Math 157 Fall 2013 — Homework 9

Problem 1. Give a rigorous (epsilon-delta) proof that the function f defined by

$$f(x) = 3x - 1$$

is continuous at 2.

Answer. Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{3}$. Then, if $|x-2| < \delta$ we have

$$|x-2| < \frac{\epsilon}{3} \Rightarrow |3x-6| < \epsilon \Rightarrow |3x-1-5| < \epsilon.$$

Problem 2. Give a rigrorous (epsilon-delta) proof that $\lim\limits_{x\to a} x^2 = a^2$.

Answer. Let $\epsilon > 0$ be given. Choose $\delta = \min\left\{1, \dfrac{\epsilon}{2|a|+1}\right\}$ and suppose $|x-a| < \delta$. So, we have $|x-a| < 1 \Rightarrow |x| < |a|+1$ We also have $|x-a| < \dfrac{\epsilon}{2|a|+1}$. Now, look at

$$|x^2 - a^2| = |x-a||x+a| < \frac{\epsilon}{2|a|+1}(|x|+|a|) < \frac{\epsilon}{2|a|+1}(|a|+1+|a|) = \epsilon.$$

Problem 3. Give a rigrorous (epsilon-delta) proof that $\lim\limits_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

Answer. Let $\epsilon > 0$ be given. Choose $\delta = \min\{1, \epsilon\}$. Suppose $0 < |x-2| < \delta$ and look at

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right|.$$

Now, since $|x-2| < 1$, we know $1 < x$, hence $\frac{1}{2x} < \frac{1}{2}$. So, we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| < \frac{|x-2|}{2} < \frac{\epsilon}{2} < \epsilon$$

as needed.

Problem 4. Give a rigorous proof that if f is continuous and $f(p) > 0$, there exists a neighborhood of p on which f is positive.

Answer. Suppose f is continuous at p and $f(p) > 0$. Let $\mathcal{N}$ be the neighborhood of $f(p)$ of radius $\frac{f(p)}{2} > 0$. By the definition of "f is continuous at p" there exists a neighborhood $\mathcal{O}$ of p with $f(\mathcal{O}) \subseteq \mathcal{N}$. This means that $\frac{f(p)}{2} < f(x) < \frac{3f(p)}{2}$ for all $x \in \mathcal{O}$. Since $0 < \frac{f(p)}{2}$, we see that f is positive on the neighborhood $\mathcal{O}$.

Infinite limits

Definition 1. Let f be a function defined on a neighborhood of a point p except possible at p. The expression

$$\lim_{x\to p} f(x) = \infty$$

means that for every number B, there exists a number δ so that if $0 < |x-p| < \delta$ then $f(x) > B$. The expression

$$\lim_{x\to p} f(x) = -\infty$$

means that for every number B, there exists a number δ so that if $0 < |x-p| < \delta$ then $f(x) < B$.

Definition 2. Let f be a function defined on an interval (B, ∞). The expression

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\epsilon > 0$, there exists a number B so that if $x > B$ then $|f(x) - L| < \epsilon$. The expression

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\epsilon > 0$, there exists a number B so that if $x < B$ then $|f(x) - L| < \epsilon$.

Problem 5. There are many variations of infinite and one-sided limits. It would be tedious to give them all, but it is a very good exercise to carefully state a few of them. Give precise definitions of $\lim\limits_{x\to p^+} f(x) = -\infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$.

Answer. The expression

$$\lim_{x\to p^+} f(x) = -\infty$$

means that for every number B there exists a number $\delta > 0$ so that $0 < x - p < \delta \Rightarrow f(x) < B$.

The expression

$$\lim_{x\to\infty} f(x) = \infty$$

means that for every number B there exists a number N so that $x > N \Rightarrow f(x) > B$.

Problem 6. Give an $\epsilon - \delta$ proof that $\lim\limits_{x\to\infty} \dfrac{\sin(x)}{\sqrt{x}} = 0$.

Answer. Let $\epsilon > 0$ be given. Choose $B > \frac{1}{\epsilon^2}$ and suppose $x > B$. We have

$$x > \frac{1}{\epsilon^2} \Rightarrow \frac{1}{x} < \epsilon^2 \Rightarrow \frac{1}{\sqrt{x}} < \epsilon \Rightarrow \left|\frac{\sin(x)}{\sqrt{x}}\right| < \epsilon.$$

The last implication follows from the fact that $|\sin(x)| \leq 1$.

Problem 7. Give an $\epsilon - \delta$ proof that $\lim_{x\to 2^+} \dfrac{1}{4-x^2} = -\infty$.

Answer. Let B be given. If $B > 0$, any choice of $\delta > 0$ works since $\frac{1}{4-x^2} < 0$ for all numbers $x > 2$. So assume $B < 0$. Choose $\delta = -\frac{1}{4B} > 0$.

$$\begin{aligned}
0 < x - 2 < \delta &\Rightarrow x - 2 < -\frac{1}{4B} \\
&\Rightarrow 2 - x > \frac{1}{4B} \\
&\Rightarrow 2 - x > \frac{1}{B(2+x)} \qquad \text{because } \frac{1}{B(2+x)} < \frac{1}{4B} \text{ when } x > 2 \\
&\Rightarrow (2-x)(2+x) > \frac{1}{B} \\
&\Rightarrow 4 - x^2 > \frac{1}{B} \\
&\Rightarrow \frac{1}{4-x^2} < B
\end{aligned}$$

Problem 8. Compute:

Answer. You should show the main steps (but not all the scratchwork!) on an exam or problem set, but it's tedious for me to type all that algebraic manipulation. So, here are just the answers, which are useful for grading your own work and finding errors.

(a) $\lim_{t\to 4} \dfrac{\sqrt{t+5}-3}{\sqrt{2t+1}-3} = \dfrac{1}{2}$

(b) $\lim_{x\to 2^-} \dfrac{x^2-3x}{x^2-4} = \infty$

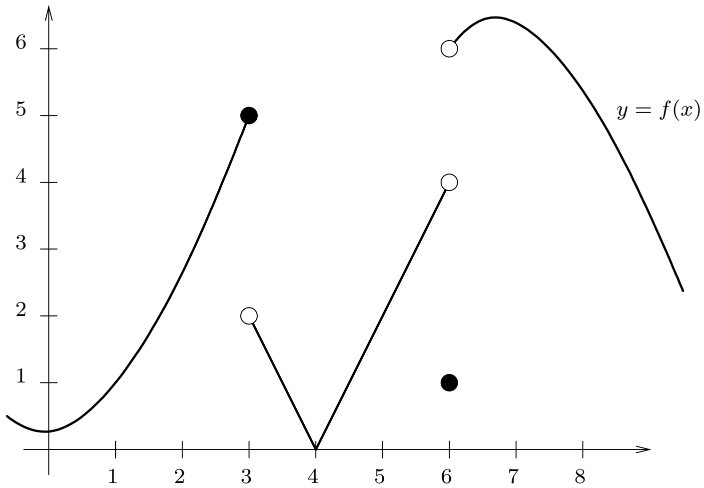
(c) $\lim_{x\to 2} \dfrac{x^4-16}{x^2-4} = 8$

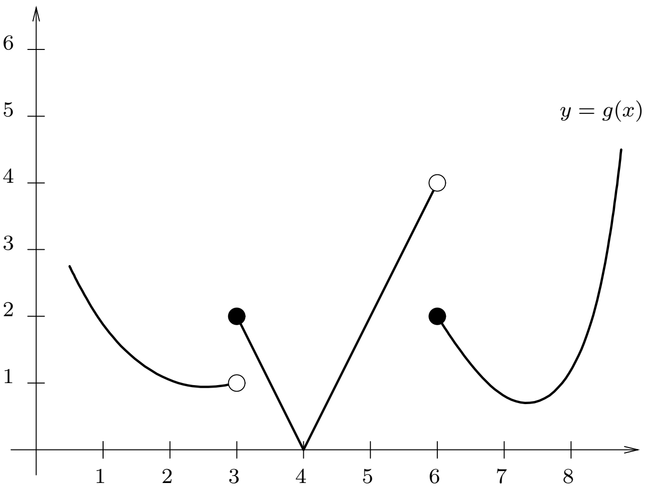
(d) $\lim_{x\to 6} \dfrac{x-6}{\sqrt{2x-3}} = 0$

(e) $\lim_{t\to\infty} \dfrac{3t^3+t-5}{4t^3+t^2+6} = \dfrac{3}{4}$ To see this, note that $\dfrac{3t^3+t-5}{4t^3+t^2+6} = \dfrac{3+\frac{1}{t^2}-\frac{5}{t^3}}{4+\frac{1}{t}+\frac{6}{t^3}}$.

(f) $\lim_{x\to\infty} \dfrac{x^2-x}{x+10\sqrt{x}} = \infty$

Problem 9. Use the picture.





(a) $\lim_{x\to 3^+} g(x) = 2$

(b) $f(3) = 5$

(c) $\lim_{x\to 3} f(x) = DNE$

(d) $\lim_{x\to 6} (f+g)(x) = 8$

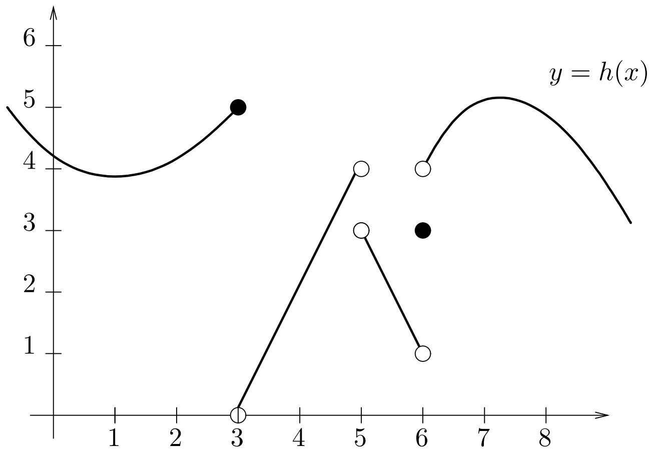
(e) $(f+g)(6) = 3$

(f) $\lim_{x\to 4} \frac{f(x)}{g(x)} = 1$

(g) Is the function $f+g$ continuous at $x = 6$? Explain.

Answer. No. Since $(f+g)(6) = 3$ and $\lim_{x\to 6}(f+g)(x) = 8$, $f+g$ is not continuous at 6.

Problem 10. Use the picture.



(a) $\lim_{x\to 6^+} h(x) = 4$

(b) $h(6) = 3$

(c) $\lim_{x\to 6} h(x)$ DNE

(d) $h(5)$ DNE

(e) $\lim_{x\to 3^-} h(x) = 5$

(f) $\lim_{x\to 3^+} \dfrac{1}{h(x)} = +\infty$

(g) $\lim_{x\to 3^+} \dfrac{x-3}{h(x)} = \dfrac{1}{2}$ (note that for $x > 3$ near 3, $h(x) = 2(x-3)$.)

(h) $\lim_{r\to 0} \dfrac{h(4+r)-2}{r} = 2$ (You can get this thinking geometrically, or analytically. Here's the analytic argumtent: since there's a nbhd of 4 on which $h(x) = 2(x-3)$ we have $\lim_{r\to 0} \frac{h(4+r)-2}{r} = \lim_{r\to 0} \frac{2((4+r)-3)-2}{r} = \lim_{r\to 0} \frac{2r}{r} = 2$.)

(i) $\lim_{x\to 3^-} h(2x) = 1$ (since $\lim_{x\to 6^-} g(x)$.)

Problem 11. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| < |x|$ for all $x \in \mathbb{R}$. Prove that f is continuous at 0.

Answer. This is false as stated, since for f to be continuous, $f(0)$ must be defined, but the condition $|f(x)| < |x|$ for all $x \in \mathbb{R}$ implies that $|f(0)| < |0| = 0$, which is impossible.

However, if $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$, then f is continuous at 0. To prove it, first note that $f(0)$ satisfies $|f(0)| \leq |0| = 0$, so $f(0) = 0$. We have to prove that for all $\epsilon > 0$, there exists a $\delta > 0$ so that $|x| < \delta \Rightarrow |f(x)| < \epsilon$. So, let $\epsilon > 0$ be given. Choose $\delta = \epsilon$ and suppose $|x| < \delta$. We have $|f(x)| < |x| < \delta = \epsilon$ as needed.

Problem 12. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at 0 and is discontinuous at every other point.

Answer. The function f defined by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = x$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ is continuous at 0 but not continuous at any other point.

Problem 13. Extending functions.

(a) Suppose $f(x) = \dfrac{x^2 - 9}{x - 3}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. Yes. The domain of f is $\mathbb{R} \setminus \{3\}$ and though f is undefined at 3, $\lim_{x\to 3} f(x) = 6$. Therefore, the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = f(x)$ for $x \neq 3$ and $F(3) = 6$ is defined and continous on $\mathbb{R}$.

(b) Suppose $f(x) = \dfrac{|x|}{x}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. No. Note that $f(x) = 1$ for $x > 0$ and $f(x) = -1$ for $x < 0$. While f is undefined at 0, there is no way to define it to make it continuous since $\lim_{x\to 0} f(x)$ does not exist.

(c) Suppose $f(x) = \dfrac{\sin(x)}{x}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. Yes. The domain of f is $\mathbb{R} \setminus \{0\}$ and though f is undefined at 0, $\lim_{x\to 0} f(x) = 1$. Therefore, the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = f(x)$ for $x \neq 0$ and $F(0) = 1$ is defined and continous on $\mathbb{R}$.

(d) Suppose $f(x) = \sin\left(\frac{1}{x}\right)$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. No. Since f is defined for all $x \neq 0$, in order to extend it to a continuous function on $\mathbb{R}$, we'd have to define $F(0) = \lim_{x\to 0} f(x)$. But this is impossible since $\lim_{x\to 0} f(x)$ does not exist.

(e) Suppose $f(x) = x \sin\left(\frac{1}{x}\right)$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. Yes. The domain of f is $\mathbb{R} \setminus \{0\}$ and though f is undefined at 0, $\lim_{x \to 0} f(x) = 0$. Therefore, the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = f(x)$ for $x \neq 0$ and $F(0) = 0$ is defined and continous on $\mathbb{R}$.

(f) Suppose

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x < 1, \\ 5x - 1 & \text{if } x > 1 \end{cases}$$

Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. Yes. The domain of f is $\mathbb{R} \setminus \{1\}$ and though f is undefined at 1, $\lim_{x \to 1} f(x) = 4$. Therefore, the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = f(x)$ for $x \neq 0$ and $F(1) = 4$ is defined and continous on $\mathbb{R}$.

(g) Suppose $f : \mathbb{Q} \to \mathbb{R}$ is given by $f(r) = 0$ for all $r \in \mathbb{Q}$. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. Sure. Let $F(x) = 0$ for all $x \in \mathbb{R}$. Then $F(x) = f(x)$ for all $x \in \mathbb{Q}$ and F is continuous at every point.

(h) Suppose $f : \mathbb{Q} \to \mathbb{R}$ is given by $f(r) = \frac{1}{q}$ if $r = \frac{p}{q}$ in lowest terms. Does there a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f?

Answer. No. Any function F satisfying $F(x) = f(x)$ cannot be continuous. To see, for example, that F is not continuous at $\frac{1}{2}$, observe that every neighborhood of $\frac{1}{2}$ contains numbers from the set

$$\left\{ \frac{1}{2} + \frac{1}{p} : p \text{ is prime} \right\} = \left\{ \frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{3}, \frac{1}{2} + \frac{1}{5}, \frac{1}{2} + \frac{1}{7}, \frac{1}{2} + \frac{1}{11}, \frac{1}{2} + \frac{1}{13}, \ldots \right\}.$$

The value of f on these numbers is

$$\left\{ \frac{1}{(2)(2)}, \frac{1}{(2)(3)}, \frac{1}{(2)(5)}, \frac{1}{(2)(7)}, \ldots \right\}$$

none of which lie in the neighborhood of $f\left(\frac{1}{2}\right) = \frac{1}{2}$ of radius $\frac{1}{4}$.

consider the neighborhood of $f(\frac{1}{2}) = \frac{1}{2}$ of radius $\frac{1}{4}$. Note that in every neighborhood of $\frac{1}{2}$ contains numbers from the set

$$\left\{ \frac{1}{2} + \frac{1}{p} : p \text{ is prime} \right\} = \left\{ \frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{3}, \frac{1}{2} + \frac{1}{5}, \frac{1}{2} + \frac{1}{7}, \frac{1}{2} + \frac{1}{11}, \frac{1}{2} + \frac{1}{13}, \ldots \right\}.$$

The value of f on these numbers is

$$\left\{ \frac{1}{(2)(2)}, \frac{1}{(2)(3)}, \frac{1}{(2)(5)}, \frac{1}{(2)(7)}, \dots \right\}$$

all of which are outside the neighborhood of $\frac{1}{2}$ of radius $\frac{1}{4}$.

(i) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous at every point of $[a, b]$. Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f (in fact, there are infinitely many such F).

Answer. There are many ways to extend f continuously (just think of continuing the graph). The only requirement is that F be continuous at $x < a$ and $x > b$ and that $\lim_{x \to a^-} F(x) = f(a)$ and $\lim_{x \to b^+} F(x) = f(b)$. Here's one possibility:

$$F(x) = \begin{cases} f(a) & \text{for } x < a \\ f(x) & \text{for } a \leq x \leq b \\ f(b) & \text{for } x > b. \end{cases}$$

(j) Give an example to show that if $f : (a, b) \to \mathbb{R}$ is continuous at every point of (a, b) there need not exist a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f.

Answer. Let $f : (0, 1) \to \mathbb{R}$ be given by $\frac{1}{x}$. There does not exist a function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for all x in the domain of f that is continuous at 0 since if F were continuous at 0 it would satisfy

$$F(0) = \overset{+}{\lim_{x \to 0}} F(x) = \overset{+}{\lim_{x \to 0}} f(x) = +\infty.$$