Math 157 Fall 2013 **Homework 9**

Problem 1. Give a rigorous (epsilon-delta) proof that the function f defined by

$$f(x) = 3x - 1$$

is continuous at 2.

Problem 2. Give a rigrorous (epsilon-delta) proof that $\lim_{x \to a} x^2 = a^2$.

Problem 3. Give a rigrorous (epsilon-delta) proof that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

Problem 4. Give a rigorous proof that if f is continuous and $f(p) > 0$, there exists a neighborhood of p on which f is positive.

Infinite limits

Definition 1. Let f be a function defined on a neighborhood of a point p except possible at p. The expression

$$\lim_{x \to p} f(x) = \infty$$

means that for every number B, there exists a number δ so that if $0 < |x-p| < \delta$ then $f(x) > B$. The expression

$$\lim_{x \to p} f(x) = -\infty$$

means that for every number B, there exists a number δ so that if $0 < |x-p| < \delta$ then $f(x) < B$.

Definition 2. Let f be a function defined on an interval (B, ∞). The expression

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\epsilon > 0$, there exists a number B so that if $x > B$ then $|f(x) - L| < \epsilon$. The expression

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\epsilon > 0$, there exists a number B so that if $x < B$ then $|f(x) - L| < \epsilon$.

Problem 5. There are many variations of infinite and one-sided limits. It would be tedious to give them all, but it is a very good exercise to carefully state a few of them. Give precise definitions of $\lim_{x \to p^+} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.

Problem 6. Give an $\epsilon - \delta$ proof that $\displaystyle\lim_{x\to\infty} \frac{\sin(x)}{\sqrt{x}} = 0$.

Problem 7. Give an $\epsilon - \delta$ proof that $\displaystyle\lim_{x\to 2^+} \frac{1}{4-x^2} = -\infty$.

Problem 8. Compute:

(a) $\displaystyle\lim_{t\to 4} \frac{\sqrt{t+5}-3}{\sqrt{2t+1}-3}$

(b) $\displaystyle\lim_{x\to 2^-} \frac{x^2-3x}{x^2-4}$

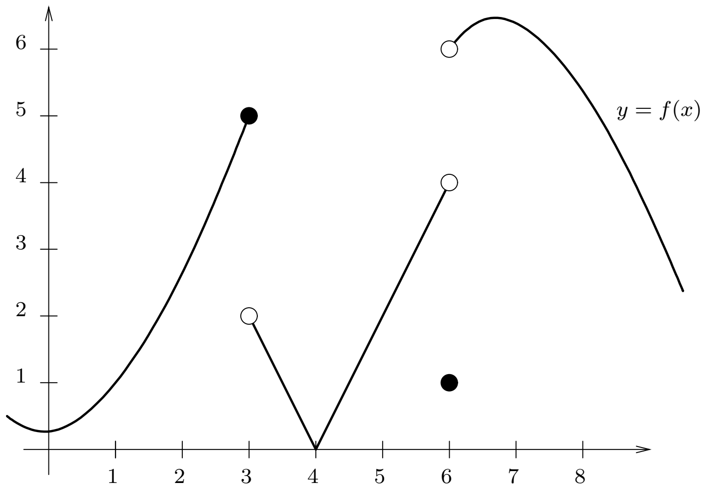
(c) $\displaystyle\lim_{x\to 2} \frac{x^4-16}{x^2-4} =$

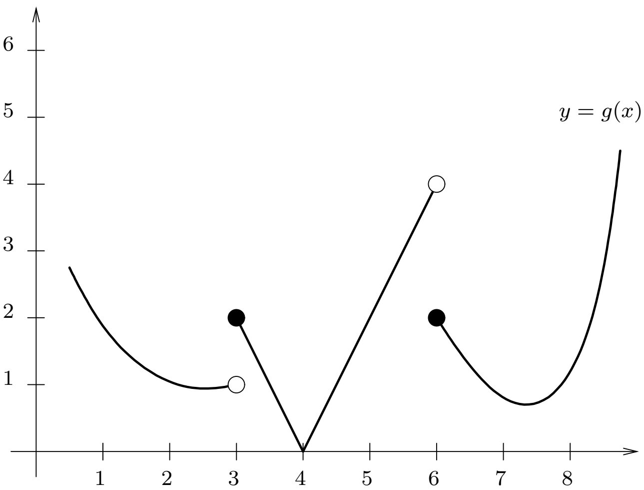
(d) $\displaystyle\lim_{x\to 6} \frac{x-6}{\sqrt{2x-3}} =$

(e) $\displaystyle\lim_{t\to\infty} \frac{3t^3+t-5}{4t^3+t^2+6} =$

(f) $\displaystyle\lim_{x\to\infty} \frac{x^2-x}{x+10\sqrt{x}} =$

Problem 9. Use the picture.





(a) $\lim_{x\to 3^+} g(x) =$

(b) $f(3) =$

(c) $\lim_{x\to 3} f(x) =$

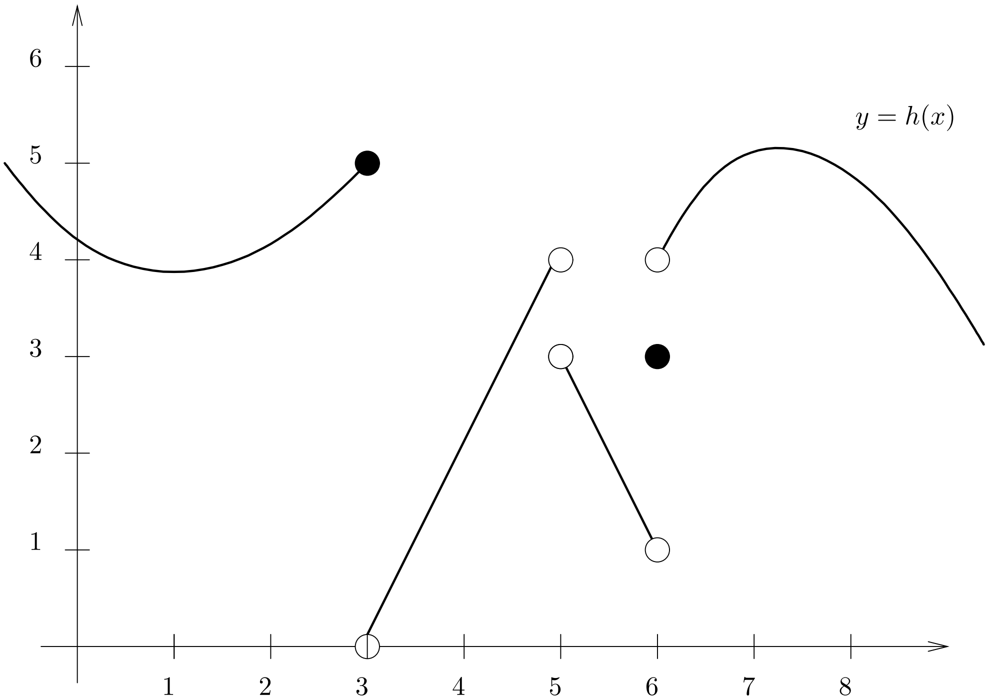
(d) $\lim_{x\to 6} (f+g)(x) =$

(e) $(f+g)(6) =$

(f) $\lim_{x\to 4} \dfrac{f(x)}{g(x)} =$

(g) Is the function $f + g$ continuous at $x = 6$? Explain.

Problem 10. Use the picture.



(a) $\lim_{x\to 6^+} h(x)$

(b) $h(6)$

(c) $\lim_{x\to 6} h(x)$

(d) $h(5)$

(e) $\lim_{x\to 3^-} h(x)$

(f) $\lim_{x\to 3^+} \frac{1}{h(x)}$

(g) $\lim_{x\to 3^+} \frac{x-3}{h(x)}$

(h) $\lim_{r\to 0} \frac{h(4+r)-2}{r}$

(i) $\lim_{x\to 3^-} h(2x)$

Problem 11. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| < |x|$ for all $x \in \mathbb{R}$. Prove that f is continuous at 0.

Problem 12. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at 0 and is discontinuous at every other point.

Problem 13. Extending functions.

(a) Suppose $f(x) = \dfrac{x^2 - 9}{x - 3}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(b) Suppose $f(x) = \dfrac{|x|}{x}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(c) Suppose $f(x) = \dfrac{\sin(x)}{x}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(d) Suppose $f(x) = \sin\left(\frac{1}{x}\right)$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(e) Suppose $f(x) = x \sin\left(\frac{1}{x}\right)$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(f) Suppose

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x < 1, \\ 5x - 1 & \text{if } x > 1 \end{cases}$$

Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(g) Suppose $f : \mathbb{Q} \to \mathbb{R}$ is given by $f(r) = 0$ for all $r \in \mathbb{Q}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(h) Suppose $f : \mathbb{Q} \to \mathbb{R}$ is given by $f(r) = \frac{1}{q}$ if $r = \frac{p}{q}$ in lowest terms. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

(i) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous at every point of $[a, b]$. Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f (in fact, there are infinitely many such F).

(j) Give an example to show that if $f : (a, b) \to \mathbb{R}$ is continuous at every point of (a, b) there need not exist a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f.