Math 157 Fall 2013 — Homework 6

Problem 1. Read Chapter 1, through section 1.8. (pages 48-61).

(a) In Section 1.5, do exercises 2,3,4,8,9,10, and 11.

(b) In Section 1.7, do exercises 1,2,3, and 6.

Functions

Definition 1. We say that a set of ordered pairs $f \subseteq X \times Y$ is a *function from X to Y* if and only if for all $x \in X$ there exists one and only one $y \in Y$ so that $(x, y) \in f$. We usually write $f : X \to Y$ if $f \subseteq X \times Y$ is a function and we write $y = f(x)$ if $(x, y) \in f$. The set X is called *the domain of f* and the set Y is called *the codomain of f*.

It is common to think of a function $f : X \to Y$ a "rule" that associates $x \in X$ to $y \in Y$ whenever $y = f(x)$ and to refer to the set of ordered pairs $f \subseteq X \times Y$ as the *graph* of the function. By this convention the "rule" is referred to as the function f and the set $graph(f) = \{(x, y) \in X \times Y : f(x) = y\}$ is the *graph of f*.

Definition 2. Suppose that $f : X \to Y$ is a function.

(a) For any subset $A \subseteq X$, we define the set $f(A) \subseteq Y$ by

$$f(A) = \{y \in Y : \exists x \in A \text{ with } f(x) = y\}.$$

(b) We call $f(X) \subseteq Y$ the *range of f*.

(c) For any subset $B \subseteq Y$, we define the set $f^{-1}(B) \subseteq X$ by

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Problem 2. Suppose that $f : X \to Y$ is a function.

(a) For any $A \subseteq X$ and $B \subseteq X$, compare $f(A \cup B)$ and $f(A) \cup f(B)$.

(b) For any $A \subseteq X$ and $B \subseteq X$, compare $f(A \cap B)$ and $f(A) \cap f(B)$.

(c) For any $C \subseteq Y$ and $D \subseteq Y$, compare $f^{-1}(C \cup D)$ and $f^{-1}(C) \cup f^{-1}(D)$.

(d) For any $C \subseteq Y$ and $D \subseteq Y$, compare $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$.

(e) For any $A \subseteq X$, compare $f(X \setminus A)$ and $Y \setminus f(A)$.

(f) For any $C \subseteq Y$, compare $f^{-1}(Y \setminus C)$ and $X \setminus f^{-1}(C)$.

(g) For any $A \subseteq X$, compare $f^{-1}(f(A))$ and A.

(h) For any $C \subseteq Y$, compare $f(f^{-1}(C))$ and C.

Here "compare" means decide whether $\subseteq$, $\supseteq$, $=$, or none apply.

Definition 3. Suppose that $f : X \to Y$ is a function.

(a) We say that f is *injective* or *one to one* if and only if

$$\forall x \in X \forall z \in X\, (f(x) = f(z) \Rightarrow x = z).$$

(b) We say that f is *surjective* or *onto* if and only if

$$\forall y \in Y \exists x \in X\, (f(x) = y).$$

(c) We say that f is *bijective* if f is both injective and surjective.

We may call an injective function an *injection*, a surjective function a *surjection*, and a bijective function a *bijection*.

Problem 3. Which apply: injective, surjective, or bijective?

(a) Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = 2n$ for every $n \in \mathbb{N}$.

(b) Define $g : \mathbb{N} \setminus \{0, 1\} \to \mathbb{N}$ by $g(n) = n - 1$ for every $n \in \mathbb{N}$.

(c) Let $X = \{\text{functions } \phi : \mathbb{N} \to \mathbb{N}\}$. Define a function $G : X \to \mathbb{N}$ by $G(\phi) = \phi(3)$ for all $\phi \in X$.

(d) Let $X = \{\text{functions } \phi : \mathbb{N} \to \{0, 1\}\}$ and let $Y = \{\text{subsets of } \mathbb{N}\}$. Define a function $H : X \to Y$ by $H(f) = f^{-1}(\{1\})$.