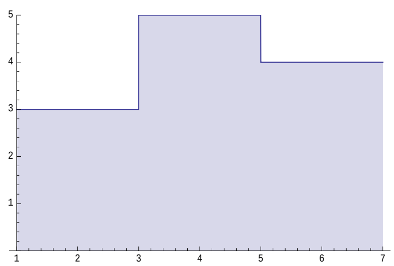
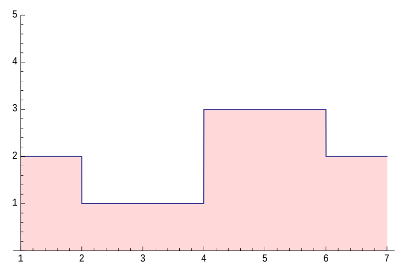
**Problem 1.** Let $s : [1,7] \to \mathbb{R}$ and $t : [1,7] \to \mathbb{R}$ be step functions given by

$$s(x) = \begin{cases} 3 & \text{if } 1 < x < 3 \\ 5 & \text{if } 3 < x < 5 \\ 4 & \text{if } 5 < x < 7 \end{cases} \qquad t(x) = \begin{cases} 2 & \text{if } 1 < x < 2 \\ 1 & \text{if } 2 < x < 4 \\ 3 & \text{if } 4 < x < 6 \\ 2 & \text{if } 6 < x < 7 \end{cases}$$

(a) Sketch the graphs of $s$ and $t$.

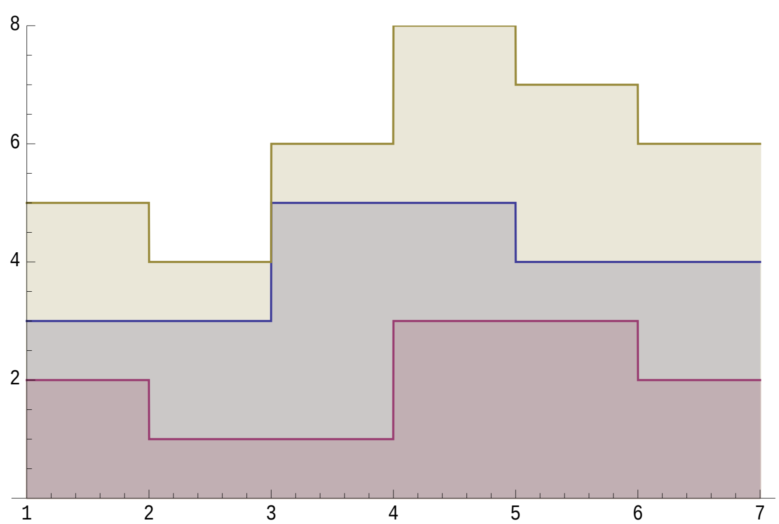
**Answer.**




(b) Describe $s + t$ and sketch its graph.

$$(s+t)(x) = \begin{cases} 5 & \text{if } 1 < x < 2 \\ 4 & \text{if } 2 < x < 3 \\ 6 & \text{if } 3 < x < 4 \\ 8 & \text{if } 4 < x < 5 \\ 7 & \text{if } 5 < x < 6 \\ 6 & \text{if } 6 < x < 7 \end{cases}$$



(c) Find a partition of $[1,7]$ for which $s, t$, and $s+t$ are constant on its open subintervals.

**Answer.** $\mathcal{P} = \{1,2,3,4,5,6,7\}$.

(d) Compute the integrals $\int_1^7 s$, and $\int_1^7 t$, $\int_1^7 s+t$.

**Answer.**

$$\int_1^7 s = 3(2) + 5(2) + 4(2) = 24$$
$$\int_1^7 t = 2(1) + 1(2) + 3(2) + 2(1) = 12$$
$$\int_1^7 s+t = 5(1) + 4(1) + 6(1) + 8(1) + 7(1) + 6(1) = 36.$$

(e) You should have found that $\int_1^7 s+t = \int_1^7 s + \int_1^7 t$. Give a step by step arithmetic argument to show explicitly that

$$\int_1^7 s+t = \int_1^7 s + \int_1^7 t.$$

**Answer.** We use the partition $\mathcal{P} = \{1,2,3,4,5,6,7\}$ and compute

$$\begin{aligned}
&\int_1^7 s + \int_1^7 t \\
&= (3(2-1) + 3(3-2) + 5(4-3) + 5(5-4) + 4(6-5) + 4(7-6)) \\
&+ (2(2-1) + 1(3-2) + 1(4-3) + 3(5-4) + 3(6-5) + 2(7-6)) \\
&= ((3+2)(2-1) + (3+1)(3-2) + (5+1)(4-3) + (5+3)(5-4) + (4+3)(6-5) + (4+2)(7-6)) \\
&= \int_a^7 s+t
\end{aligned}$$

**Problem 2.** Read sections 1.16-1.17 in Apostol.

**Problem 3.** Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 2 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

Compute $\overline{I}(f)$ and $\underline{I}(f)$, the upper and lower integrals of $f$ over the interval $[0,1]$.

**Answer.** Let

$$S = \left\{ \int_0^1 s : s \text{ is a step function with } s \leq f \right\}.$$

Suppose $s$ is a step function satisfying $s \leq f$. Since there are rational numbers in every open interval, $s \leq 1$ so $\int_0^1 s = 1$. So, 1 is an upper bound for $S$. Since the constant function 1 is a step function less than $f$ and $\int_0^1 s = 1$, $1 \in S$. We conclude that $\underline{I}(f) = \sup S = 1$.

Let

$$T = \left\{ \int_0^1 t : t \text{ is a step function with } f \leq t \right\}.$$

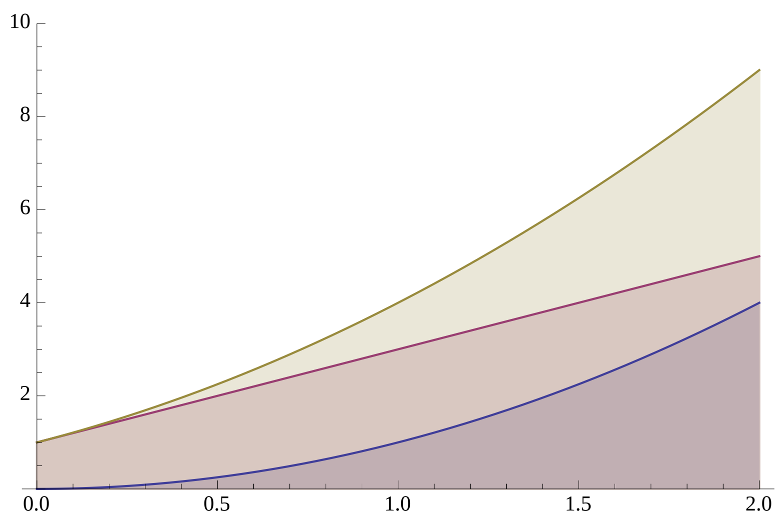
Suppose $t$ is a step function satisfying $f \leq t$. Since there are irrational numbers in every open interval, $2 \leq t$ so $\int_0^1 t \geq 2$. Thus 2 is a lower bound for $T$. Since the constant function 2 is a step function satisfying $2 \leq f$, the number $2 = \int_0^1 2 \in T$. We conclude that $\overline{I}(f) = \inf T = 2$.

**Problem 4.** Read sections 1.18-1.27 in Apostol.

**Problem 5.** Let $f(x) = x^2$ and $g(x) = 2x + 1$.

(a) Sketch a picture of the graphs of $f$, $g$, and $f + g$ over the interval $[0, 2]$.

**Answer.**




(b) Compute $\int_0^2 f$, $\int_0^2 g$, and $\int_0^2 f + g$.

**Answer.** For $\int_0^2 x^2$, we use the fact that for any $p \in \mathbb{N}$, we have $\int_0^b x^p = \frac{b^{p+1}}{p+1}$. So,

$$\int_0^2 x^2 = \frac{2^3}{3} = \frac{8}{3}.$$

To compute $\int_0^2 2x+1$, use the fact that the integral computes the area of the ordinate set and observe that the ordinate set is comprised of a triangle of area height 4 and width 2 on top of a rectangle of width 2 and height 1. So,

$$\int_0^2 2x+1 = 6.$$

For the last integral, we use the fact that the integral is additive over sums:

$$\int_0^2 x^2+2x+1 = \int_0^2 x^2 + \int_0^2 2x+1 = \frac{8}{3} + 6.$$