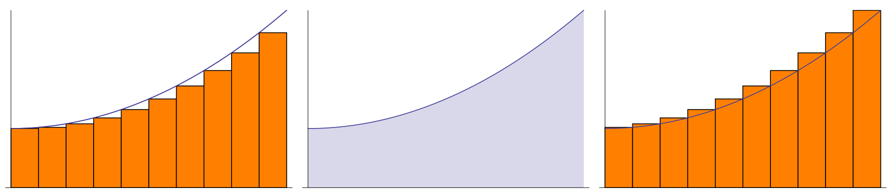
Math 157 Fall 2013 Homework 1 - selected answers

Problem 1 (Problem 1d in Section 1 page 8)**.** Compute the area of the region defined as follows. The base is a horizontal line segment of length b. If we choose an arbitrary point on the base of this figure distance x from the left hand corner, then the vertical distance from this point to the curve is $2x^2 + 1$.

Answer. Bound the region in question by two rectangular regions obtained by inscribing n rectangles of equal width and circumscribing n rectangles of equal width, as in the pictures below:



Let s_n be the area of the inscribed rectangular region, let A be the area of the curved region, and S_n be the area of the circumscribed rectangular region. We have

$$s_n < A < S_n.$$

Note,

$$\begin{aligned}
s_n &= \frac{b}{n}(2(0)^2+1) + \frac{b}{n}\left(2\left(\frac{b}{n}\right)^2+1\right) + \frac{b}{n}\left(2\left(\frac{2b}{n}\right)^2+1\right) + \cdots + \frac{b}{n}\left(2\left(\frac{(n-1)b}{n}\right)^2+1\right) \\
&= \frac{b}{n}\left(1 + 2\left(\frac{b}{n}\right)^2 + 1 + 2\left(\frac{2b}{n}\right)^2 + 1 + \cdots + 2\left(\frac{(n-1)b}{n}\right)^2 + 1\right) \\
&= \frac{b}{n}\left(2\left(\frac{b}{n}\right)^2 + 2\left(\frac{2b}{n}\right)^2 + \cdots + 2\left(\frac{(n-1)b}{n}\right)^2 + n\right) \\
&= \frac{b}{n}\left(2\left(\frac{b}{n}\right)^2 + 2\left(\frac{2b}{n}\right)^2 + \cdots + 2\left(\frac{(n-1)b}{n}\right)^2\right) + b \\
&= 2\frac{b^3}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) + b
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_n &= \frac{b}{n}\left(2\left(\frac{b}{n}\right)^2+1\right) + \frac{b}{n}\left(2\left(\frac{2b}{n}\right)^2+1\right) + \cdots + \frac{b}{n}\left(2\left(\frac{nb}{n}\right)^2+1\right) \\
&= 2\frac{b^3}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) + b
\end{aligned}$$

By the inequalities previously established, we know

$$1^2 + 2^2 + \cdots + (n-1)^2 < \frac{n^3}{3} < 1^2 + 2^2 + \cdots n^2$$

so

$$s_n < 2\frac{b^3}{n^3}\frac{n^3}{3} + b = 2\frac{b^3}{3} + b \text{ and } S_n > 2\frac{b^3}{n^3}\frac{n^3}{3} + b = 2\frac{b^3}{3} + b.$$

Therefore, we have both

$$s_n < 2\frac{b^3}{3} + b < S_n \text{ and } s_n < A < S_n.$$

Since $S_n - s_n$ can be made arbitrarily small, by choosing n large enough, there is only one number that lies between s_n and S_n for all n. We conclude that $A = 2\frac{b^3}{3} + b$.

Problem 2 (Problem 20 in Section 2 page 16)**.** (a) Prove that one of the following two formulas about sets is always right and the other is sometimes wrong:

$$A - (B - C) = (A - B) \cup C$$
$$A - (B \cup C) = (A - B) - C$$

Answer. Here is a proof that $(A - B) - C = A - (B \cup C)$ for any sets A, B, and C.

Proof. Let $a \in A - (B \cup C)$. This means $a \in A$ and $a \notin B \cup C$. Since $a \notin B \cup C$, $a \notin B$, so $a \in A - B$. Since $a \notin B \cup C$, $a \notin C$. So, $a \in (A - B) - C$. This proves $A - (B \cup C) \subseteq (A - B) - C$.

Now let $a \in (A - B) - C$. This means that $a \in A - B$ and $a \notin C$. Since $a \in A - B$, $a \in A$ and $a \notin B$. It follows from $a \notin C$ and $a \notin B$ that $a \notin B \cup C$. The fact that $a \in A$ and $a \notin B \cup C$ implies that $a \in A - (B \cup C)$. This shows that $(A - B) - C \subseteq A - (B \cup C)$.

From the two statements $A - (B \cup C) \subseteq (A - B) - C$ and $(A - B) - C \subseteq A - (B \cup C)$, it follows that $(A - B) - C = A - (B \cup C)$. $\square$

Sometimes, it's not true that $A - (B - C) = (A - B) \cup C$. For example, let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 5, 6\}$, $C = \{2, 3, 6, 7\}$. Then $B - C = \{1, 5\}$ and

$$A - (B - C) = \{2, 3, 4\}.$$

One the other hand $A - B = \{3, 4\}$ and

$$(A - B) \cup C = \{2, 3, 4, 6, 7\}.$$

(b) State some additional necessary and sufficient condition for the formula which is sometimes incorrect to be always right.

Answer. First note that for all sets A, B, and C, we have

$$A - (B - C) \subset (A - B) \cup C.$$

To prove this, let $a \in A - (B - C)$. Then $a \in A$ and $A \notin B - C$. If $A \notin B - C$, we have either $a \notin B$ or $a \in C$. If $a \in C$, then $a \in (A - B) \cup C$. If $a \notin B$, then $a \in A$ and $a \notin B$ imply that $a \in A - B$, hence $a \in (A - B) \cup C$.

Now, we prove that $C - A = \emptyset$ is necessary and sufficient for

$$A - (B - C) = (A - B) \cup C.$$

Proof. To see that the condition $C - A = \emptyset$ is *necessary* for $A - (B - C) = (A - B) \cup C$, suppose that $C - A \neq \emptyset$ then there is an element $c \in C - A$. Then $c \in (A - B) \cup C$. But since $c \in C - A \Rightarrow c \notin A$, it follows that $c \notin A - (B - C)$. Therefore, $(A - B) \cup C \neq A - (B - C)$.

To see that $C - A = \emptyset$ is sufficient for $A - (B - C) = (A - B) \cup C$, suppose that $C - A = \emptyset$. Since it's already been shown that we always have $A - (B - C) \subset (A - B) \cup C$, it remains to prove that $(A - B) \cup C \subseteq A - (B - C)$. So, let $a \in (A - B) \cup C$. This means that $a \in A - B$ or $a \in C$. If $a \in A - B$, we have $a \in A$ and $a \notin B$. If $a \notin B$, we have $a \notin B - C$, so we have $a \in A - (B - C)$. On the other hand if $a \in C$, then $a \in A$ (since $C - A = \emptyset$) and $a \notin B - C$, so $a \in A - (B - C)$. $\square$