Math 157 Fall 2013 Homework 1 - selected answers

Problem 1. (a) Prove that one of the following two formulas about sets is always right and the other is sometimes wrong:

$$A - (B - C) = (A - B) \cup C$$
$$A - (B \cup C) = (A - B) - C$$

Answer. Here is a proof that $(A - B) - C = A - (B \cup C)$ for any sets A, B, and C.

Proof. Let $a \in A - (B \cup C)$. This means $a \in A$ and $a \notin B \cup C$. Since $a \notin B \cup C$, $a \notin B$, so $a \in A - B$. Since $a \notin B \cup C$, $a \notin C$. So, $a \in (A - B) - C$. This proves $A - (B \cup C) \subseteq (A - B) - C$.

Now let $a \in (A - B) - C$. This means that $a \in A - B$ and $a \notin C$. Since $a \in A - B$, $a \in A$ and $a \notin B$. It follows from $a \notin C$ and $a \not inB$ that $a \notin B \cup C$. The fact that $a \in A$ and $a \notin B \cup C$ implies that $a \in A - (B \cup C)$. This shows that $(A - B) - C \subseteq A - (B \cup C)$.

Since $A - (B \cup C) \subseteq (A - B) - C$ and $(A - B) - C \subseteq A - (B \cup C)$ together imply that $(A - B) - C = A - (B \cup C)$. □

Sometimes, it's not true that $A - (B - C) = (A - B) \cup C$. For example, let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 5, 6\}$, $C = \{2, 3, 6, 7\}$. Then $B - C = \{1, 5\}$ and

$$A - (B - C) = \{2, 3, 4\}.$$

One the other hand $A - B = \{3, 4\}$ and

$$(A - B) \cup C = \{2, 3, 4, 6, 7\}.$$

(b) State some additional necessary and sufficient condition for the formula which is sometimes incorrect to be always right.

Answer. First note that for all sets A, B, and C, we have

$$A - (B - C) \subset (A - B) \cup C.$$

To prove this, let $a \in A - (B - C)$. Then $a \in A$ and $A \notin B - C$. If $A \notin B - C$, we have either $a \notin B$ or $a \in C$. If $a \in C$, then $a \in (A - B) \cup C$. If $a \notin B$, then $a \in A$ and $a \notin B$ imply that $a \in A - B$, hence $a \in (A - B) \cup C$.

Now, we prove that if $C - A \neq \emptyset$ then

$$A - (B - C) = (A - B) \cup C.$$

Proof. Since it's already been shown that we always have $A - (B - C) \subset (A - B) \cup C$, it remains to prove that if $C - A = \emptyset$ then $(A - B) \cup C \subseteq A - (B - C)$. So, assume that $C - A = \emptyset$ and let $a \in (A - B) \cup C$. This means that $a \in A - B$ or $a \in C$. If $a \in A - B$, we have $a \in A$ and $a \notin B$. If $a \notin B$, we have $a \notin B - C$, so we have $a \in A - (B - C)$. On the other hand if $a \in C$, then $a \in A$ (since $C - A = \emptyset$) and $a \notin B - C$, so $a \in A - (B - C)$. $\square$

Finally, note that the statement

$$C - A \neq \emptyset \text{ if and only if } A - (B - C) = (A - B) \cup C.$$

We've already shown that if $C - A = \emptyset$, then $A - (B - C) = (A - B) \cup C$. On the other hand if $C - A \neq \emptyset$ then there is an element $c \in C - A$. Then $c \in (A - B) \cup C$. But since $c \in C - A \Rightarrow c \notin A$, it follows that $c \notin A - (B - C)$. Therefore, $(A - B) \cup C \neq A - (B - C)$.