Math 157 Fall 2013 Homework 5 - selected answers

Problem 1. Read the rest of chapter I (through page 43).

Problem 2. Find the least upper bound of the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$. Prove your answer.

Answer. 1 is the least upper bound for this set. Every element of this set has the form $\frac{n}{n+1}$ for some $n \in \mathbb{N}$. Since $n < n+1$, $\frac{n}{n+1} < \frac{n+1}{n+1} = 1$, so 1 is an upper bound. To see that 1 is the least upper bound, let $x < 1$. We will show that x is not an upper bound for this set. Note that $\frac{x}{1-x}$ is a real number, so by the Archimidean property, there is a natural number

$$n > \frac{x}{1-x} = \frac{1}{1-x} - \frac{1-x}{1-x} = \frac{1}{1-x} - 1.$$

Then

$$n+1 > \frac{1}{1-x} \Rightarrow \frac{1}{n+1} < 1-x.$$

Subtracting a smaller number from yields a larger result so

$$1 - \frac{1}{n+1} > 1 - (1-x) = x.$$

Since $1 - \frac{1}{n+1} = \frac{n}{n+1}$, we've shown that

$$\frac{n}{n+1} > x.$$

Thus, any number $x < 1$ is not an upper bound for the given set.

Problem 3. True or False:

(a) $p \Rightarrow (q \vee r) \equiv (p \wedge \neg q) \Rightarrow r$

(b) $(p \wedge q) \Rightarrow r \equiv (p \wedge \neg r) \Rightarrow \neg q$.

Problem 4. Prove that

If $a, b, c > 0$ and $a + b + c = 1$ then $(1-a)(1-b)(1-c) \geq 8abc$.

Answer. First note that for any $x, y \in \mathbb{R}$, we have $(x+y)^2 \geq 4xy$ since $(x+y)^2 - 4xy = (x-y)^2 \geq 0$. Dividing by 4 and taking square roots yields

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Now, begin with $(1-a)(1-b)(1-c)$ and use $a+b+c = 1$ to get

$$\begin{aligned}
(1-a)(1-b)(1-c) &= (b+c)(a+c)(a+b) \\
&= 8\left(\frac{b+c}{2}\right)\left(\frac{a+c}{2}\right)\left(\frac{a+b}{2}\right) \\
&\geq 8\sqrt{bc}\sqrt{ac}\sqrt{ab} \\
&= 8abc
\end{aligned}$$

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Problem 5. $n!$ may be defined inductively for $n = 0, 1, 2, \ldots$ by

- $0! = 1$ and
- $n! = n \times (n-1)!$

For $n, k = 0, 1, 2, \ldots$, define $\binom{n}{k}$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Answer. We give a direct proof using the definition:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k!)k!} \\
&= \frac{(n!)(k) + (n!)(n-k+1)}{(n-k+1)!k!} \\
&= \frac{(n!)(n+1)}{(n-k+1)!k!} \\
&= \frac{(n+1)!}{n+1-k)!(k!)} \\
&= \binom{n+1}{k!}.
\end{aligned}$$

Problem 6. Prove that for all $x \in \mathbb{R}$, $0 \le |x| - x \le 2|x|$.

Problem 7. Prove that for all $n \in \mathbb{N}$, $\sum_{k=1}^{n} 2k - 1 = n^2$.

Answer. We use a proof by induction. For $n = 1$, the statement is that $2(1) - 1 = 1^2$, which is true.

Now assume that $\sum_{k=1}^{n} 2k - 1 = n^2$. Consider $\sum_{k=1}^{n+1} 2k - 1$:

$$\begin{aligned}
\sum_{k=1}^{n+1} 2k - 1 &= \left(\sum_{k=1}^{n} 2k - 1\right) + (2(n+1) - 1) \\
&= n^2 + 2(n+1) - 1 \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

This completes a proof by mathematical induction.