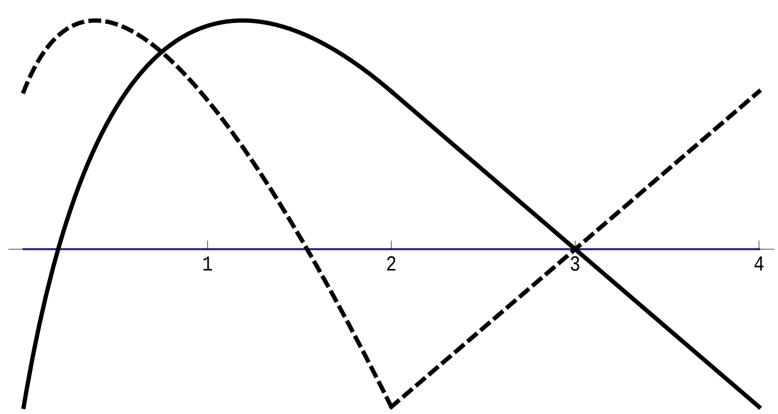
(b) $4\pi^2$

(c) $\frac{25}{4}\pi^2$

(d) $\sqrt{2\pi}$

(e) $\frac{1}{4}\pi^2$

Problem 7. The graphs of two functions are sketched below.



The graph of f is solid and the graph of g is dashed. $\lim_{x\to 3} \frac{f(x)}{g(x)} =$

(a) -1

(b) 1

(c) 2

(d) does not exist

(e) 0

Problem 8. Which statement is false?

(a) There exists a continuous surjection $f : (0,1) \to [0,1]$.

(b) There exists a continuous injection $f : (0,1) \to [0,1]$.

(c) There exists a continuous surjection $f : [0,1] \to (0,1)$.

(d) There exists a continuous injection $f : [0,1] \to (0,1)$.

Problem 9. Which statements about a function $f : X \to Y$ could be false?

(a) If f is injective, then for all sets $A, B \subseteq X$ we have $f(A \cap B) \subseteq f(A) \cap f(B)$.

(b) If f is surjective then for all sets A we have $f(X \setminus A) \subseteq Y \setminus f(A)$.

(c) If f is injective, then for all sets $A, B \subseteq X$ we have $f(A) \cap f(B) \subseteq f(A \cap B)$.

(d) If $f(A \cap B) = f(A) \cap f(B)$ for all sets $A, B \subseteq X$ then f is injective.

(e) If f is not injective then there exist sets $A, B \subseteq X$ with $f(A \cap B) \neq f(A) \cap f(B)$

Problem 10. Let

$$g(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ -x & \text{if } x \text{ is rational.} \end{cases}$$

Which of the following statements is false?

(a) g is piecewise monotonic

(b) g is continuous at 0

(c) $\lim_{x \to 0} g(x) = 0$

(d) g is invertible

(e) g is bounded on $[-1, 1]$

Problem 11. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x \text{ is rational and } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Which of the following statements is false?

(a) f is integrable on $[0, 1]$

(b) $f(0) = 1$

(c) $\lim_{x \to p} f(x) = 0$ for every number p

(d) f is continuous at every irrational number

(e) f is invertible

Problem 12. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function satisfying $f(0) = \frac{1}{2}$ and $f(1) = \frac{1}{2}$. Which of the following statements *must be* false?

(a) there is a number c with $f(c) = c$

(b) $0 < \int_0^1 f < 1$

(c) f is invertible

(d) there is a number $c \in [0, 1]$ with $f(c) = 0$

(e) f is bounded

Problem 13. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function satisfying $f(0) = \frac{1}{2}$ and $f(1) = \frac{1}{2}$. Which of the following statements *might be* false?

(a) there is a number c with $f(c) = c$

(b) $0 < \int_0^1 f < 1$

(c) f is invertible

(d) there is a number $c \in [0, 1]$ with $f(c) = 0$

(e) f is bounded

Matching comutations [1 point each]

14. $\lim_{h\to 0}\frac{1}{h}\left(\cos\left(\frac{\pi}{6}+h\right)-\frac{\sqrt{3}}{2}\right)$

15. $\int_0^{\pi}\left|\cos(t)+\frac{1}{2}\right|dt$

16. $\int_0^{6}[\sqrt{x}]dx$

17. $\sin\left(\frac{\pi}{12}\right)$

18. $\int_0^{\pi^2}\sqrt{x}dx$

The answers (out of order) are:

(a) 7

(b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

(c) $\sqrt{3}+\frac{\pi}{6}$

(d) $-\frac{1}{2}$

(e) $\frac{2\pi^3}{3}$

Bonus [1 point]

Let $f(x) = 2x^3$. Give a rigorous, epsilon-delta proof that the function f is continuous at 1.

Answer:

EXAM

Final Exam

Math 157

Tuesday, December 17, 2013

- Make sure your solutions are clearly and carefully written. Proofread.
- There are 20 points and 1 bonus points (a total of 21 possible).

May the Force be with you!