Step functions

Problem 1. Read sections 1.8-1.14 (pages 60-70) in Apostol and do the following exercises:

(a) Exercises 1, 3 in 1.11 on page 63.

(b) Exercises 1, 2, 5, 11, 13-17 in section 1.15 on pages 70-72.

More on abstract functions

Definition 1. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. We define the composition $g \circ f$ to be the function

$$g \circ f : X \to Z$$

given by $(g \circ f)(x) = g(f(x))$.

Definition 2. For any set X, define the function $\mathrm{id}_X : X \to X$ by $\mathrm{id}_X(x) = x$ for all $x \in X$.

Problem 2. Prove that $\circ$ is associative. That is, if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Answer. We check: for $x \in X$, we have

$$(h \circ g) \circ f(x) = (h \circ g)(f(x)) = h(g(f(x))) \text{ and } h \circ (g \circ f)(x) = h(g \circ f)(x) = h(g(f(x))).$$

Problem 3. Prove that for any function $f : X \to Y$, we have

$$f \circ \mathrm{id}_X = f \text{ and } \mathrm{id}_Y \circ f = f.$$

Answer. To see that f and $f \circ \mathrm{id}_X$ are the same functions, we check that they assign the same values to every element in the domain. For any $x \in X$, we have,

$$f \circ \mathrm{id}_X(x) = f(\mathrm{id}_X(x)) = f(x).$$

Similarly, we check that f and $\mathrm{id}_Y \circ f$ assign the same values to every element in the domain. For any $x \in X$, we have

$$\mathrm{id}_Y \circ f(x) = \mathrm{id}_Y(f(x)) = f(x).$$

Definition 3. Let $f : X \to Y$ be a function. We say that a function $g : Y \to X$ is a *left inverse* of f if $g \circ f = \mathrm{id}_X$. We say that a function $g : Y \to X$ is a *right inverse* of f if $f \circ g = \mathrm{id}_Y$. We say that a function $g : Y \to X$ is *an inverse* of f if g is both a left and a right inverse of f.

Problem 4. Prove that $f : X \to Y$ has a left inverse if and only if f is injective.

Answer. To prove that if $f : X \to Y$ has a left inverse, it must be injective suppose that $g : Y \to X$ satisfies $gf = \mathrm{id} : X \to X$ and let $x, x' \in X$ satisfy $f(x) = f(x')$. Apply g to get $g(f(x)) = g(f(x'))$. Since $gf = \mathrm{id} : X \to X$, we have $x = x'$ as needed.

Conversely, if f is injective, choose a fixed element $x_0 \in X$ and define a $g : Y \to X$ by

$$g(y) = \begin{cases} x & \text{if } g(x) = y \\ x_0 & \text{if } y \text{ is not in } f(X) \end{cases}$$

The function g is well defined because f is injective. By construction, we have $gf = \mathrm{id}_A$.

Problem 5. Prove that $f : X \to Y$ has a right inverse if and only if f is surjective.

Answer. To see that if f has a right inverse, then it must be surjective, suppose that $g : Y \to X$ and $fg = \mathrm{id}_Y$. For every $y \in Y$, $y = fg(y) = f(g(y))$. Therefore, for every $y \in Y$, there exists an $x \in X$ (namely, $x = g(y)$) with $f(x) = y$. This says f is surjective.

Conversely, if f is surjective, define $g : Y \to X$ as follows: for every $y \in Y$, choose an element $x \in X$ with $f(x) = y$. This is possible since f is surjective. Define $g(y) = x$. Then, $fg(y) = f(x) = y$, so g is a right inverse for f.

Problem 6. Prove that if $f : X \to Y$ has a left inverse $g : Y \to X$ and a right inverse $h : Y \to X$, then $g = h$.

Answer. To see that if f has both a left and a right inverse, then they are the same, suppose that $g, h : Y \to X$ and $fg = \mathrm{id}_Y$ and $hf = \mathrm{id}_X$. We have $hf = \mathrm{id}_X \Rightarrow (hf)g = g \Rightarrow h(fg) = g \Rightarrow h(\mathrm{id}_Y) = g \Rightarrow h = g$.